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Introduction to Dynamical Systems and Ergodic Theory

PART 1

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$f(x) = \sqrt{x}$

$f(2) = \sqrt{2} = 1.41421356...$
$f \circ f(2) = f^2(2) = \sqrt{\sqrt{2}} = 2^{1/4} = 1.189207...$
$f^3(2) = 1.090507...$
$f^4(2) = 1.04427...$
...
$f^{100}(2) = 1.0000... + \varepsilon$

$f^n(2) = 2^{1/2^n} \to 1$, and $f(1) = \sqrt{1} = 1$,
1 is a fixed point of f.

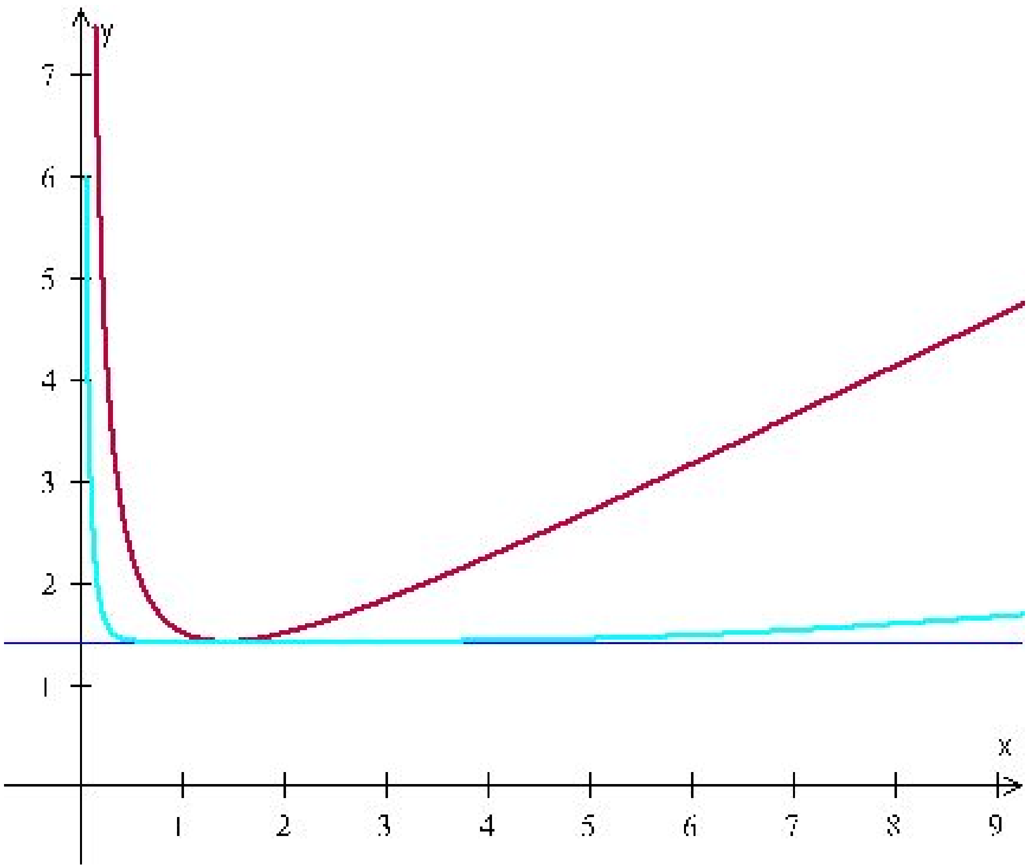
Recall Banach's fixed point theorem:

D.: Suppose (X, d) is a metric space. If for $f : X \to X$ there exists $\gamma \in (0, 1)$ such that for all $x, y \in X$
$d(f(x), f(y)) \leq \gamma d(x, y)$ then f is a contraction.

T.: *Suppose that f is a contraction defined on the complete metric space (X, d). Then f has exactly one fixed point, x_∞ and for any $x_0 \in X$ we have $f^n(x_0) \to x_\infty$.*

$f(x) = \sqrt{x}$ is NOT a contraction on $X = (1, +\infty)$. (Prove it.)

Find an interval $I \subset (1, +\infty)$ such that f maps I into itself, $2 \in I$ and f is a contraction on I.



Greek method of computing $\sqrt{2} \approx 1.414213562...$:

$1.2^2 = 1.44 < 2 < (\frac{2}{1.2})^2 = 2.77...$

$\Rightarrow \dfrac{1.2 + \frac{2}{1.2}}{2} = 1.433...$

is a better approximation,

$\dfrac{(1.433...) + \frac{2}{1.433...}}{2} = 1.414341085$

is even better.

We take the sequence $x_1 = 1.2$,

$x_{k+1} = f(x_k) = \dfrac{x_k + \frac{2}{x_k}}{2}$.

$x_k \to \sqrt{2}$.

$f(\sqrt{2}) = \sqrt{2}$ is a globally attracting fixed point in $(0, +\infty)$.

(On the figure

$f(x)$, $y = \sqrt{2}$, $f^3(x)$.)

Given X a “phase space”, “state space” and
a transformation of X into itself “law of nature” $f : X \to X$
we would like to understand the dynamics, the long term behavior of this (dynamical) system.

If f is a contraction of a complete metric space everything is very simple. Every point converges to this fixed point. We will see that things can get much more complicated.

i.) If X is a differentiable manifold and T is a (sufficiently smooth) diffeomorphism (or at least a differentiable transformation) then we speak about differentiable (smooth) dynamics.
ii.) If X is a topological, or metric space and T is a homeomorphism (or at least a continuous transformation) then we speak about topological dynamics.
iii.) If X is a measure space $(X, \mathcal{B}, \mu)$ and T is a measure preserving transformation ($\mu(T^{-1}A) = \mu(A)$) then we speak about Ergodic theory.
ergod+odos=energy-path

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Sometimes the same system can be an example of all three types.

Example: Circle rotations: Let $X = \mathbb{T}$ be the circle of unit length $= [0, 1)$ $= \mathbb{R}/\mathbb{Z} =$ the reals modulo 1.
Given $\alpha \in \mathbb{R}$ let $T_\alpha : \mathbb{T} \to \mathbb{T}$, or $[0, 1) \to [0, 1)$ be
$T_\alpha(x) = x + \alpha$ (modulo 1) $= \{x + \alpha\}$.
If we think of $\mathbb{T}$ as the normalized unit circle in $\mathbb{C}$ then $T_\alpha e^{2\pi i\phi} = e^{2\pi i(\phi+\alpha)}$.
T_α clearly smooth (and hence continuous) on the manifold $\mathbb{T}$.
If we consider $(\mathbb{T}, \mathcal{L}, \lambda)$, where λ=Lebesgue-measure and $\mathcal{L}$=Lebesgue measurable sets, then
T_α is measure preserving $(\lambda(T_\alpha^{-1}(A)) = \lambda(A - \alpha) = \lambda(A)$ for $\forall A \in \mathcal{L})$.

Ergodic theory = study of actions of (semi)groups on measure spaces

$T : X \to X$, we consider $\{T^n : n \in \mathbb{Z}_{\geq 0}\}$ the semi-group action of $\mathbb{Z}_{\geq 0}$.
We have $(\star)\quad T^{n+m} = T^n T^m,\ \forall n, m \in \mathbb{Z}_{\geq 0}$.
If T is invertible we can consider the group action $\{T^n : n \in \mathbb{Z}\}$, having $(\star)$ for all $n, m \in \mathbb{Z}$.
In these cases we have "discrete time", "snapshots" of the system. We work with discrete dynamical systems.

One can consider continuous dynamical systems, flows (coming usually from autonomous differential equations).
These are semigroup actions of $\mathbb{R}_{\geq 0}$, or in the invertible case of $\mathbb{R}$:
$T_t : X \to X$, $T_t : t \in \mathbb{R}$, $T_{s+t} = T_s T_t$ for all $s, t \in \mathbb{R}$.

One can consider other group actions
for example Z^2-actions, $\{T_g : g \in \mathbb{Z}^2\}$,
or in general Z^d-actions, $\{T_g : g \in \mathbb{Z}^d\}$.
If $\alpha \neq \beta$ one can consider the $\mathbb{Z}^2$-action, $T_{\alpha,\beta}^{(n,m)} : \mathbb{T} \to \mathbb{T}$, $(n, m) \in \mathbb{Z} \times \mathbb{Z}$
$T_{\alpha,\beta}^{(n,m)} x = \{x + n\alpha + m\beta\}$.

Origin from Physics

k particles in $\mathbb{R}^3$,
positions (in generalized coordinates) q_i, momenta p_i, $i = 1, ..., k$.
Phase space $X = \mathbb{R}^{6k}$.
The Hamilton function $H(p, q) = K(p) + U(q)$
where $K(p)$ is the kinetic energy, and $U(q)$ is the potential energy.
Hamilton's equations:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}.$$

Connection to high-school Physics:

$$mv = \frac{\partial \frac{1}{2} mv^2}{\partial v} = \frac{\partial K(p) + U(q)}{\partial p}, \text{ and } F = ma = \frac{\partial mv}{dt} = -\frac{\partial K(p) + U(q)}{\partial q}.$$

Energy surface $H^{-1}(e)$, Hamiltonian H is constant on solution curves (preservation of energy).
Liouville's theorem: The Hamiltonian flow, T_t (the solution flow from the H. equations) preserves the Lebesgue-measure on $\mathbb{R}^{6k}$.
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Boltzmann's ergodic hypothesis: "$\{T_t(x) : t \in \mathbb{R}\}$ "equals" the energy surface $H^{-1}(e)$."

Boltzmann gave the name Ergodic, recall ergon=work, energy, odos=path in Greek.

Boltzmann's ergodic hypothesis is false.

We can only hope for density of $\{T_t(x) : t \in \mathbb{R}\}$ on the energy surface.

Boltzmann also conjectured the hypothesis for the equality of time means and phase (space) means.

D.: A measure space $(X, \mathcal{B}, \mu)$ is the triple consisting of the phase space X, the σ-algebra of the measurable sets $\mathcal{B}$ and a probability measure μ, (this means $\mu(X) = 1$).
Sometimes we work with finite measures $\mu(X) < +\infty$, but these can always be normalized.
Infinite Ergodic theory is different in that case σ-finite measure spaces like $(\mathbb{R}, \mathcal{L}, \lambda)$ can be considered.

D.: Given two measure spaces $(X_1, \mathcal{B}_1, \mu_1)$ and $(X_2, \mathcal{B}_2, \mu_2)$ the transformation $T : X_1 \to X_2$ is measure preserving if
$\mu_1(T^{-1}A) = \mu_2(A)$ holds for all $A \in \mathcal{B}_2$.

T.: (Poincaré's Recurrence Theorem) Let $T : X \to X$ be meas. pres. on the prob. space $(X, \mathcal{B}, \mu)$. If $\mu(A) > 0$ then μ almost every $x \in A$ returns to A.

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Proof.: $F \stackrel{\text{def}}{=} A \setminus \bigcup_{k=1}^{\infty} T^{-k} A$

(these are those points which never return to A).

$F = A \cap T^{-1}(X \setminus A) \cap T^{-2}(X \setminus A) \cap \ldots$

$F \cap T^{-n} F = \emptyset$ for all $n \geq 1$ $\Rightarrow T^{-k} F \cap T^{-(n+k)} F = \emptyset$ for all $n \geq 1$, $k \geq 0$

$\Rightarrow F$, $T^{-1}F$, $T^{-2}F$, ... are pairwise disjoint.

T is measure preserving $\Rightarrow \mu(T^{-k}F) = \mu(F)$

$\mu(X) < +\infty \Rightarrow \mu(F) = 0$. ■

$(\mathbb{R}, \mathcal{L}, \lambda)$ with $Tx = x + 1$ gives an example that Poincaré's Recurrence Theorem is not true on σ-finite measure spaces.

No point returns to say $A = [0, 1)$.

The three and n-body problem

The problem of finding the general solution to the motion of more than two orbiting bodies in the solar system known originally as the three-body problem and later the n-body problem ($n \geq 2$). In honour of his 60th birthday, Oscar II, King of Sweden, advised by Gösta Mittag-Leffler, established a prize for anyone who could find the solution to the problem. The prize was finally awarded to Poincaré, even though he did not solve the original problem. (The first version of his contribution even contained a serious error). The version finally printed contained many important ideas which led to the theory of chaos. He found that there can be orbits that are nonperiodic, and yet not forever increasing nor approaching a fixed point. (source Wikipedia)

Poincaré called the recurrence theorem: "the stability theorem à la Poisson".

Ex.1.: Suppose our space X is the disjoint union of two circles $X = \mathbb{T}_1 \cup \mathbb{T}_2$.
We consider the normalized Lebesgue measure on the union.

Suppose $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$. Define $Tx = \begin{cases} x + \alpha & \text{if } x \in \mathbb{T}_1 \\ x + \beta & \text{if } x \in \mathbb{T}_2. \end{cases}$

Then $T^{-1}(\mathbb{T}_1) = \mathbb{T}_1$ is an invariant set of of measure 1/2.

Ex.2.: Suppose $X = \mathbb{T} \times [0, 1]$ with the Lebesgue measure on the product.
Let $T(x, \alpha) = (x + \alpha, \alpha)$.
Then we have continuum many T invariant sets.
The invariant sets $X_\alpha = \{(x, \alpha) : x \in \mathbb{T}\}$ are all of zero measure, but one can find invariant sets of positive but not of full measure as well, for example $X^* = \mathbb{T} \times [0, 1/2]$ is also invariant and is of measure 1/2.

D.: Suppose $(X, \mathcal{B}, \mu)$ is a prob. space. A meas. pres. tr. T of $(X, \mathcal{B}, \mu)$ is ergodic if for all $A \in \mathcal{B}$, $T^{-1}A = A$ implies $\mu(A) = 0$, or $\mu(A) = 1$.

Example 1 is not an ergodic tr. but the space can be split into two components on which T is ergodic.
Example 2 is more delicate. This space splits into continuum many “ergodic” components each of measure zero and one needs to “disintegrate” the original masure to obtain suitable ergodic measures on the components.

T.: (L^p Ergodic Thm. of von Neumann) *Let* $1 \le p < \infty$, T *be a meas. pres. tr. on the prob. space* $(X, \mathcal{B}, \mu)$. *If* $f \in L^p(\mu)$ *then there exists* $f^* \in L^p(\mu)$ *such that* $f^* \circ T = f^*$ *a.e. and*
$$||\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) - f^*(x)||_p \to 0.$$

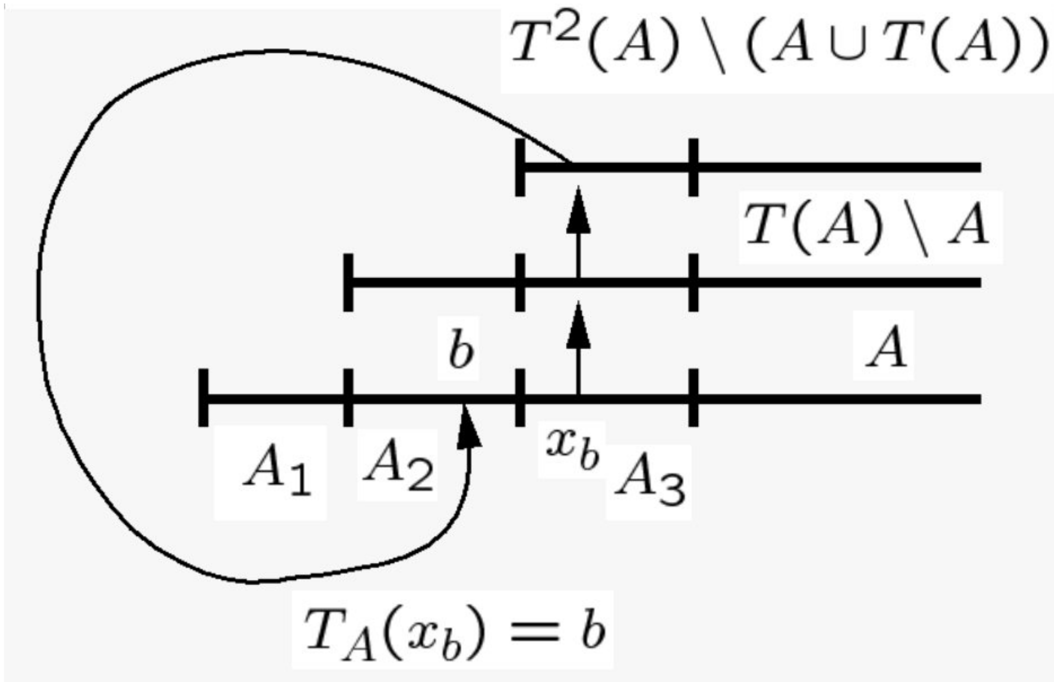
T.: (Birkhoff's Ergodic Theorem) *Suppose* $(X, \mathcal{B}, \mu)$ *is a prob. meas. space and* $T : X \to X$ *is a meas. pres. tr., moreover* $f \in L^1(\mu)$. *Then*
$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) \to f^*(x) \in L^1(\mu) \text{ a.e.}$$
$f^* \circ T = f^*$ *a.e. and* $\int_X f^* d\mu = \int f d\mu$.
If T *is ergodic then* $(\star)$ $\quad f^* = \int f d\mu$ *a.e.*

In the ergodic case $(\star)$ means that the "Boltzmann time average" $\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x)$ converges a.e. to the "space average" $\int_X f(x) d\mu(x)$.

Next we suppose that T is an invertible measure preserving transformation on the prob. meas space $(X, \mathcal{B}, \mu)$.
For invertible transfomations $\mu(T(A)) = \mu(T^{-1}(T(A))) = \mu(A)$, which means that T^{-1} is also measure preserving.

D.: Let T be a meas. pres. tr., and $A \in \mathcal{B}$ with $\mu(A) > 0$ be fixed.
By Poincaré's recurrence theorem
$n_A(x) = \inf\{n \geq 1 : T^n x \in A\}$ is finite for μ a.e. $x \in A$.
Consider $(X, \mathcal{B}|_A, \mu|_A)$ where $\mu_A(B) = \frac{\mu(B)}{\mu(A)}$, for any $B \in \mathcal{B}|_A = \{B' \cap B : B' \in \mathcal{B}\}$.
The induced ("derivative") transformation $T_A : A \to A$ is given by
$T_A(x) = T^{n_A}(x)$.

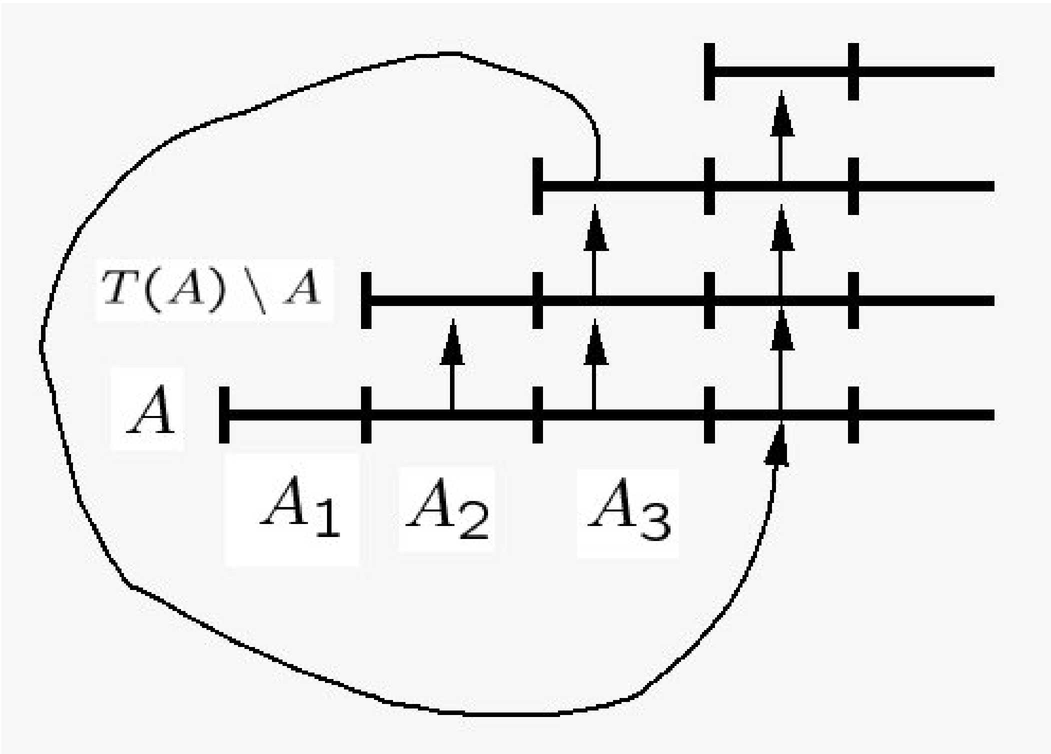
Most of the time we ignore sets of measure zero so it is not a problem that T_A is defined only $\mu|_A$ a.e.
E.g. the a.e. version of the definition of ergodicity is this:
D.: Suppose $(X, \mathcal{B}, \mu)$ is a prob. space. A meas. pres. tr. T of $(X, \mathcal{B}, \mu)$ is ergodic if for all $A \in \mathcal{B}$,
$\mu(T^{-1}A \Delta A) = \mu((T^{-1}A \setminus A) \cup (A \setminus T^{-1}A)) = 0$
implies $\mu(A) = 0$, or $\mu(A) = 1$.



Prop.: T_A is measure preserving.

Proof.: $A_n \stackrel{\text{def}}{=} \{x \in A : n_A(x) = n\}$
Suppose $B \subset A$,
for a.e. $b \in B$ select x_b such that $T^{n_A(x_b)}(x_b) = b$,
since T is meas. pres. and invertible T^{-1} is also meas pres. and invertible,
hence for a.e. $b \in B$ there is x_b.
Set $B_n = \{b \in B : n_A(x_b) = n\}$.

Then $\mu(T_A^{-1}(B_n)) = \mu(B_n)$ for all n.
If $n \neq m$ then $T_A^{-1}(B_n)$ and $T_A^{-1}(B_m)$ are disjoint.
$\mu(T_A^{-1}(B)) = \mu(T_A^{-1}(\cup B_n)) = \sum \mu(T_A^{-1}(B_n)) = \sum \mu(B_n) = \mu(B)$. ■



T.: Kac Lemma Suppose $(X, \mathcal{B}, \mu)$ is a prob. meas. sp. and T is an invertible ergodic meas. pres. tr. If $A \in \mathcal{B}$ with $\mu(A) > 0$ then $\int_A n_A(x) d\mu(x) = 1$.

Remark: The expected recurrence time of a point to A:

$$\int_A n_A d\mu|_A = \frac{1}{\mu(A)} \int_A n_A d\mu = \frac{1}{\mu(A)}.$$

Proof.: We use again the Kakutani skyscraper. Let $A_\infty \stackrel{\text{def}}{=} \cup_{k=0}^{\infty} T^n(A) = A \cup (TA \setminus A) \cup (T^2 A \setminus (TA \cup A)) \cup \ldots$.

Obviously, $TA_\infty \subset A_\infty$, since T^{-1} is meas. pres. $\mu(TA_\infty) = \mu(A_\infty)$

$\Rightarrow A_\infty = TA_\infty$, (modulo set of meas. zero) $\Rightarrow T^{-1}A_\infty = A_\infty$ a.e.

Since $\mu(A_\infty) > \mu(A) > 0$, by ergodicity $A_\infty = X$ a.e.

$A_n = \{x \in A : n_A(x) = n\}$.

$$\int_A n_A d\mu = \sum_{n=1}^{\infty} n \cdot \mu(A_n) = \mu(X) = 1. \blacksquare$$

Next we turn to topological dynamical systems.

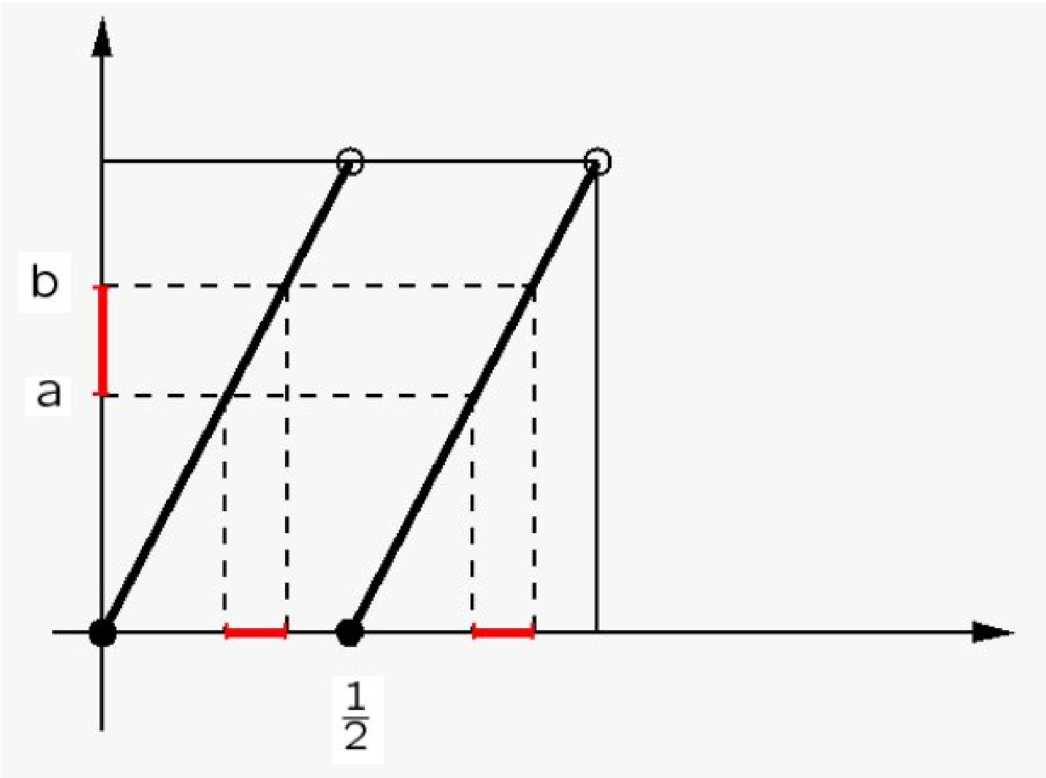
Suppose X is a metric (or a topological) space and $T : X \to X$ is a homeomorphism, (or continuous in the non-invertible case).

D.: The T-orbit , or trajectory of $x \in X$ is $\mathcal{O}_T(x) \stackrel{\text{def}}{=} \{T^n x : n \in \mathbb{Z}\}$.
In case of non-invertible T we can talk about the positive semiorbit $\mathcal{O}_T^+(x) \stackrel{\text{def}}{=} \{T^n x : n \in \mathbb{Z}_{\geq 0}\}$. In this case $\mathcal{O}_T^+(x)$ used in the next definitions instead of $\mathcal{O}_T(x)$.
D.: A $T : X \to X$ topological dynamical system is topologically transitive if $\exists x \in X$ such that its orbit, $\mathcal{O}_T(x)$ is dense in X.

D.: A $T : X \to X$ topological dynamical system is minimal if $\forall x \in X$ its orbit, $\mathcal{O}_T(x)$ is dense in X.
Exercise: Show that for irrartional α the rotation $T_\alpha : \mathbb{T} \to \mathbb{T}$, is minimal.

D.: For $T : X \to X$ denote by $P_n(T)$ the number of the set of those $x \in X$, for which $T^n x = x$. (n is not necessarily the minimal/prime period.)

The doubling map



$E_2(x) = \{2x\}$

$$= \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{2} \\ 2x - 1 & \text{if } \frac{1}{2} \le x < 1. \end{cases}$$

In complex notation $E_2(z) = z^2$, since $(e^{2\pi ix})^2 = e^{2\pi i2x}$.

E_2 is non-invertible but it preserves the Lebesgue measure, $\lambda(E_2^{-1}(A)) = \lambda(A)$ for all $A \in \mathcal{L}$. One can see it on the figure for intervals, and they generate the σ-algebra $\mathcal{L}$.

Prop.: $P_n(E_2) = 2^n - 1$, the periodic points of E_2 are dense in $\mathbb{T}$ and E_2 is topologically transitive (and obviously non-minimal).

This shows that E_2 has much more complicated dynamics, than T_α.

Prop.: $P_n(E_2) = 2^n - 1$, the periodic points of E_2 are dense in $\mathbb{T}$ and E_2 is topologically transitive (and obviously non-minimal).

Proof.: Using the complex representation of E_2:
$E_2^n(z) = z \Leftrightarrow z^{2^n} = z \Leftrightarrow z^{2^n-1} = 1$
$\Rightarrow$each $(2^n - 1)$st root of unity corresponds to a point with $z^{2^n} = z$ there are $2^n - 1$ such equally spaced points $\Rightarrow$the result about number and density.
Topological transitivity: Consider $x \in [0,1) = \mathbb{T}$ in base-2,
$x = \sum_{i=1}^{\infty} a_i 2^{-i} = \Xi[a_1 a_2 ...]$,
where $a_i \in \{0,1\}$ and $\forall N > 0$, $\exists n > N$ s.t. $a_n = 0$ (this way we have unique repr.).
Then $E_2(x) = \left\{ a_1 + \sum_{i=2}^{\infty} a_i 2^{-i+1} \right\} = \sum_{i=1}^{\infty} a_{i+1} 2^{-i} = \Xi[a_2 a_3 ...]$.
$\Rightarrow E_2$ acts on the binary digits of x as the one sided shift: delete the first entry and then move each entry to the left. Notation $\sigma[a_1 a_2 ...] = [a_2 a_3 ...]$.
(From this approach one can see the periodic points as well, there are $2^p - 1$ many $0-1$-sequences of length p which are allowed, $[\underbrace{1...1}_{p}...]$ is not allowed.)

Topological transitivity: Consider $x \in [0,1) = \mathbb{T}$ in base-2,
$x = \sum_{i=1}^{\infty} a_i 2^{-i} = \Xi[a_1 a_2 ...]$,
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For the top. transitivity we need x with a dense orbit:
$x \overset{\text{def}}{=} \Xi[\underbrace{0\,1}_{\text{len.}1}\ \underbrace{0\,0\,0\,1\,1\,0\,1\,1}_{\text{all str. of length }2}\ \underbrace{0\,0\,0\,0\,0\,1\,...\,1\,1\,1}_{\text{all strings of length }3}...] = \Xi[\omega]$, this x is allowed and for any binary "base interval" $J = \Xi[a_1 ... a_j *)$, $\exists k$ s.t. the first j entries of $E_2^k(x) = \Xi(\sigma^k[\omega])$ equal $a_1 ... a_j$, i.e. $E_2^k(x) \in J$. ■

D.: Given a $T: X \to X$ top. dyn. sys. and $x \in X$ the ω-limit set of x (and the α-limit set) is

$$\omega(x) \stackrel{\text{def}}{=} \{y \in X : \ \exists n_i \to +\infty \text{ s.t. } T^{n_i}x \to y\} = \bigcap_{n=0}^{\infty} \text{cl}\left(\bigcup_{m \geq n} T^m x\right)$$

$$\alpha(x) \stackrel{\text{def}}{=} \{y \in X : \ \exists n_i \to -\infty \text{ s.t. } T^{n_i}x \to y\} = \bigcap_{n=0}^{-\infty} \text{cl}\left(\bigcup_{m \leq n} T^m x\right).$$

It is clear that $\omega(x)$ and $\alpha(x)$ are closed.

So far we have seen examples when $\omega(x)$ is

one point when we have an attracting fixed point;

union of finitely many points if x is a periodic point;

the whole space X if T is topologically transitive and $\mathcal{O}^+(X)$ is dense in X.

Other cases are also possible: It is possible that $\omega(x)$ is the Cantor-triadic set C_3.

Exercise: $x \in C_3$ iff x has an expansion in base-3 that do not contain the digit 1, in fact $\forall x_3 \in C_3$ there is unique triadic expansion $0.a_1a_2...$ with $a_i \in \{0,2\}$.

Let $E_3 : \mathbb{T} \to \mathbb{T}$ be given by
$E_3(x) = \{3x\}$.

Prop.: *For E_3, C_3 is E_3 invariant, $E_3(C_3) \subset C_3$, and C_3 contains a dense orbit* $\Rightarrow \exists x \in \mathbb{T}$ *s.t.* $\omega(x) = C_3$.

Proof.: E_3 acts on the trenary digits of x as the shift $\Rightarrow E_3(C_3) \subset C_3$

$$x \overset{\text{def}}{=} 0.\underbrace{0\,2}_{\text{len.}1}\ \underbrace{0\,0\,0\,2\,2\,0\,2\,2}_{\text{all str. of length }2}\ \underbrace{0\,0\,0\,0\,0\,2 \ldots 2\,2\,2}_{\text{all strings of length }3}\ldots \quad \blacksquare$$

Symbolic Dynamical Systems

Suppose $N \geq 2$,

$\Omega_N = \{\omega = (..., \omega_{-1}, \omega_0, \omega_1, ...) : \omega_i \in \{0, 1, ..., N-1\},\ i \in \mathbb{Z}\}$,

the space of bi-infinite sequences on N symbols.

$\Omega_N^R = \{\omega = (\omega_0, \omega_1, ...) : \omega_i \in \{0, 1, ..., N-1\},\ i \in \mathbb{Z}_{\geq 0}\}$,

the space of (right)-infinite sequences on N symbols.

Topology on Ω_N (and on Ω_N^R) take $\{0, 1, ..., N-1\}$ with the discrete topology and consider on $\{0, 1, ..., N-1\}^{\mathbb{Z}}$, (or on $\{0, 1, ..., N-1\}^{\mathbb{Z}_{\geq 0}}$) the product topology.

(More structure: If we think of $\{0, 1, ..., N-1\}$ as a finite Abelian group $\mathbb{Z}/N\mathbb{Z}$ then Ω_N and Ω_N^R are copmpact Abelian (product) topological groups.)

Given $n_1 < n_2 < ... < n_k$ and $\alpha_1, ..., \alpha_k \in \{0, 1, ..., N-1\}$ the sets $C_{\alpha_1,...,\alpha_k}^{n_1,...,n_k} = \{\omega \in \Omega_N : \omega_{n_i} = \alpha_i,\ i = 1, ..., k\}$ are the cylinder sets, (similar def. for Ω_N^R).

One can define the topology on Ω_N, (or on Ω_N^R) by saying that the cylinder sets are open and form the base for the topology.

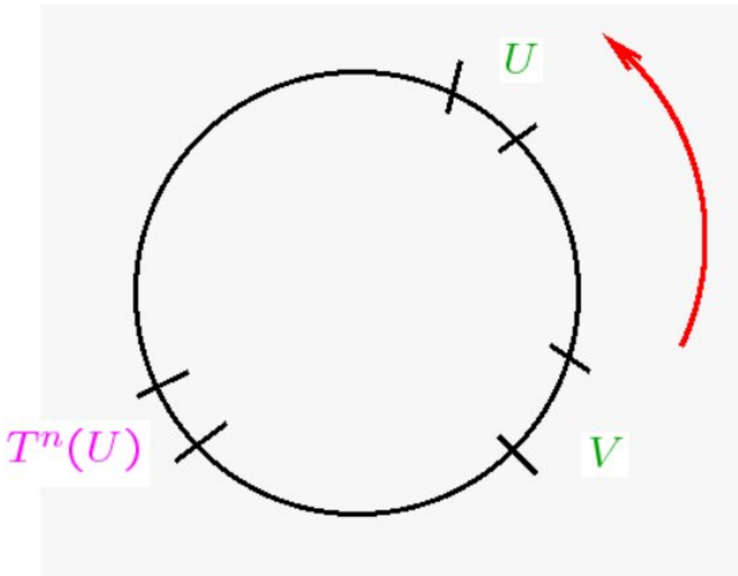
(The cylinder sets are also closed, since their complement is the union of finitely many cylinder sets.)

With $t > 1$, the metric $d_t(\omega, \omega') = \sum_{n=-\infty}^{\infty} \frac{|\omega_n - \omega_n'|}{t^{|n|}}$ generates this top.

Shift:

$\sigma_N : \Omega_N \to \Omega_N$, $\sigma_N(\omega) = (..., \omega'_0, \omega'_1, ...)$, where $\omega'_n = \omega_{n+1}$ for $\forall n$.
σ_N is one-to-one and cylinders are mapped onto cylinders $\Rightarrow \sigma_N$ is a homeomorphism.
(Ω_N, σ_N) is the topological Bernoulli shift.
The right-N-shift $\sigma_N^R : \Omega_N^R \to \Omega_N^R$ is given by
$\Omega_N^R(\omega_0, \omega_1, ...) = (\omega_1, \omega_2, ...)$.
It is a continuous, but a non-invertible map of Ω_N^R into itself.

D.: A top. dyn. sys. $T : X \to X$ is topologically mixing if for any open (non-empty) $U, V \subset X$ there exists an integer $N = N(U, V)$ such that for $\forall n > N$, $T^n(U) \cap V \neq \emptyset$.



Example 1. Irrational rotations of $\mathbb{T}$ are not top. mixing.

D.: A top. dyn. sys. $T : X \to X$ is topologically mixing if for any open (non-empty) $U, V \subset X$ there exists an integer $N = N(U, V)$ such that for $\forall n > N$, $T^n(U) \cap V \neq \emptyset$.

Prop.: *The periodic points of σ_N (and of σ_N^R) are dense in Ω_N (or in Ω_N^R), $P_n(\sigma_N) = P_n(\sigma_N^R) = N^n$ moreover σ_N and σ_N^R are top. mixing.*

Proof.: $\sigma_N^n \omega = \omega \Leftrightarrow \omega_{n+m} = \omega_m$ for $\forall m \in \mathbb{Z}$, for $\forall m \in \mathbb{Z}_{\geq 0}$ for σ_N^R.

For the density wee need to find in each cylinder set $C_{\alpha_1,\dots,\alpha_k}^{n_1,\dots,n_k}$ a periodic point.

Each cylinder in Ω_N contains symmetric cylinders $C_{\beta_{-m},\dots,\beta_m}^{-m,\dots,m} = C_{\underline{\beta}}^m$ with $\underline{\beta} = \beta_{-m}, \dots, \beta_m$.

$$\omega = (\dots \underbrace{\dots}\underbrace{\beta_{-m}, \dots, \beta_m}_{\substack{\uparrow \\ 0}}\underbrace{\beta_{-m}, \dots, \beta_m}\underbrace{\dots})$$ is a periodic point in $C_{\underline{\beta}}^m$.

(The case of Ω_N^R is similar.)

Each ω periodic by n is determined by the entries $\omega_0, \dots, \omega_{n-1}$ and these can be chosen N^n many ways.

Prop.: *The periodic points of σ_N (and of σ_N^R) are dense in Ω_N (or in Ω_N^R), $P_n(\sigma_N) = P_n(\sigma_N^R) = N^n$ moreover σ_N and σ_N^R are top. mixing.*

Topological mixing: Each cylinder contains symmetric cylinders. $\Rightarrow$
it is sufficient to show that for any $\underline{\alpha} = \alpha_{-m}, ..., \alpha_m$ and $\underline{\beta} = \beta_{-m}, ..., \beta_m$
for sufficiently large n we have $\sigma_N^n(C_{\underline{\alpha}}^m) \cap C_{\underline{\beta}}^m \neq \emptyset$.
If $n > 2m + 1$, $n = 2m + k + 1$ with $k > 0$ then let

$$\omega = (* \underbrace{\alpha_{-m}, ..., \alpha_m}_{-m \quad \overset{\uparrow}{0} \quad m} * \underbrace{\beta_{-m}, ..., \beta_m}_{n-m \quad \overset{\uparrow}{n} \quad n+m} *)$$

Then $\omega_i = \alpha_i$ if $|i| \leq m$ and
$\omega_i = \beta_{i-n}$ if $|i-n| \leq m$, that is $i = m+k+1, ..., 3m+k+1 = n-m, ..., n+m$.
Then $\omega \in C_{\underline{\alpha}}^m$ and $\sigma_N^n(\omega) \in C_{\underline{\beta}}^m$, since $\sigma_N^n(\omega) \in \sigma_N^n(C_{\underline{\alpha}}^m)$ $\Rightarrow$
$\sigma_N^n(C_{\underline{\alpha}}^m) \cap C_{\underline{\beta}}^m \neq \emptyset$.
The argument for σ_N^R is similar. ■

D.: If $T : X \to X$ are $S : Y \to Y$ are two top. dyn. sys. and there exists a homeomorphism $h : X \to Y$ such that $h \circ T = S \circ h$ then the two systems are called topologically conjugate.

Prop.: (Ω_2^R, σ_2^R) and (C_3, E_3) are topologically conjugate.

Proof.: Set $\phi(0) = 0$ and $\phi(1) = 2$.
For points in C_3 we will use again the triadic expansion.
Define $h : \Omega_2^R \to C_3$ by $h(\omega_0, \omega_1, ...) = 0.\phi(\omega_0)\phi(\omega_1)...$.
It is not difficult to see that h is a homeomorphism and $h \circ \sigma_2^R = E_3 \circ h$.
■

D.: A symbolic dynamical system, or a shift space, is the restriction of σ_N, (or of σ_N^R) onto a closed shift invariant subspace of Ω_N, (or of Ω_N^R).