TEC Estimation Using GNSS

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Workshop:

African School on Space Science: Related Applications and Awareness for Sustainable Development of the Region

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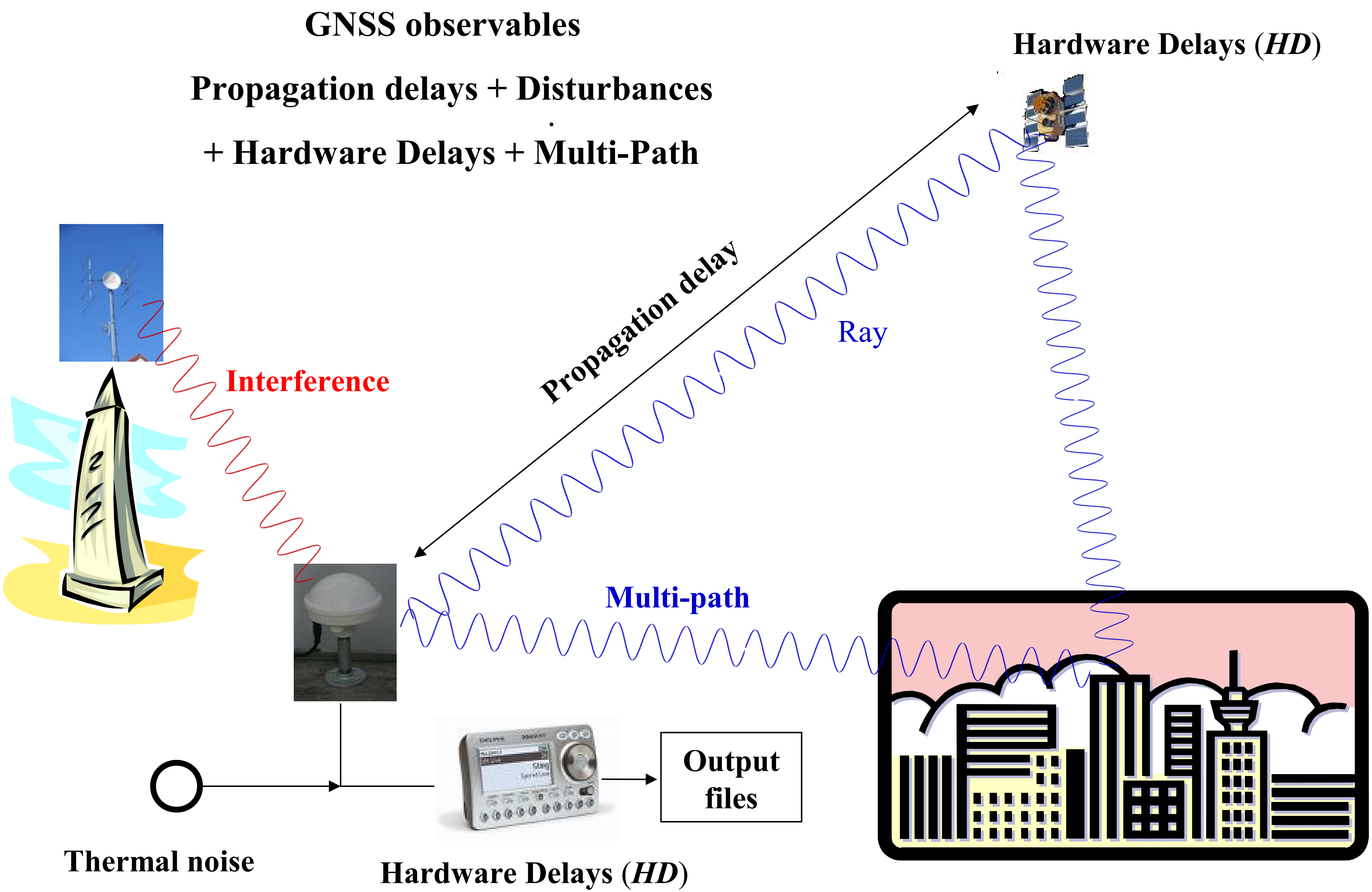
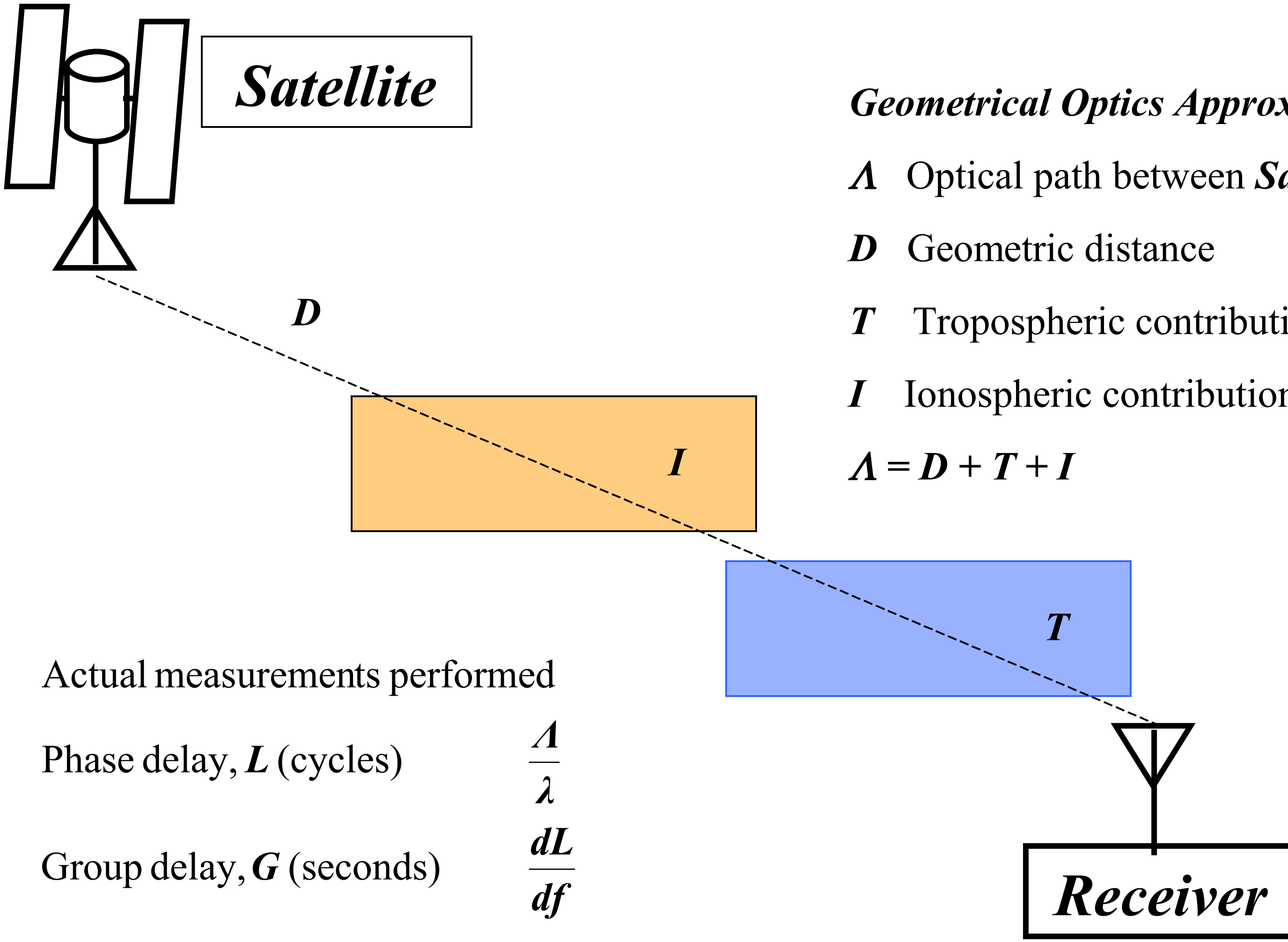

GNSS observables
Propagation delays + Disturbances
+ Hardware Delays + Multi-Path
Hardware Delays (HD)
Propagation delay
Ray
Interference
Multi-path
Output files
Thermal noise
Hardware Delays (HD)

Propagation delays are derived by the Optical path



Geometrical Optics Approximation

Λ Optical path between ***Sat*** and ***Rec***

D Geometric distance

T Tropospheric contribution

I Ionospheric contribution

$\Lambda = D + T + I$

Actual measurements performed

Phase delay, L (cycles) $\quad \dfrac{\Lambda}{\lambda}$

Group delay, G (seconds) $\quad \dfrac{dL}{df}$

Propagation Delays

Propagation and Atmospheric contributions to optical path $\boldsymbol{\Lambda}$**:**

Geometric (Distance), **T**ropospheric, **I**onospheric

$$\Lambda \quad = \quad D + T + I$$

Equivalent Group Path $\boldsymbol{P}$ = Group delay $\boldsymbol{G}$ × speed of light

$$P = G \cdot c = D + T - I$$

Refractivity $\boldsymbol{R = n - 1}$, $\boldsymbol{n}$ Index of Refraction

$$T = \int R_{atm}(s)ds \qquad I = \int R_{Iono}(s)ds \qquad R_{Iono} = -\frac{40.3 \cdot N_e}{f^2},$$

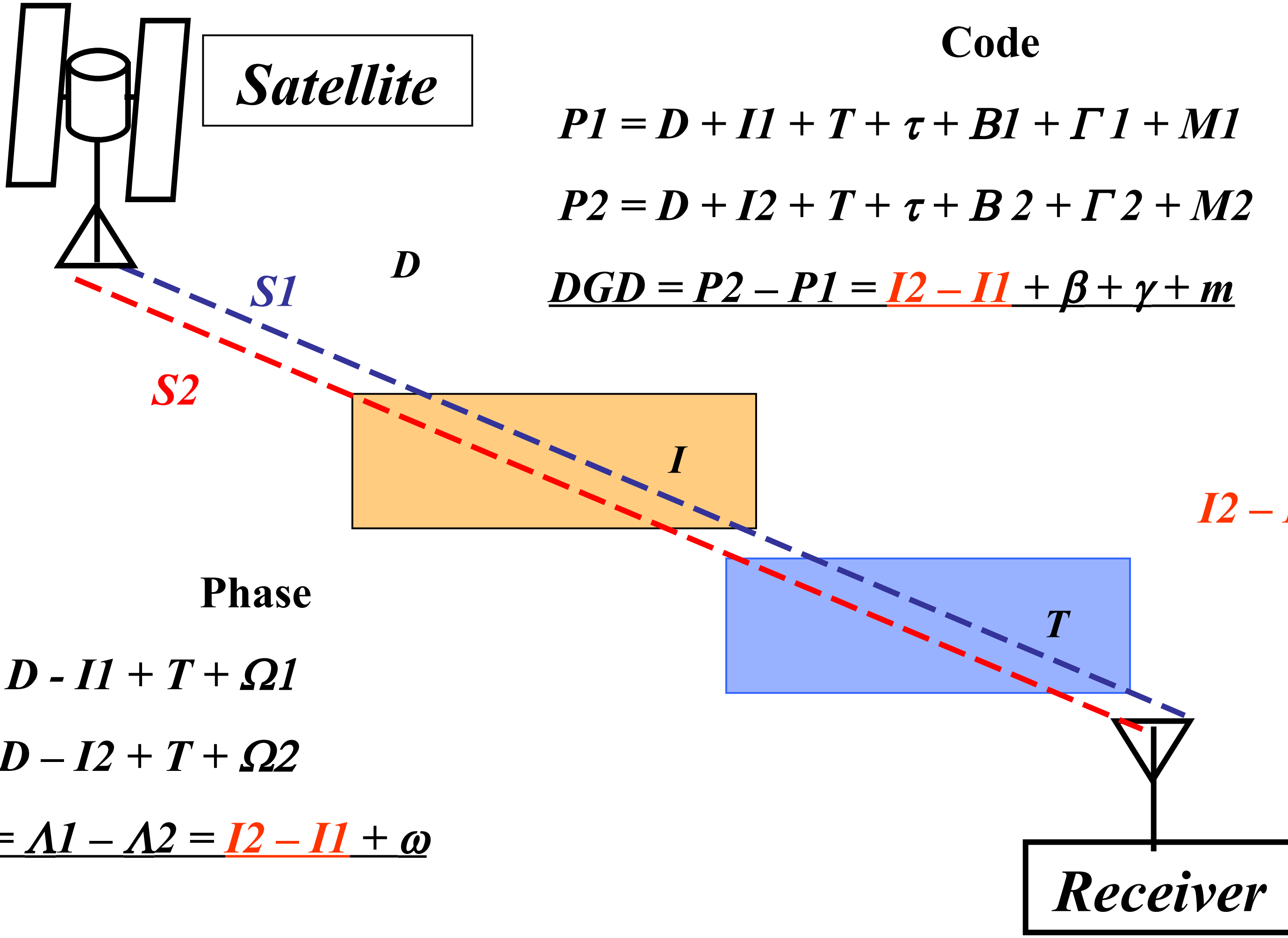
$$TEC = \int N_e(s)ds, \qquad I = -\frac{40.3 \cdot TEC}{f^2}$$

$$L = \frac{D + T + I}{\lambda} = \frac{f}{c}(D + T) - \frac{40.3TEC}{cf}$$

$$G = \frac{dL}{df} = \frac{D + T}{c} + \frac{40.3TEC}{cf^2}$$

Ionospheric Observables

Forming the *Differential Phase* (***DPD***) and the *Differential Group* (***DGD***) delays, any non dispersive term is canceled out, apart the difference of biasing terms and multi-paths (***Q1 – Q2 => q***)



Code

$$P1 = D + I1 + T + \tau + B1 + \Gamma 1 + M1$$

$$P2 = D + I2 + T + \tau + B2 + \Gamma 2 + M2$$

$$\underline{DGD = P2 - P1 = I2 - I1 + \beta + \gamma + m}$$

$$I2 - I1 = TEC$$

Phase

$$\Lambda 1 = D - I1 + T + \Omega 1$$

$$\Lambda 2 = D - I2 + T + \Omega 2$$

$$\underline{DPD = \Lambda 1 - \Lambda 2 = I2 - I1 + \omega}$$

How do ionospheric observable look like

Source of data : **(*Daily*) RINEX Files**

Typical: One data point each 30th second per satellite

For a given satellite and epoch get the phases (***L1, L2***) and the pseudo-ranges (***P1, P2***) at two carriers

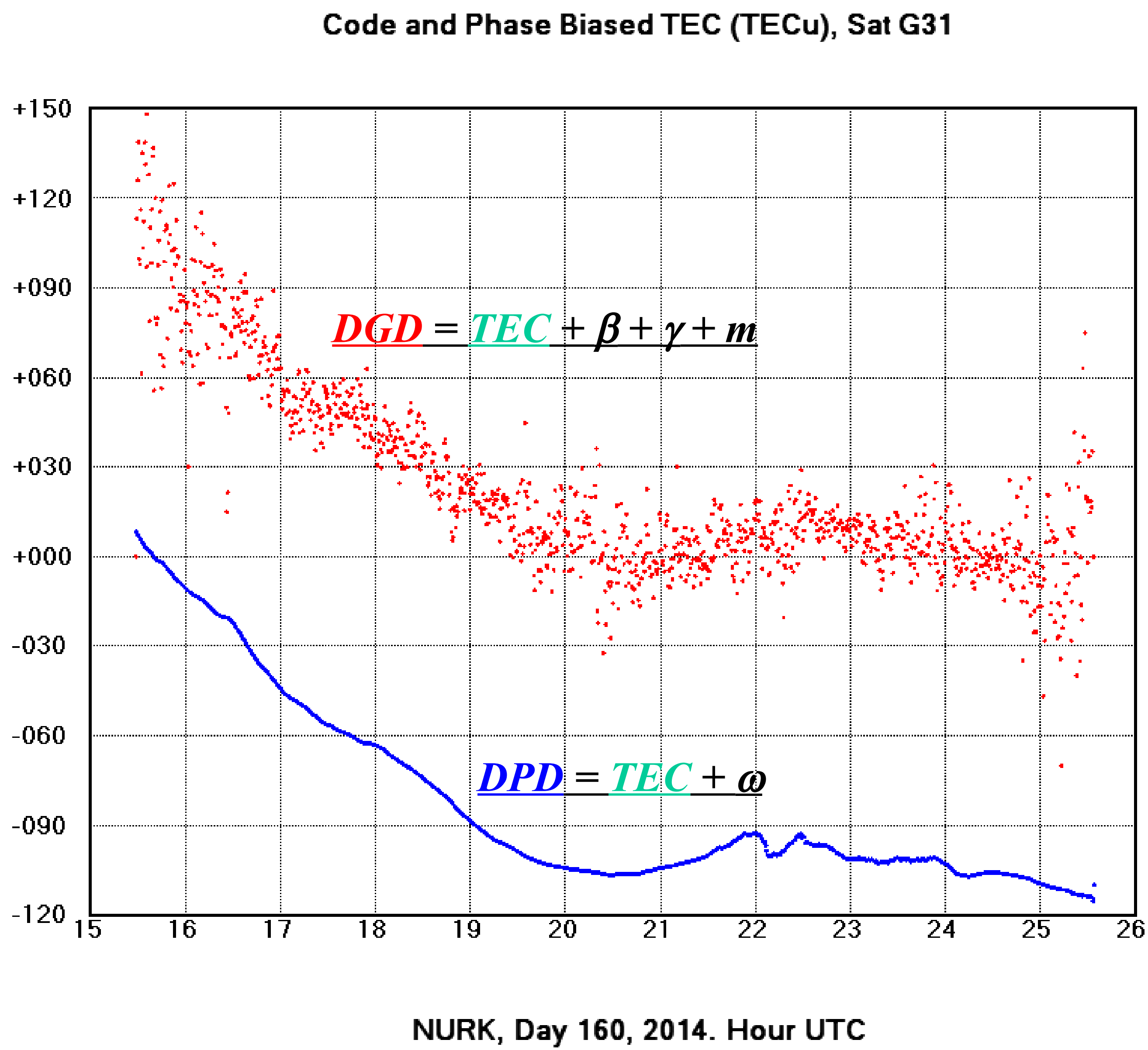
Compute $\boldsymbol{DGD = P2 - P1}$ (meters)

Compute the optical paths $\boldsymbol{\Lambda 1 = L1 \cdot \lambda 1, \Lambda 2 = L2 \cdot \lambda 2}$

Compute $\boldsymbol{DPD = \Lambda 1 - \Lambda 2}$ (meters)

Transform in ***TEC*** units (10^{16} e/m^2) getting the

Slants (Phase and Code)



Code and Phase Biased TEC (TECu), Sat G31
+150
+120
+090
+060
+030
+000
-030
-060
-090
-120
15
16
17
18
19
20
21
22
23
24
25
26
$DGD = TEC + \beta + \gamma + m$
$DPD = TEC + \omega$
NURK, Day 160, 2014. Hour UTC

Offset ω is an arbitrary quantity: can we set it in some useful way?

A new set of observables: Phase slants leveled to Code

Operator <·> is a properly selected weighted (possibly robust) average

Build, arc by arc, the <u>leveled</u> slants S_L

$$S_L = S_P - < S_P - S_C >$$

$$< S_P - S_C > = \omega - < m > - b - g$$

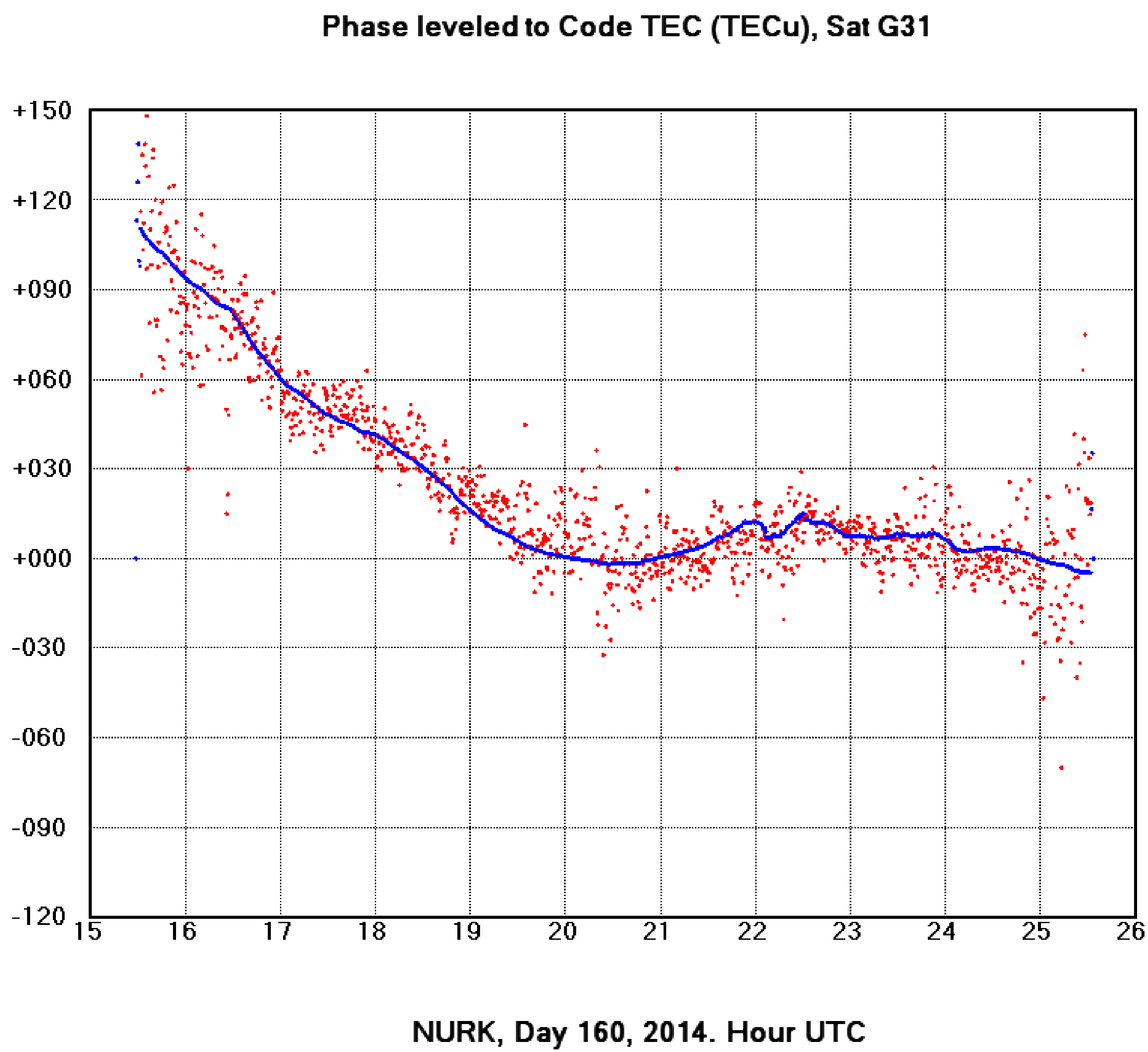
$$S_L = TEC + < m > + b + g$$

Properties of S_L

Noise is the same (neglected) of phase slants

Biased as code slants, **But**

an arc dependent constant leveling error $\lambda = < n > + < m >$


Phase leveled to Code TEC (TECu), Sat G31
+150
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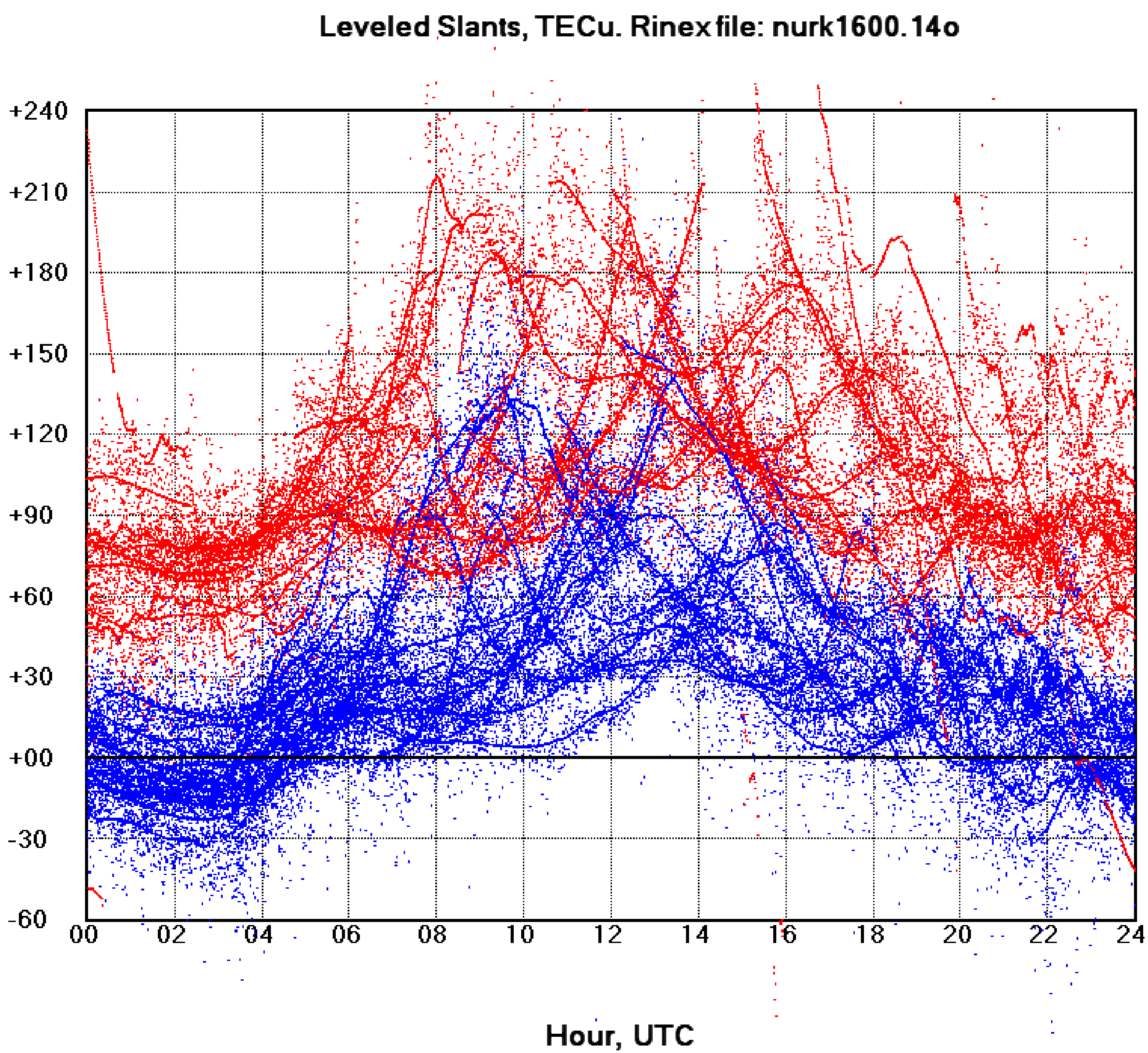
NURK, Day 160, 2014. Hour UTC

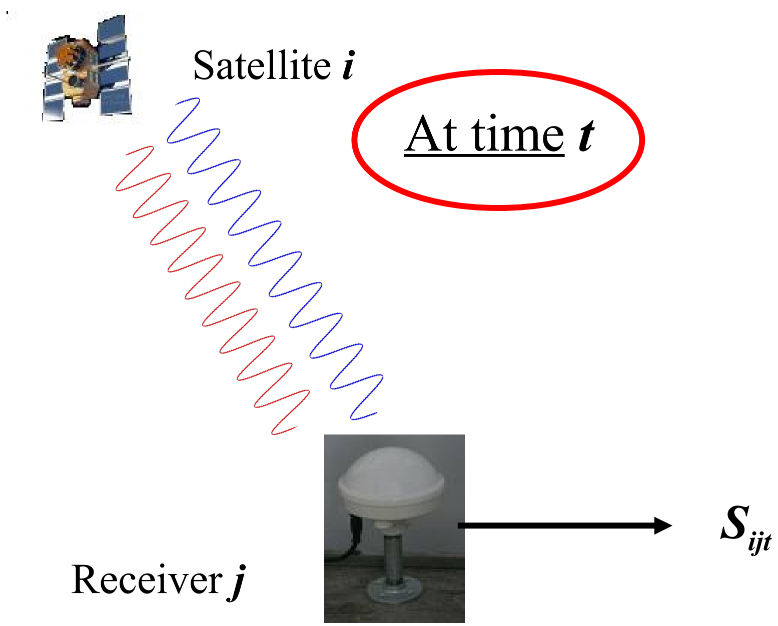

Leveled Slants, TECu. Rinex file: nurk1600.14o
+240
+210
+180
+150
+120
+90
+60
+30
+00
-30
-60
00
02
04
06
08
10
12
14
16
18
20
22
24
Hour, UTC

CALIBRATION

Rewriting the full set of observations

As already shown, properly processing GPS measurements, forming differential delays (dual frequency receiver), combining them to obtain '**<u>leveled slants</u>**', one gets slant Total Electron Content (***TEC)*** measurements affected by biasing terms $\beta_i\,,\ \gamma_j\,,\ (\lambda_{Arc})$

$$S_{ijt} = TEC_{ijt} + \beta_i + \gamma_j + (\lambda_{Arc})$$



i = 1, 2 , …,32 available GPS satellites

j = 1, .., available receivers

t all the available observation epochs (in one day or fraction, or many days)

Arc = common to all continuous observations performed by receiver j on satellite i at times contiguos to t

We bracket λ_{Arc} *because this term is disregarded in the traditional approach but basic for the proposed "arc offset" solution.*

How performing calibration

Calibration can be performed writing ***TEC*** s as functions of a set of unknown parameters $|Z|$, forming the residuals

$$\varepsilon_{ijt} = S_{ijt} - TEC_{ijt}(\,|Z|\,) + \beta_\iota + \gamma_j + (\lambda_k)$$

and estimating together the parameters $|Z|$ and unknown biases β_ι, γ_j, (λ_k) in order to minimize $\Sigma_{ijt}\,\varepsilon_{ijt}^2$

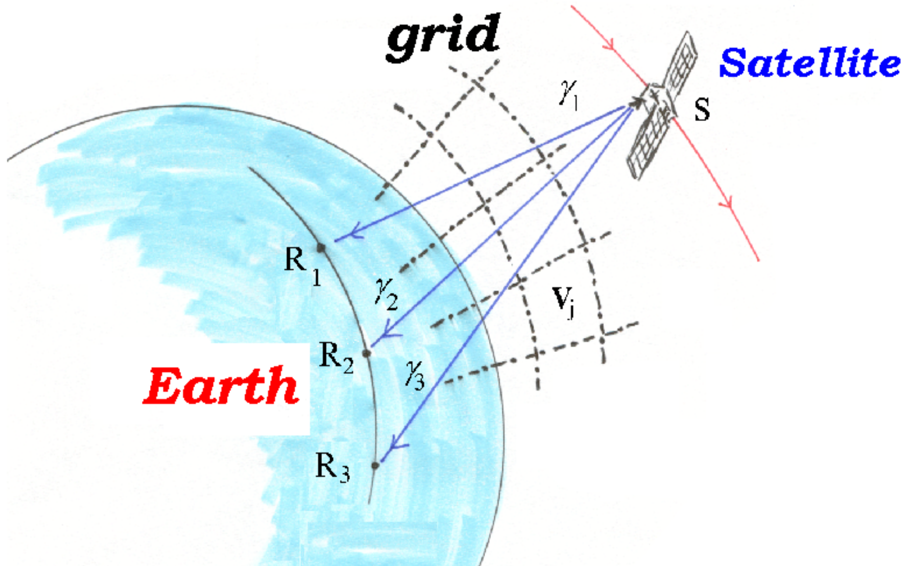
Given that

$$TEC = \int N_e(P, Q, t, Z_1, Z_{2,\dots})\,ds$$

some possible approaches are shown

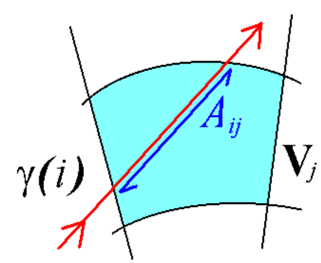
Tomography: |***Z***| parameters are the electron densities of *voxels*.

The ionosphere is divided in elements of volume (voxels) inside which N_e is constant. N_e of voxels are the unknowns. Evolution with time of N_e is considered to improve the budget unknowns/observations. Vertical behaviour of N_e is expanded in Empirical Orthogonal Functions (EOF)



$$d_i = \int_{\gamma(i)} N(\mathbf{r},t)\,ds = \sum_j x_j \int_{\gamma(i)} b_j(\mathbf{r})\,ds \equiv \sum_j A_{ij} x_j$$

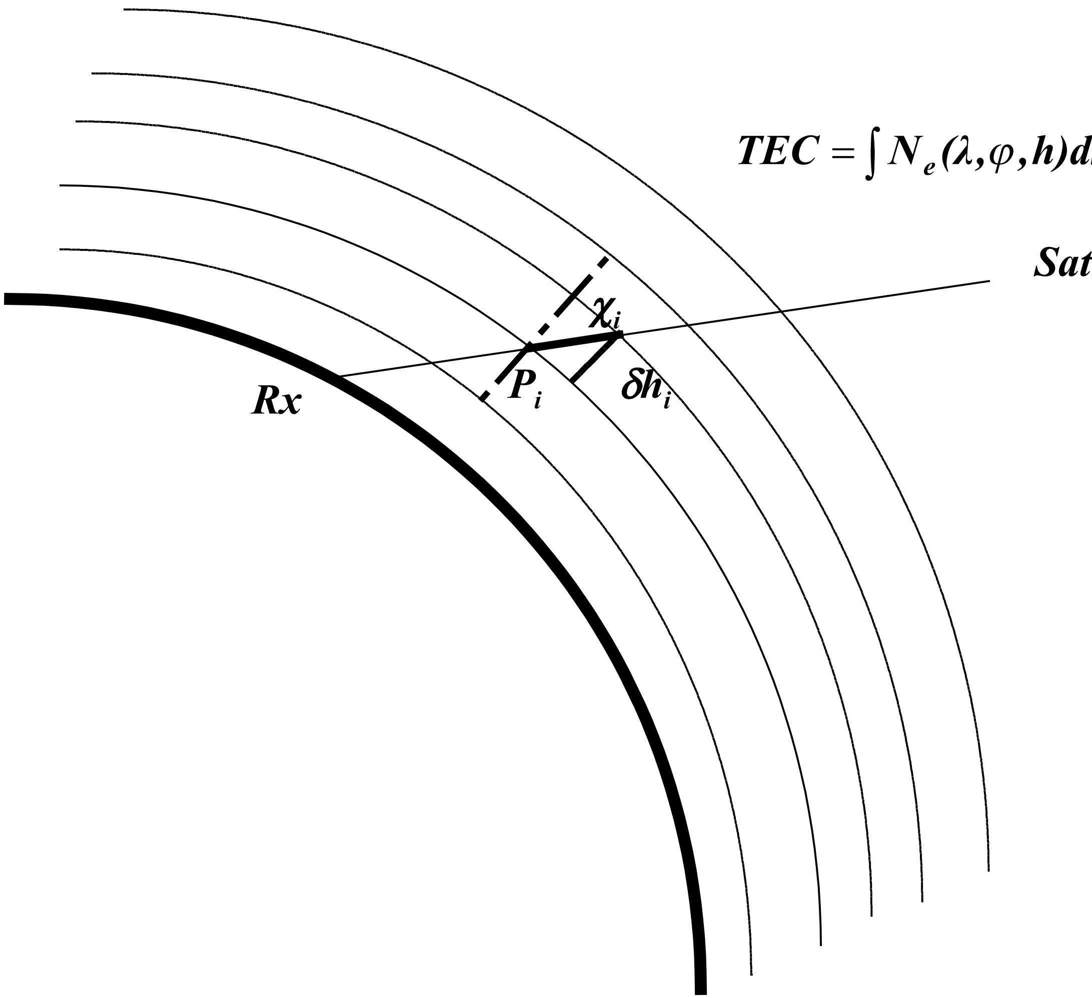
$$\underline{d} = A\underline{x} \quad \Rightarrow \quad \underline{x} = A_{SVD}^{-1}\underline{d}$$



The multi-shell method

This is the method by which numerical integration is carried out. For each shell, a suitable 2D expansion in horizontal coordinates is assumed.

$$TEC = \int N_e(\lambda,\varphi,h)ds \approx \sum N_e(P_i)\delta s_i = \sum N_e(P_i) sec\chi_i \delta h_i$$



P_i, point on the generic i^{th} shell

δh_i increment in height

δs_i increment in arc length

$\delta s_i = \delta h_i \, sec \, \chi_i$

The classical thin shell model

Reducing down the number of shells, ***and in principle the expected accuracy***,

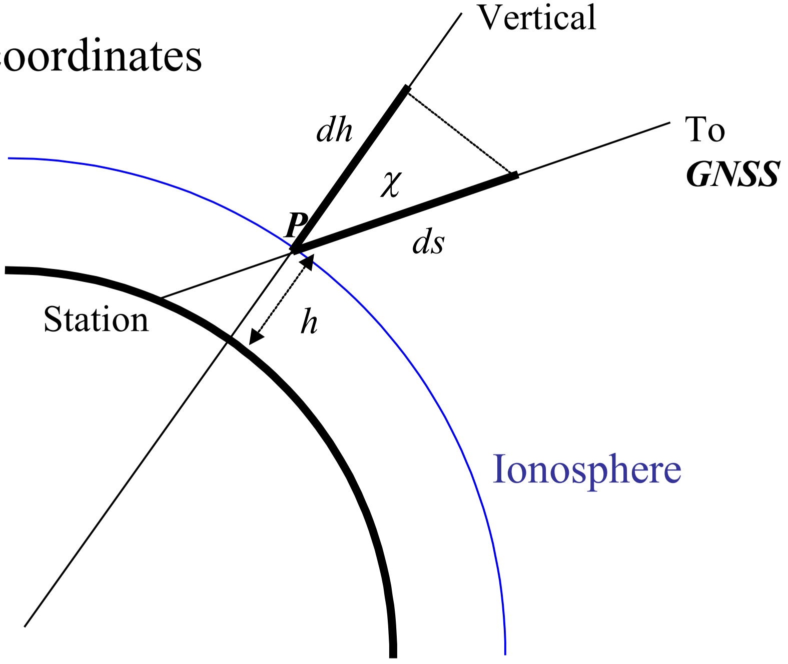
take only one **(thin) shell at some reference height *h***

$$\boldsymbol{TEC = V(P)\ sec\ \chi}$$

V(P) is the ***TEC*** along the vertical of the ionospheric point ***P***

(**V**ertical **E**lectron **C**ontent, ***VEC***)

VEC is a **2D** function of horizontal coordinates



In the following, the thin shell method will be the only one considered

The expansion of *VEC*

$$V(P_{ijt}) = \Sigma_n c^{(t)}{}_n \Psi_n\,(\Phi_{ijt}, \Lambda_{ijt})$$

Some simple example

Single-station: assume, at time t, that *VEC* is constant over the station horizon, $VEC = V_0^{(t)}$:

$$V(P_{ijt}) = V_0^{(t)}$$

Single-station : assume *VEC* varies linearly with latitude Φ and longitude Λ

$$V(P_{ijt}) = V^{(t)}{}_0 + a^{(t)}\,(\Phi - \Phi_0) + b^{(t)}\,(\Lambda - \Lambda_0)$$

Which can be improved to bi-linear, bi-polynomial expansion up to the full spherical harmonics expansion for global solutions.

The system of equations and its solution

Rewrite equations of observation

$$S_{ijt} = TEC_{ijt} + \beta_\iota + \gamma_j + \lambda_{Arc} = V(P_{ijt}) \sec \chi_{ijt} + \beta_\iota + \gamma_j + (\lambda_{Arc})$$

$$S_{ijt} = \sec \chi_{ijt} \, \Sigma_n c^{(t)}{}_n \Psi_n \, (\Phi_{ijt}, \Lambda_{ijt}) + \beta_\iota + \gamma_j + (\lambda_{Arc})$$

Symbolically written as

$$S = Ax$$

Unknowns x will be solved using Least Squares or equivalent (and more sophisticated) methods

$$x = (A^T A)^{-1} A^T S$$

Going back to the equations of observations, knowing solution x means knowing

The coefficients of the expansion of vertical TEC $c^{(t)}{}_n$

The biasing terms β_ι , γ_j , (λ_{Arc})

After the numerical solution

Having solved for $c^{(t)}_n, \beta_\iota, \gamma_j, (\lambda_{Arc})$, available products are

The calibrated slants

Calibrated slants will be available as $TEC_{ijt} = S_{ijt} - \beta_\iota - \gamma_j - (\lambda_{Arc})$

The Vertical TEC

In addition, as a by-product of calibration, knowledge of the coefficients $c^{(t)}_n$ of ***TEC*** expansion will enable to estimate slants along directions different from the ones of the actual observations.

$$TEC_{ijt} = \sec \chi_{ijt} \Sigma_n c^{(t)}_n \Psi_n (\Phi_{ijt}, \Lambda_{ijt})$$

The most familiar is vertical ***TEC (VEC)***, the Total Electron Content relative to the zenith of the station of coordinates Φ^*_j, Λ^*_j

$$VEC(j,t) = TEC_{jt} = \Sigma_n c^{(t)}_n \Psi_n (\Phi^*_j, \Lambda^*_j)$$

Summary

All solutions for calibration follow the reported scheme

Extraction of un-calibrated slants from GNSS observations

Solution of the system in the unknown *VEC* coefficients and biasing terms

According to the geographical distribution of stations and the time span in which observations are available, several solutions are possible getting the possible combinations of one solution per line

Hourly / Single-day / Multi-day

Single-station / Regional /Global

The traditional method: assumptions

Accept the known limitations of the thin shell approach (which enables global and regional solutions)

Accept the constancy of biases

Disregard the leveling error contribution λ

Solve the system

$$S_{ijt} = sec\, \chi_{ijt}\, \Sigma_n c^{(t)}_n \Psi_n\, (\Phi_{ijt}, \Lambda_{ijt}) + \beta_\iota + \gamma_j$$

In the unknowns $c^{(t)}_n, \beta_\iota, \gamma_j$

The $\beta + \gamma$ indetermination is avoided assuming some additional condition on the set of unknowns β_ι, γ_j

The traditional method: Advantages

$$S_{ijt} = sec\ \chi_{ijt}\ \Sigma_n c^{(t)}{}_n \Psi_n\ (\Phi_{ijt}, \Lambda_{ijt}) + \beta_\iota + \gamma_j$$

Excellent observations/unknowns budget

Unknowns: coefficients of ***VEC*** expansion plus one $\boldsymbol{\beta}$ per satellite, one $\boldsymbol{\gamma}$ per receiver, both constant.

No need to perform calibration for every new set of data:

just compute the leveled slants and subtract a set of pre-computed β_ι , γ_j

$$TEC_{ijt} = S_{ijt} - \beta_\iota - \gamma_j$$

Use pre-computed values during storm periods or at extreme latitudes (inadequacy of ***VEC*** expansion)

Use pre-computed values provided by others

Use of pre-computed values

Slants to calibrate

From a set of IGS stations (RINEX files)

Work has been already done by IGS: monthly values biases for satellites and IGS stations are available at

ftp://ftp.unibe.ch/aiub/CODE/

For users owning their own receiver

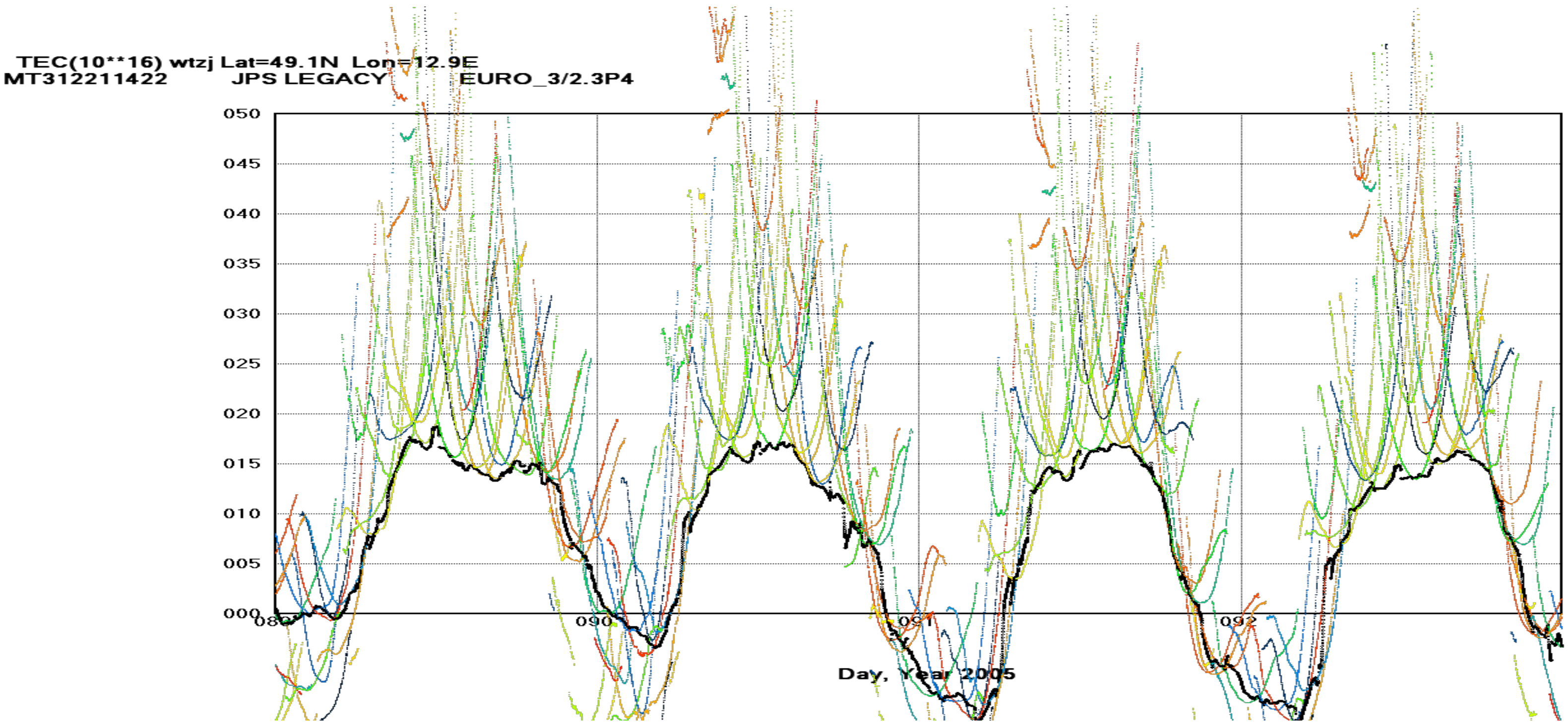
Use CODE for satellite biases, set up a calibration algorithm to estimate the bias of the receiver $\boldsymbol{\gamma}$

$$S_{ijt} - \beta_{\iota} = \sec \chi_{ijt} \, \Sigma_n c^{(t)}{}_n \Psi_n \, (\Phi_{ijt}, \Lambda_{ijt}) + \gamma$$

But it may occurr:

Slants (to the same satellite) of co-located receivers are not the same

Possible occurrence of negative ***TEC***s at middle latitudes



Factors affecting the reliability of calibration

Quality of Measurements

Modeling of observation

$$S = VEC \sec \chi + \beta + \gamma + (\lambda_{Arc})$$

Mapping function accuracy, constancy of biases, role of (λ_{Arc}), adequacy of the model used for the expansion of *VEC*

$$VEC(P, t) = \Sigma c \, \Psi(P, t) + ?$$

Conditioning of the resulting systems of equations, used algorithms

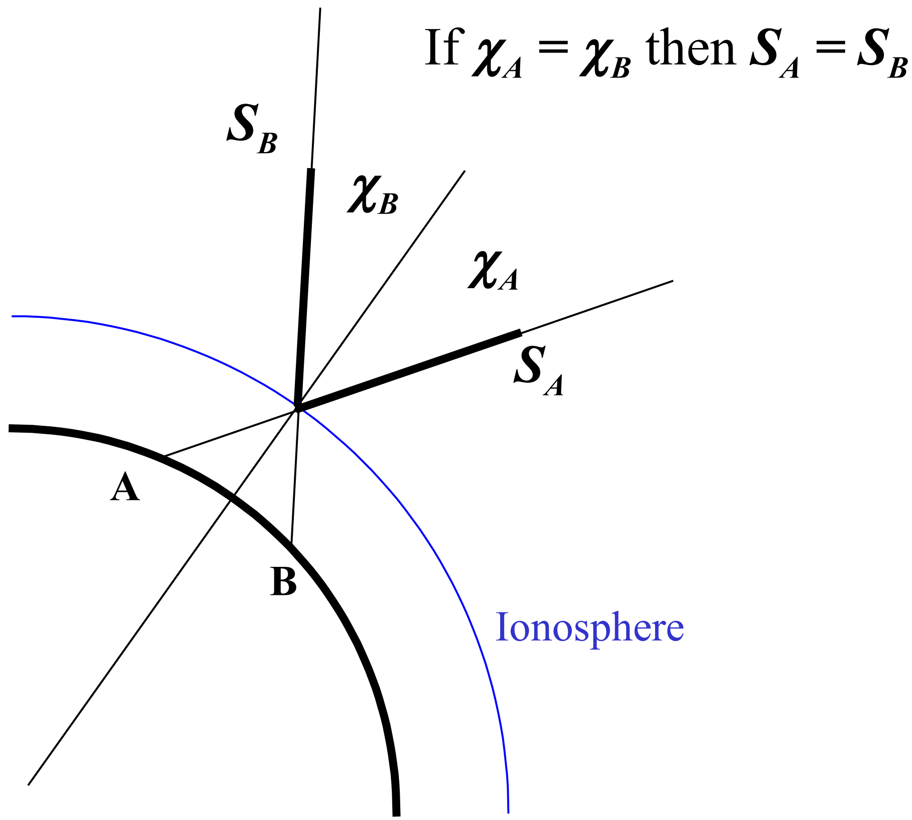
Biasing terms and ***VEC*** strongly correlated. Is the classical Least Squares method the best choice

Modeling of observations :

Limitations of the thin shell assumption

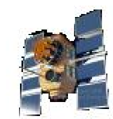
The thin shell assumption is self-evidently poor: ***TEC*** is the same for rays passing through the same ionospheric point for given χ, **disregarding at all gradients**

Errors range to few ***TECu*** in normal conditions, but up to 30-40 ***TECu*** under storm (thesis of ***Bruno Nava,*** carried out on super-truth data). This may introduce severe errors in regional and global solutions.



Modeling of observations : the role of (λ_{Arc})

The close stations experiment



β

TEC

Station 1

$$S1_{PRN} = TEC + \lambda 1 + \beta_{PRN} + \gamma 1$$

Station 2

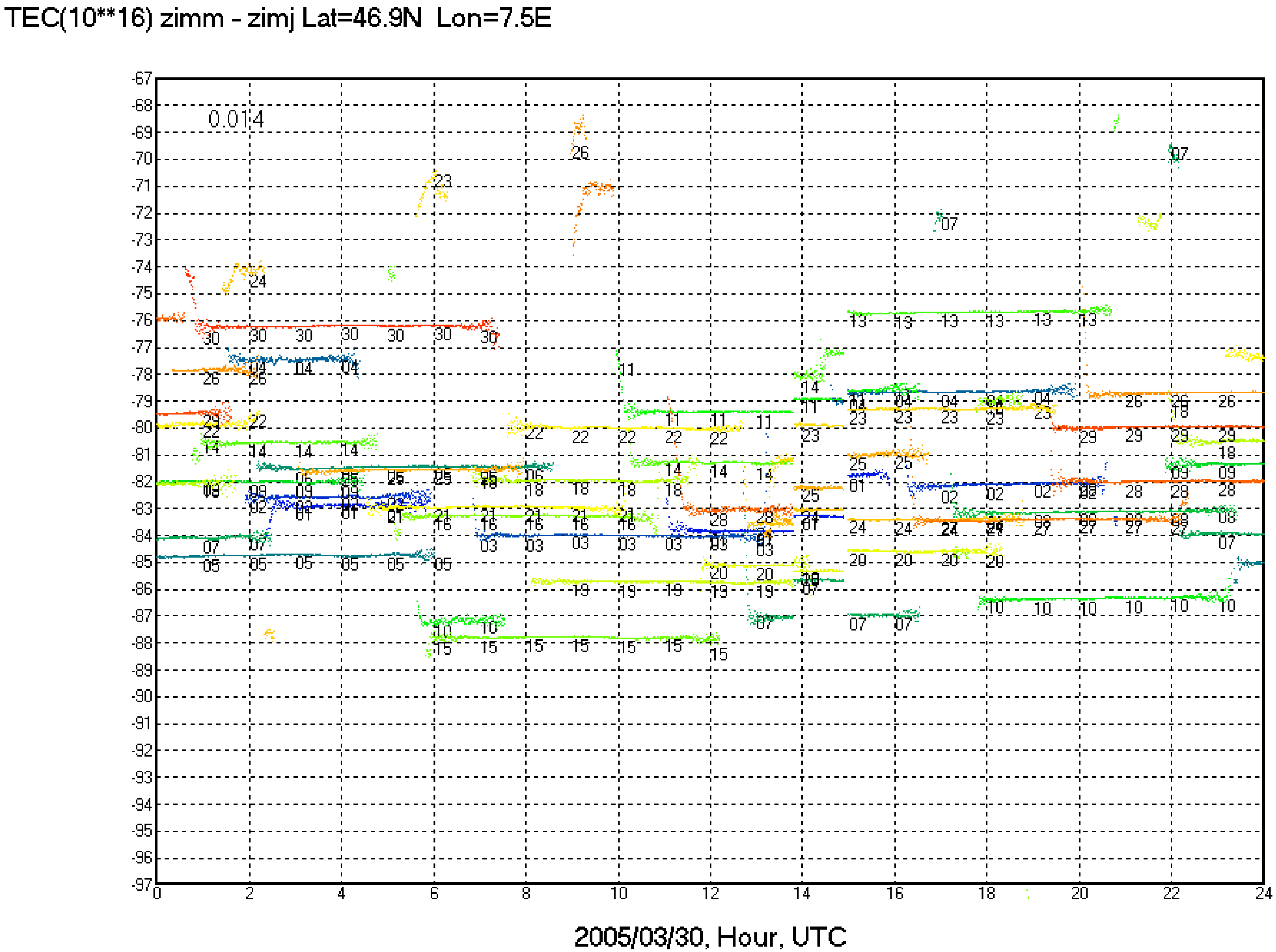
$$S2_{PRN} = TEC + \lambda 2 + \beta_{PRN} + \gamma 2$$

< 100 m

$$S1 - S2 = \gamma 1 - \gamma 2 + \lambda 1 - \lambda 2$$

Not dependent on PRN

$S_1 - S_2$, all satellites



It turns out that the λ_{Arc} term should be takent into account.

$$S = VEC \sec \chi + \beta + \gamma + \lambda_{Arc}$$

To know more about the topic: look at the recent publication on the

Journal of Geodesy

Calibration Errors on Experimental Slant Total Electron Content (TEC) Determined with GPS

L. Ciraolo, F. Azpilicueta, C. Brunini, A. Meza, S. M. Radicella
(DOI 10.1007/s00190-006-0093-1)

The alternative solution

Always perform a single station solution: the thin shell approach can be considered exact provided VEC is interpreted as a **Vertical Equivalent** (***VEq***), such that $S = VEq \sec \chi$

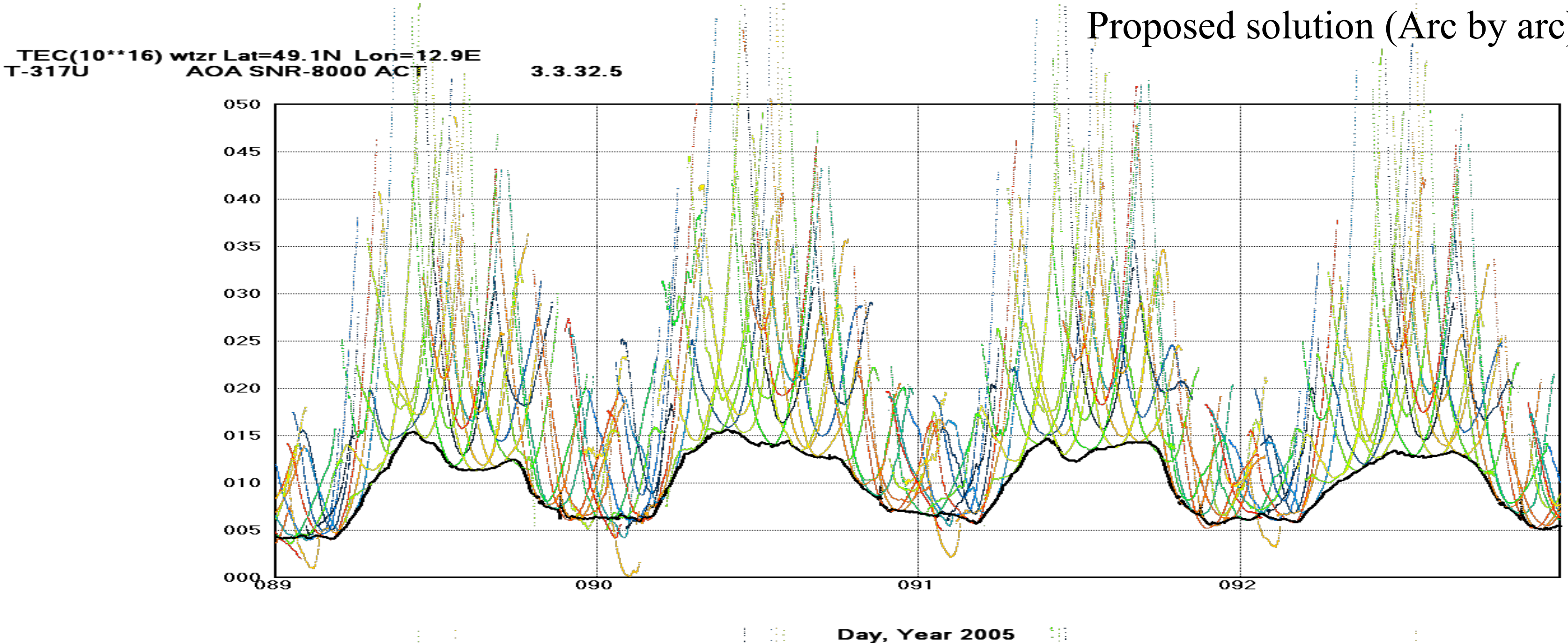
Take into account of the multi-path error λ_{Arc} considering an unknown for each arc $\Omega_{Arc} = \beta + \gamma + \lambda_{Arc}$

Observations: leveled slants (or directly phase slants)
V_{Eq} is expressed as a proper expansion of horizontal coordinates l, f with one set of coefficients at each time $V_{Eq}(l, f) = \Sigma_n c_n p_n(l,f)$

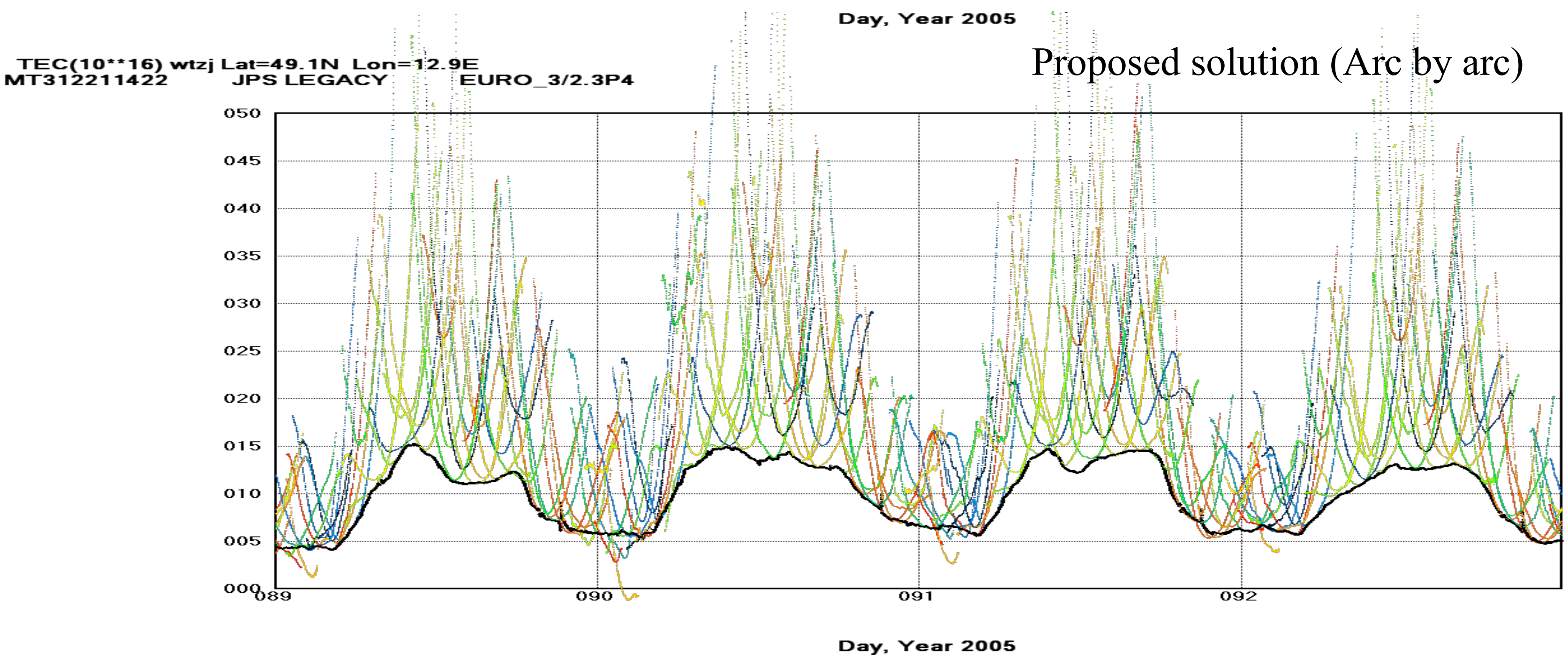
$$S_{ijt} = \Sigma_n c^{(t)}_n p_n(l_{ijt}, f_{ijt}) \sec \chi_{ijt} + \Omega_{Arc}$$

The unknowns are now the coefficients $c_n^{(t)}$ and the offsets Ω_{Arc}

Proposed solution (Arc by arc)



Proposed solution (Arc by arc)



Summary of the characteristics of the Proposed Solution

Observations

Leveled slants or directly phase slants

Assumptions

One thin shell at 400 km

Elevation mask: 10^{o} ($20°$ at low latitudes)

TEC is expressed through V_{Eq} at the ionospheric point, by the mapping function ***sec χ*** (*one station only*!)

V_{Eq} expressed as a proper expansion of horizontal coordinates ***l, f*** with one set of coefficients at each time $V_{Eq}(l, f) = \Sigma_n c_n p_n(l,f)$

$$S_{ijt} = \Sigma_n c^{(t)}{}_n p_n (l_{ijt}, f_{ijt}) \sec \chi_{ijt} + \Omega_{Arc}$$

The unknowns are now the coefficients $c_n^{(t)}$ and the offsets Ω_{Arc}

The adopted horizontal coordinates

Using as horizontal coordinates ***Modified Dip Angle (modip)*** and ***Local Time***, we can assume that for a set of adjacent epochs (up to ±15 minutes), the coefficients $c_n^{(t)}$ keep constant.

This allows also reducing computing resources during solution using commonly used standard methods for sparse systems.

After the solution of the system, we shall avail :

Calibrated slants along the observed rays $TEC_{ijt} = S_{ijt} - \Omega_{Arc}$

"Mapped slants" at given coordinates l_{ijt}, f_{ijt}

Vertical ***TEC*** above the station (ionospheric point at the its zenith)

$$VTec(t) = \sum_n c_n^{(t)} p_n\left(l_{ijt}^{Zenith}, f_{ijt}^{Zenith}\right)\sec\chi_{ijt}$$

Why multi-day solution

A multi-day solution is performed, avoiding day to day discontinuities in calibrated slants, except that at the beginning and the end of the solution.

Still, at the beginning and the end of the set of data, broken arcs occur.

Broken arcs are generally shorter implying

1. worse results during leveling
2. worse numerical conditioning for the solution

To reduce these problems, in order to calibrate ***N*** days, ***N+2*** days are actually processed: first and last day of the ***N+2*** set are discarded.

The Errors

using a single station and ***VEq*** approach avoids the mapping function problems.

solving for an unknown offset for each arc helps in taking into account of the average multi-path contribution $\boldsymbol{\lambda}_{Arc}$

Solutions look generally reliable at middle latitudes: but let's come to the ***basic question (BQ)***

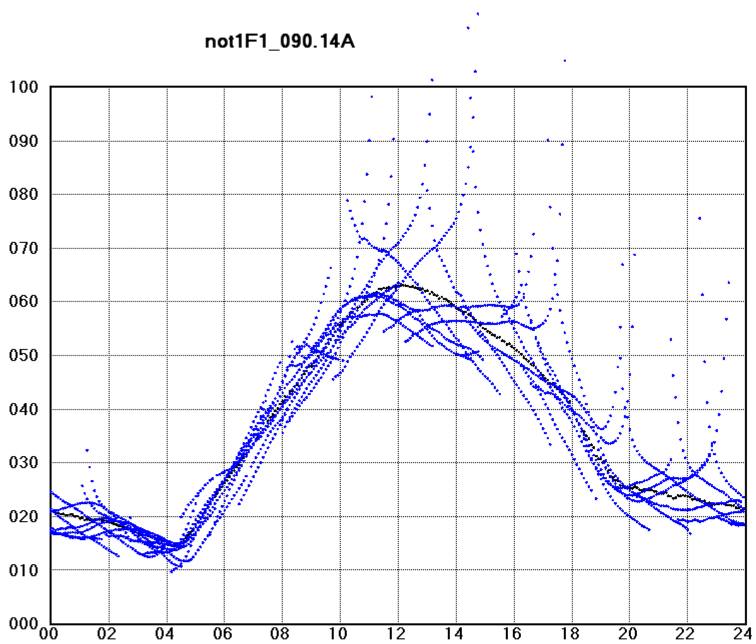
How estimating the order of magnitude of errors?

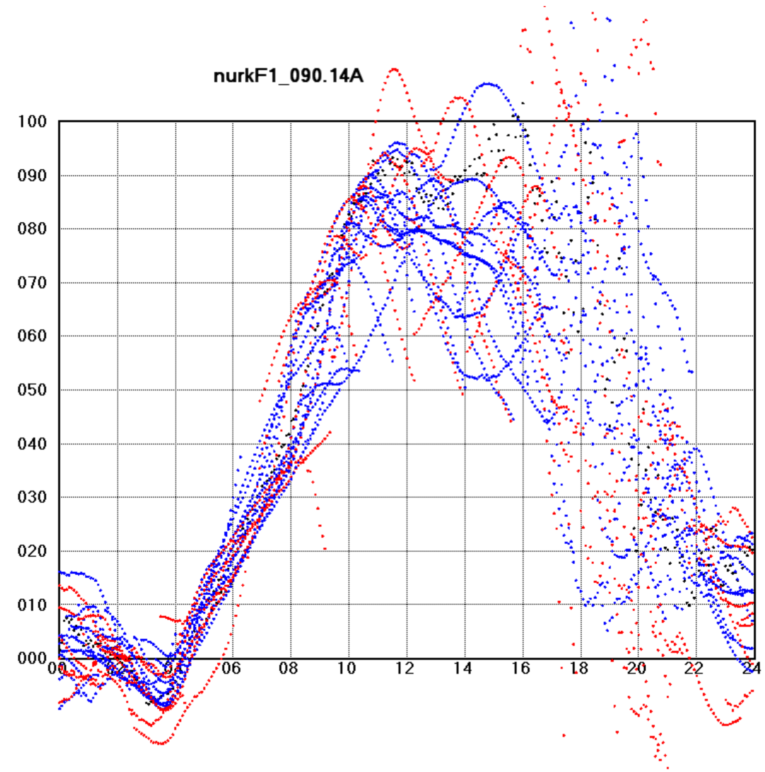
Presence of errors is evidenced only by the occurrence of negative ***TEC***s: see next slides

Left : middle latitude "*not1*" (15.0°E, 36.9°N) No errors?

Right: low latitude "*nurk*" (30.0°E, 2.0°S) Errors!

Why errors in "*nurk*"? Likely bad observations? (scintillations, …)





Estimating Errors

To answer this questions, ***truth data*** would be needed, but they are not (or at least scarcely) available.

Why not using ***artificial data*** as provided by ionospheric models? This way we shall "***calibrate***" quantities we exactly know, so getting some answer to the "***BQ***".

(But keeping in mind that agreement with artificial data is a condition **necessary but not sufficient** to validate the method).

The artificial data

Ionospheric models enable to estimate median electron density at some time at some geographic location, i.e. given date and time, latitude, longitude, height.

$$N_e = N_e(t, \phi, \lambda, h)$$

TEC is the integral of electron density along the ray-path from satellite to receiver,

$$TEC = \int N_e(P)ds$$

which will be numerically evaluated as the sum

$$TEC \approx \sum N_e(P_i)\delta s_i$$

or with any more effective numerical algorithm (Gauss, …)

Providing with the ***slants*** we need for checking calibration

Model TEC computation

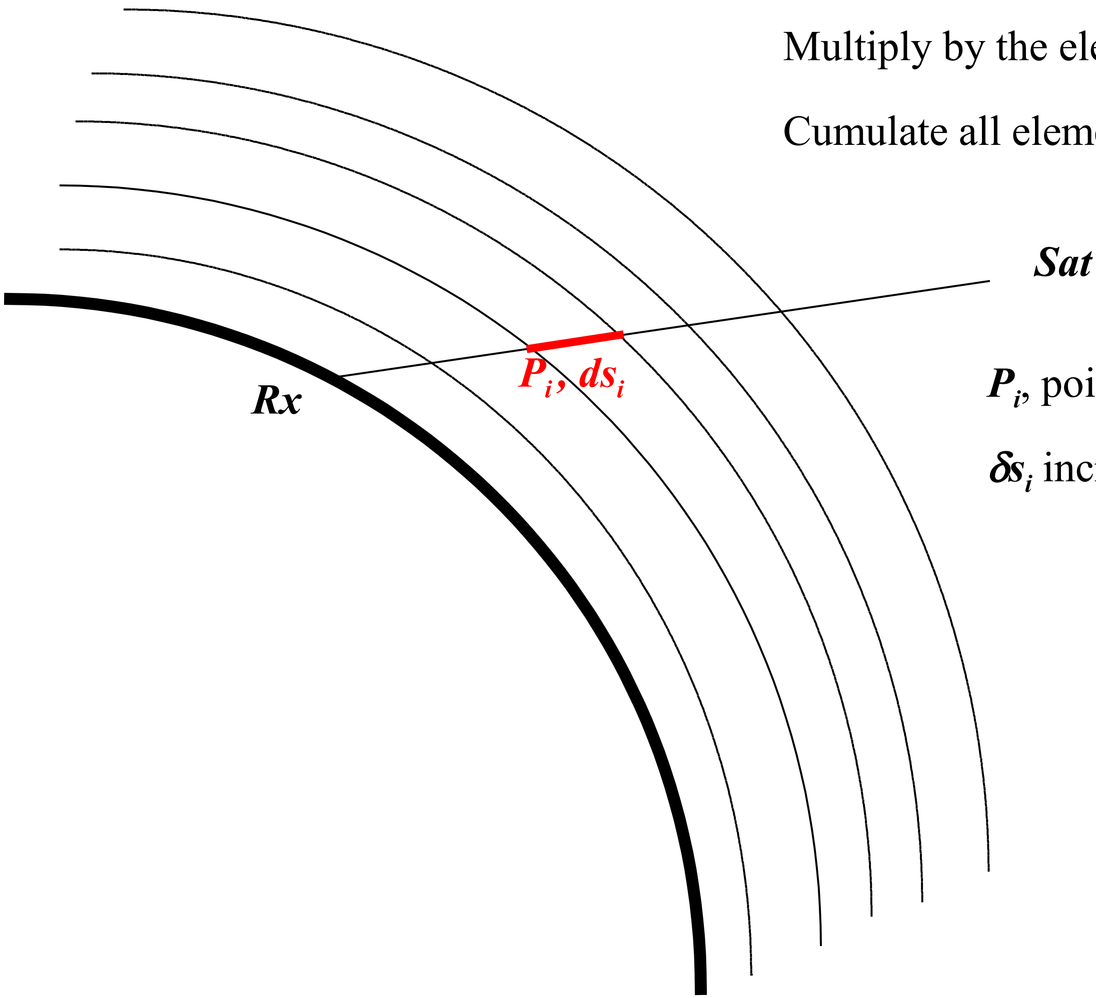
$$TEC = \int N_e(P)ds \approx \sum N_e(P_i)\delta s_i$$

Divide the path in elements δs_i

At each point P_i compute the electron density $N_e(P_i)$ provided by the model

Multiply by the element length δs_i

Cumulate all elements



P_i, point on the generic i^{th} shell

δs_i increment in arc length

Generation of artificial truth data: former approach

<u>Given all slants actually observed and archived</u> in a (quasi) complete set of IGS stations (≈ 200 per day) for year 2000, days 88-91 (March 28-31)

<u>Re-compute them using</u>

NeQuick (Az =150), integrating up to 2000 km

<u>Therefore</u>:

Not only the actual GPS constellation has been preserved for the reference period, but also the possible lack of observations (this will affect the solution)

Generation of artificial truth data: current approach

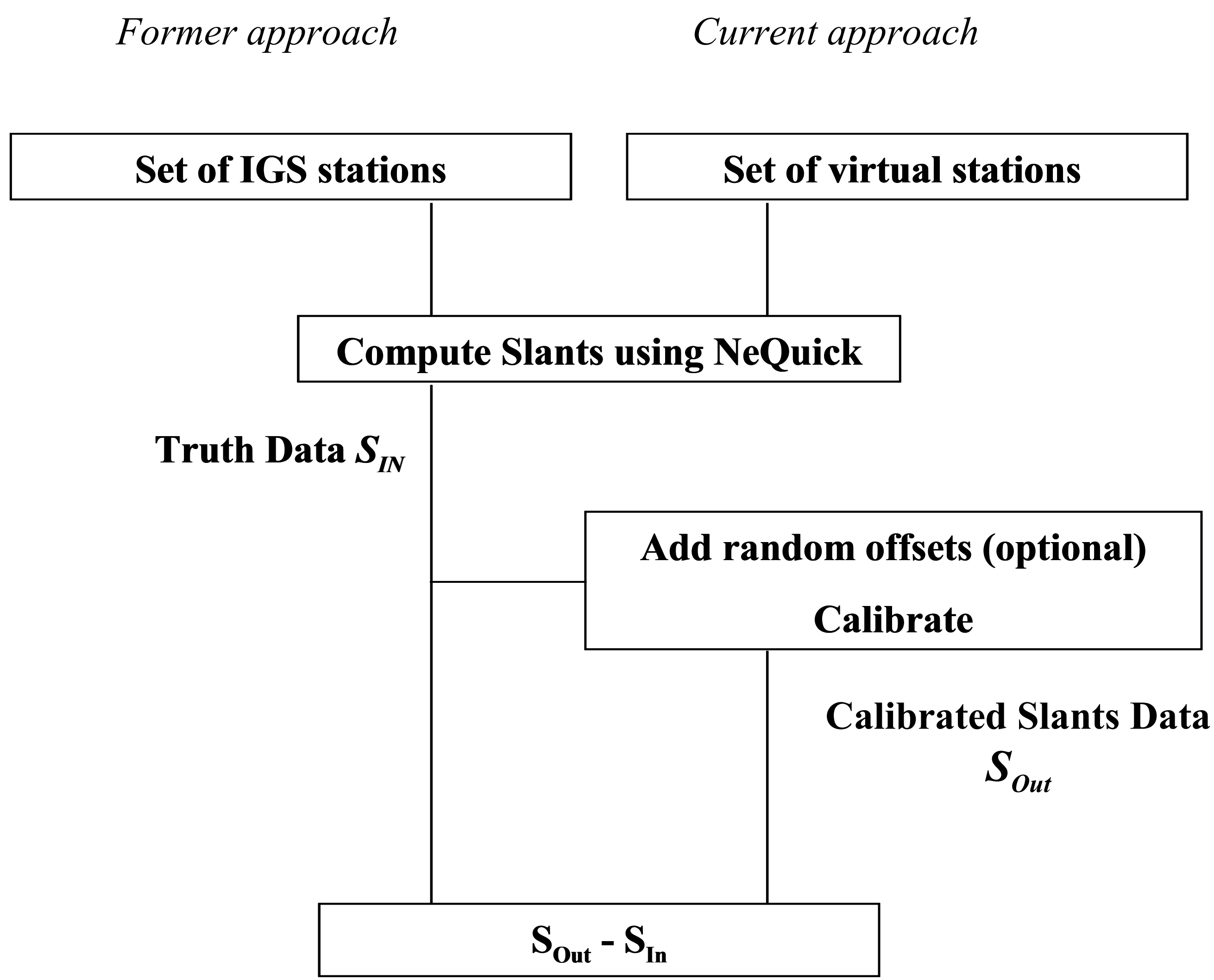
NeQuick slants are computed for a set of virtual stations distributed from 45° N to 45°S, spaced 5° degrees in latitude and longitude, for year 2012, days 79-81 (March19-21), using NeQuick (Az =200), integration up to 2000 km.

Therefore:

Also in this case the actual GPS constellation has been preserved for the reference period, keeping all available observations.

In both cases we shall avail data **free from multi-path and any other disturbance**

Testing procedure



$\boldsymbol{S}_{\boldsymbol{Out}} - \boldsymbol{S}_{\boldsymbol{In}}$ are plotted vs time

Worth (but expected) noting that errors at low latitudes are larger

Remark about highlighted arc:

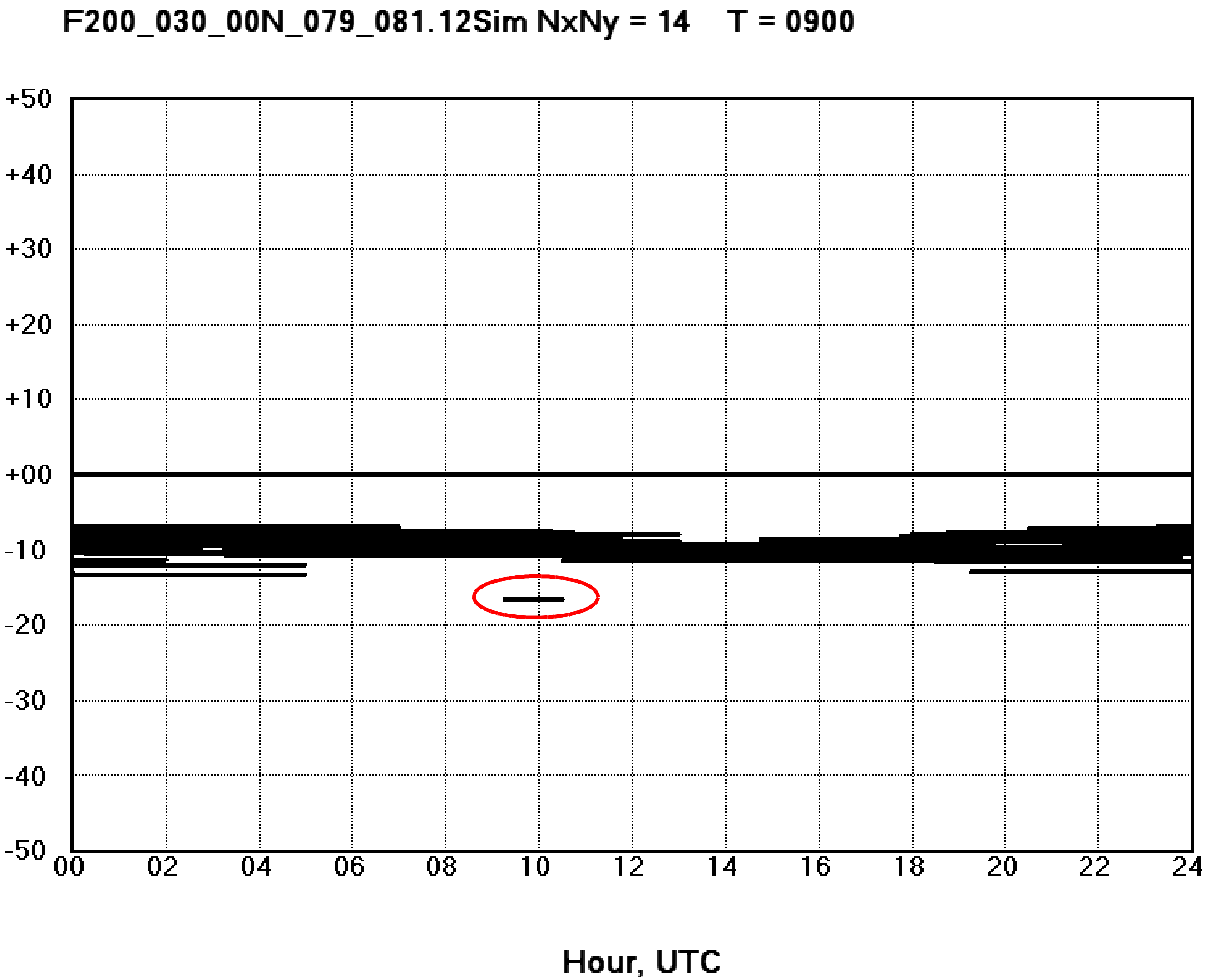
errors show a weakness of the solution**.**

These errors occur for arcs of low elevation also if, in some case, of long duration.

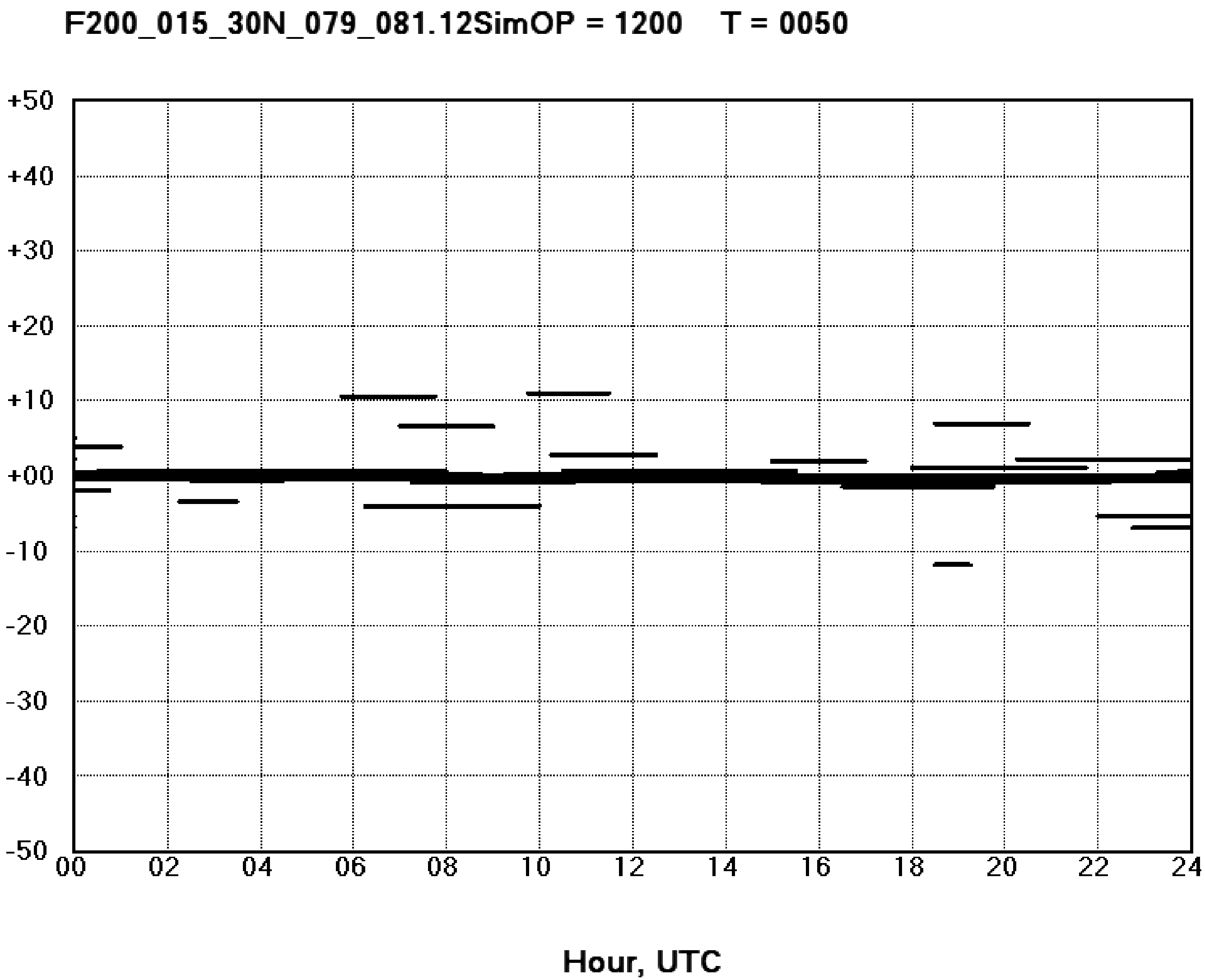
Processing real data, there is no chance to know if the subject arc is ill-calibrated (unless in presence of very strong errors)

Testing the solution with simulated data will (likely) enable to find a more effective way of avoiding such errors, or in a last instance, rejecting them

Sample $\mathbf{S}_{\mathbf{Out}} - \mathbf{S}_{\mathbf{in}}$ (Lon 30.0°E, Lat 0.0°)

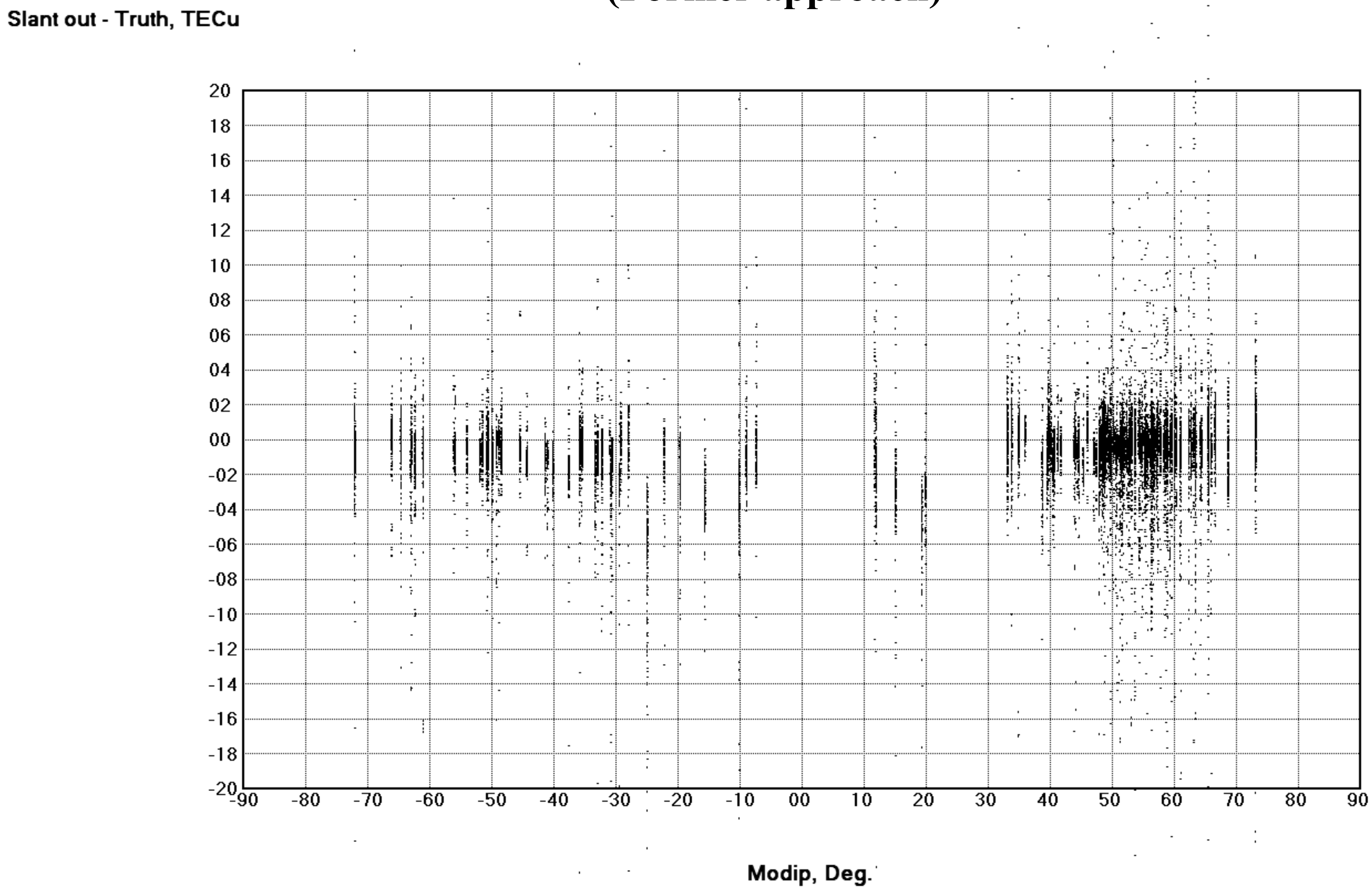


Sample $\mathbf{S}_{\mathbf{Out}} - \mathbf{S}_{\mathbf{in}}$ (Lon 15.0°E, Lat 30°N)



An overall look to the errors: $S_{Out} - S_{In}$, whole set

(Former approach)



Current approach

Using all virtual stations enables to look at the behavior of calibration errors versus geographical position.

Worth noting:

The generally satisfactory behavior at middle latitudes

The strong correlation on Equatorial Anomaly (modeled by NeQuick, through CCIR coefficients i.e. data from ionosondes)

The non continuity of error (i.e. the jump from negative to positive values) in close stations.

Results for different *VEq* bi-polynomial expansions are shown

All expansions: linear in local time displacement.

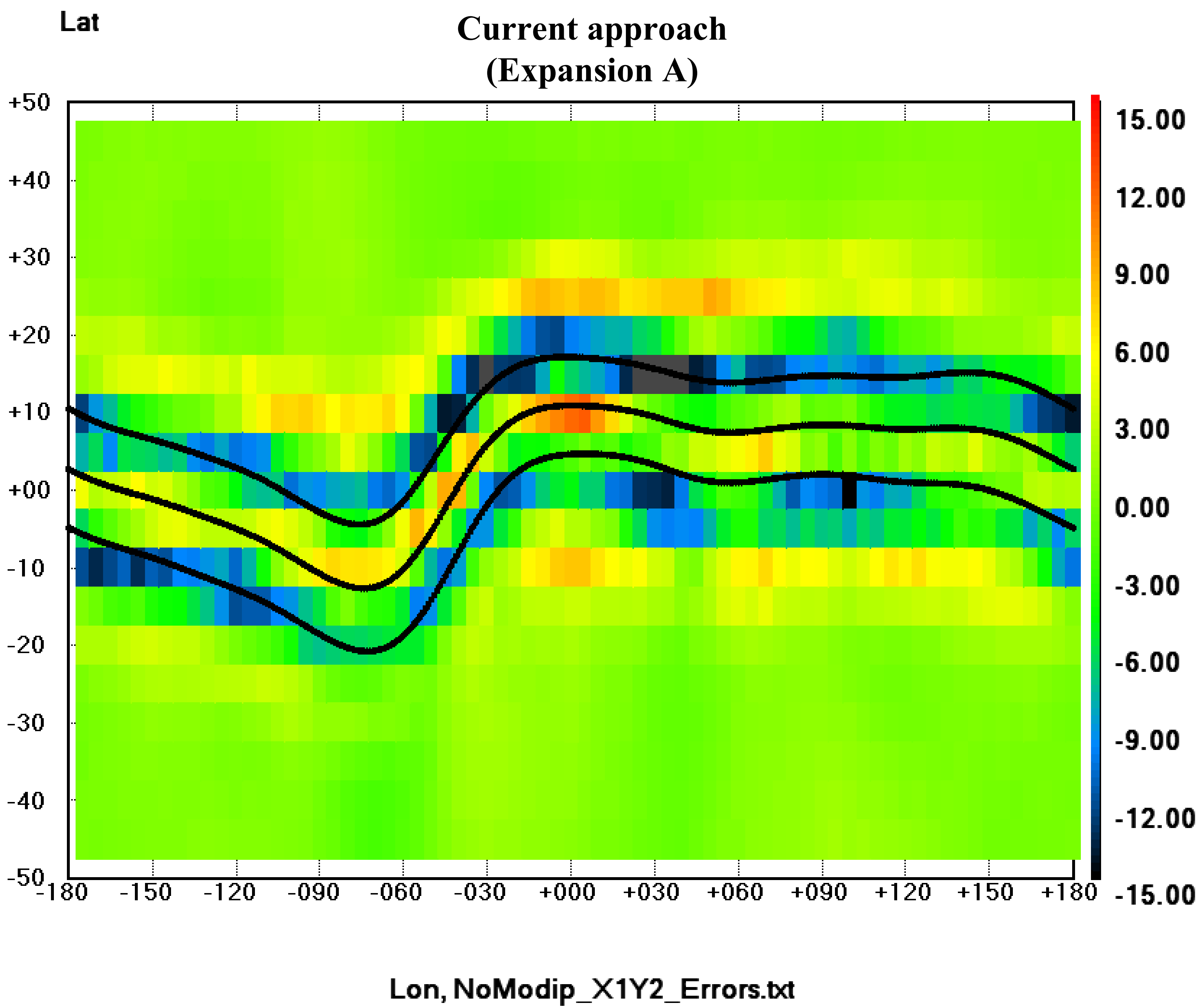
A) Quadratic in latitude displacement (actually used in the past)

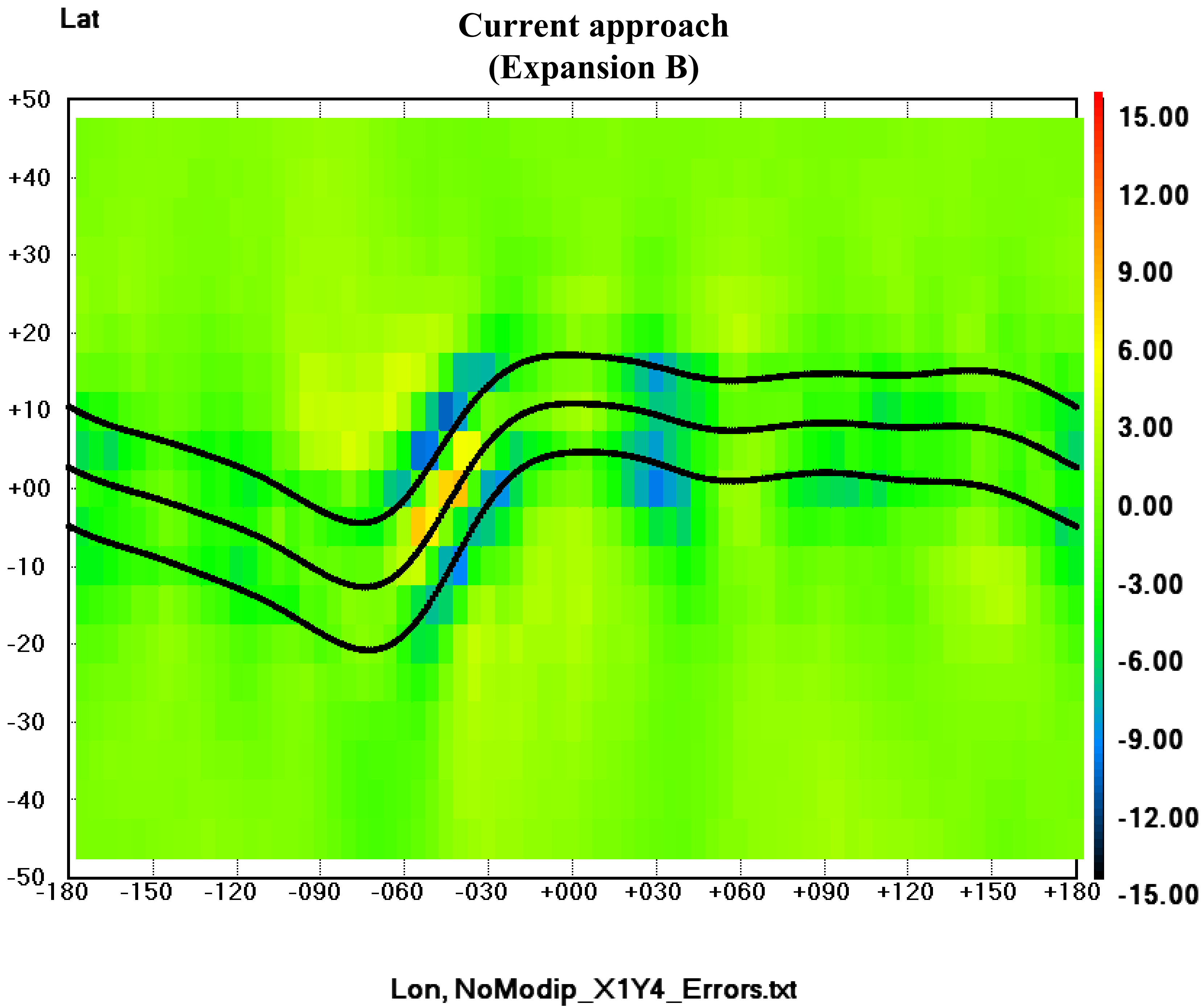
It is expected that at low latitudes a 4th order polynomial in latitude or modip displacement is more suitable

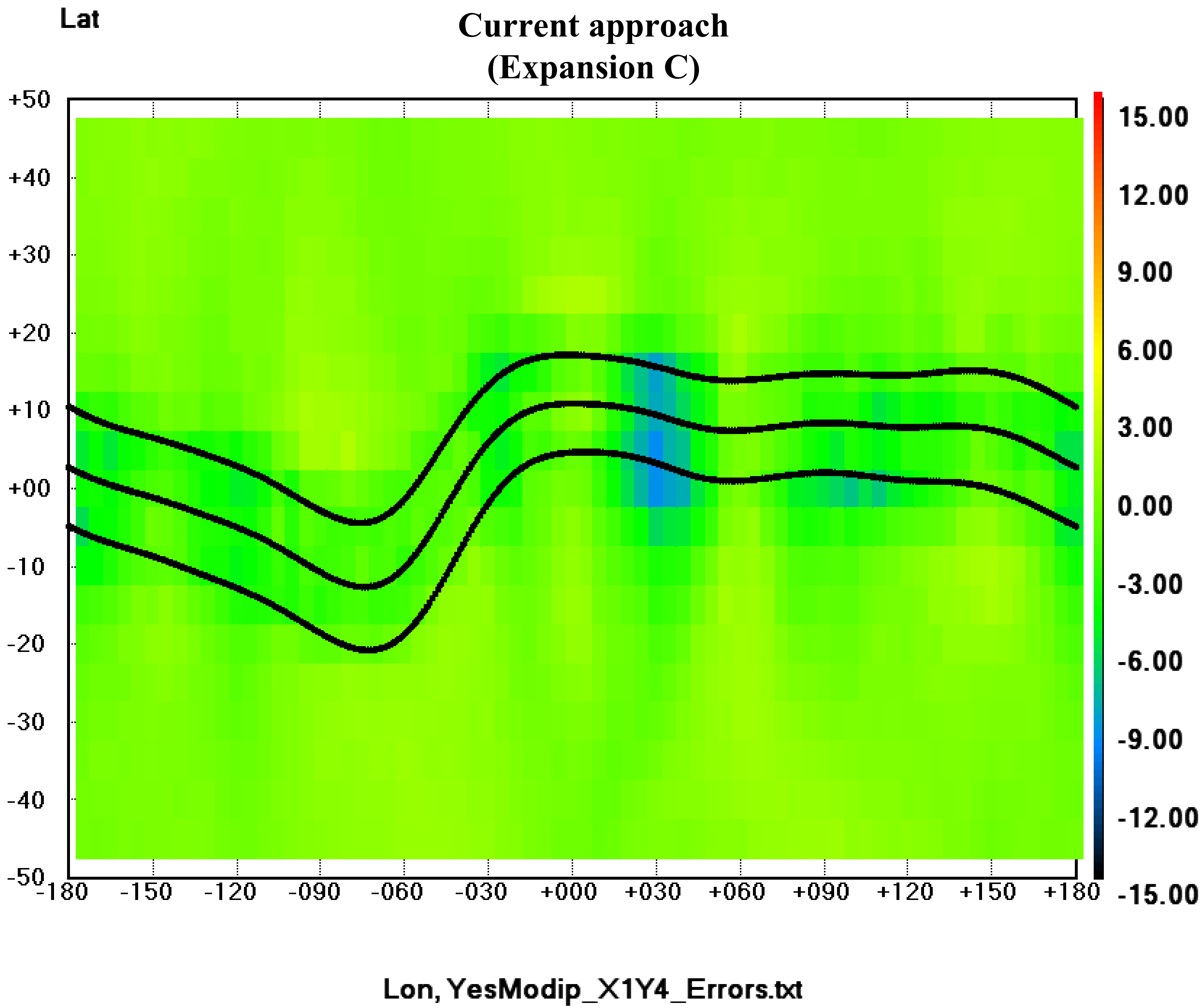
B) 4th order in latitude displacement

C) 4th order in modip displacement

The three black lines plot -15°, 0°, +15° modiip


Lat
Current approach
(Expansion A)
+50
+40
+30
+20
+10
+00
-10
-20
-30
-40
-50
-180
-150
-120
-090
-060
-030
+000
+030
+060
+090
+120
+150
+180
15.00
12.00
9.00
6.00
3.00
0.00
-3.00
-6.00
-9.00
-12.00
-15.00
Lon, NoModip_X1Y2_Errors.txt

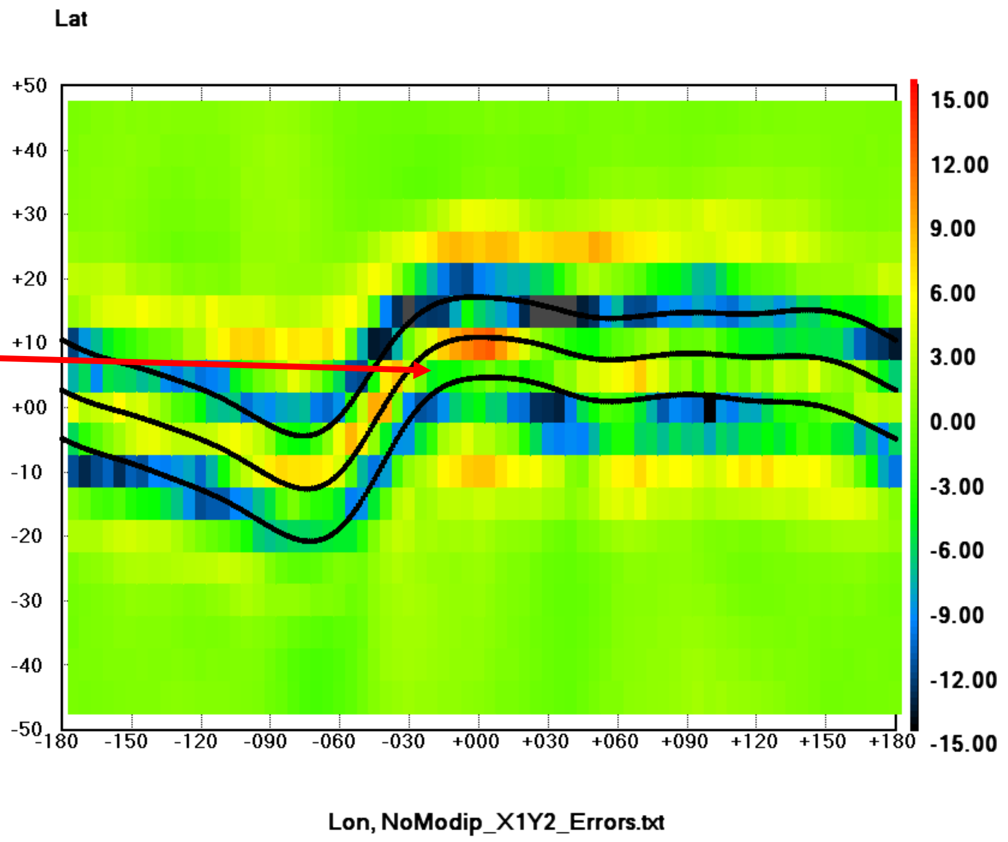

Lat
Current approach
(Expansion B)
+50
+40
+30
+20
+10
+00
-10
-20
-30
-40
-50
-180
-150
-120
-090
-060
-030
+000
+030
+060
+090
+120
+150
+180
15.00
12.00
9.00
6.00
3.00
0.00
-3.00
-6.00
-9.00
-12.00
-15.00
Lon, NoModip_X1Y4_Errors.txt


Lat
Current approach
(Expansion C)
+50
+40
+30
+20
+10
+00
-10
-20
-30
-40
-50
-180
-150
-120
-090
-060
-030
+000
+030
+060
+090
+120
+150
+180
15.00
12.00
9.00
6.00
3.00
0.00
-3.00
-6.00
-9.00
-12.00
-15.00
Lon, YesModip_X1Y4_Errors.txt

Some remark:

Why current approach is more effective?

Several locations in the anomaly area show acceptable errors (Expansion A), but in very close ones errors jump to their maximum values. Especially in the past, with a poor coverage of stations at low latitudes, one could convince himself that his calibration method was reliable also there!



Still remarks

Where the ***VEq*** expansion fits satisfactorily (*middle latitudes*) the data (*artificial*!), errors seem to be confined to the classical "*few TECu*".

At low latitudes better results are obtained using expansion "C" . Note that artificial data are free from "observation errors" and possible improvements should be based on an improvement of the ***VEq*** expansion itself. But increasing the order of the bi-polynomial expansion did not give anyway significant results.

Possible future developments

Using effective physical models able to describe the fountain effect and the winds (very difficult job)

Using **EIV** ("**E**rrors **I**n **V**ariables") methods of analysis able to take into account of the errors of the expansion (Total Least Squares)

Answer to the basic question

You can imagine now which is the answer

Thank you