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Winter College on Optics: Light a bridge between Earth and Space

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INTERFERENCE

Scalar Approximation

In a region free of charges and currents and of ferromagnetic materials, from Maxwell equations one obtain the Equation of d'Alembert for both electric field **E** and magnetic field **B**. Description of propagation of the electromagnetic field requires knowledge of three components of the electric field and three components of the magnetic field; A total of six unknown quantities.

In the case of waves at optical frequencies, *generally* only one component of the two fields is sufficient to describe all the electromagnetic field. This fact is known as "optics approximation". It is valid, for example, when the distances from the source are large with respect to the wavelength, and in this case one has the so called TEM waves. In TEM waves, the two fields, **E** and **B** are normal to the propagation direction and normal to each other, in such a way that "propagation direction **k**", **E** and **B** can be taken in the directions **i**, **j**, ***k*** of a rectangular coordinate system x, y, z.

A transverse Cartesian component, v = v(P,t) of **E** or **B** is representative of the complete e.m. field. Recall that the modulus E and B of the two vectors **E** and **B** are related by E/B= V, propagation velocity, in the empty space V=c.
The scalar approximation is also called optics approximation.

$v^2 \propto |\mathbf{S}|$ v square is proportional to modulus of Poynting vector, **S**. It is denoted by I, intensity, and is proportional to power flux.

Monochromatic radiation, central frequency $0.5\ 10^{+15}$ Hertz
Linearity,
Complete systems.

For component v(P,t), here simply denoted as v, D'Alembert equation becomes:

1) $$\nabla^2 v - \frac{1}{V^2}\frac{\partial^2 v}{\partial t^2} = 0 \qquad \text{where} \qquad \mu\varepsilon = 1/V^2$$

where ∇ denotes Laplacian; ε and μ denote dielectric constant and magnetic permeability, respectively. This equation is valid in the case of empty space and homogeneous non magnetic media. Interest here: transparent media.

Choice of coordinate system.
Separation of variables.

Method of complex exponentials

Let us remember that the e.m. field is real quantity. For instance one solution of Eq.3 is

2) $$v_1(P,t) = A(P)\cos[\varphi(P) - \omega t]$$

Use of complex exponentials helps with mathematics and allows one to find two simultaneous solutions. Let us write:

3) $$v(P,t) = A(P)\, e^{i\varphi(P)} e^{-i\omega t} = u(P) e^{-i\omega t}$$
$$u(P) = A(P)\, e^{i\varphi(P)}$$

where u(P) is called complex amplitude.
It is immediately verified that real part of v(P,t) gives above solution $v_1(P,t)$ and the coefficient of imaginary part gives another independent solution $v_2(P,t)$

4) $$v_2(P,t) = A(P)\sin[\varphi(P) - \omega t]$$

Conclusion: one can use complex exponentials method by taking into account that the **real part** and the **coefficient of the imaginary** part only **have physical meaning**.
Introduction of Eq.s 3) in D'Alembert Equation gives

5) $$\nabla^2 u(P) + k^2 u(P) = 0$$

Helmholtz Equation or
Wave Equation

Quantity u(P) is called complex amplitude

Some notations :
k = ω/ V Wavenumber
ω = 2π ν ν frequency
T =1/ν T period

In Eq. 2) quantity

6) $$\Phi(P,t) = \varphi(P) - \omega(t)$$

is called instantaneous phase, and $\varphi(P)$ simply phase.
A surfaces where

$\varphi(P) = \text{Constant}$ is "equiphase surface" called **WAVEFRONT**

Two wavefronts differing by an entire number of 2π are said to be "in phase",

$$\varphi(P_1) - \varphi(P_2) = m\,2\pi \qquad \text{m entire number}$$

If the difference is $(2m+1)\pi$, that is an odd number of π, one has opposite phases.
Intensity I(P)

7) $$I(P) = v(P,t).\,v^*(P,t) = u(P).u^*(P) = |A(P)|^2$$

Values in Optics:

$\nu \sim 0.75 \div 0.37\ 10^{15}$ Hertz
$k \sim 1.6 \div 0.8\ 10^{7}\ m^{-1}$
$\lambda \sim 4 \div 8\ 10^{-7}$ m $= 0.4 - 0.8\ \mu$m $= 400 \div 8oo$ nm
$T \sim 1.3 \div 2.7\ 10^{-15}$ s

NOTE: Laser has reached large part of the spectrum outside optics and now speaking of Optics one includes infrared and ultraviolet radiation.

PLANE WAVES

Here we remember two wave solutions which are of interest for interference.
Plane wave solution of wave Equation:

8) $$u(P) = Ae^{ik(\alpha x + \beta y + \gamma z)}, \qquad \text{where} \quad \alpha^2 + \beta^2 + \gamma^2 = 1$$

and A constant (real or complex), is called "plane wave" and can also be written as:

9) $$u(P) = A\, e^{i\mathbf{k}\cdot\mathbf{r}}$$

where $\mathbf{k} = k\,\mathbf{n} = k(\alpha\,\mathbf{i} + \beta\,\mathbf{j} + \gamma\,\boldsymbol{k})$
$\mathbf{n} = \alpha\,\mathbf{i} + \beta\,\mathbf{j} + \gamma\,\boldsymbol{k}$
$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\boldsymbol{k}$ vector from origin to a wave point P

Wavefronts are the planes, Fig 1:

10) $$\mathbf{k} \cdot \mathbf{r} = k\,(\alpha x + \beta\, y + \gamma\, z) = \text{Const}$$

where α, β, γ are real quantities and represent the "cosine directors" of the normal to the wavefront from the origin; $p = \mathbf{n} \cdot \mathbf{r}$ is the distance of wavefront from origin.

11) $$u(P) = A\, e^{ik\,\mathbf{n}\cdot\mathbf{r}} = A\, e^{ikp}$$

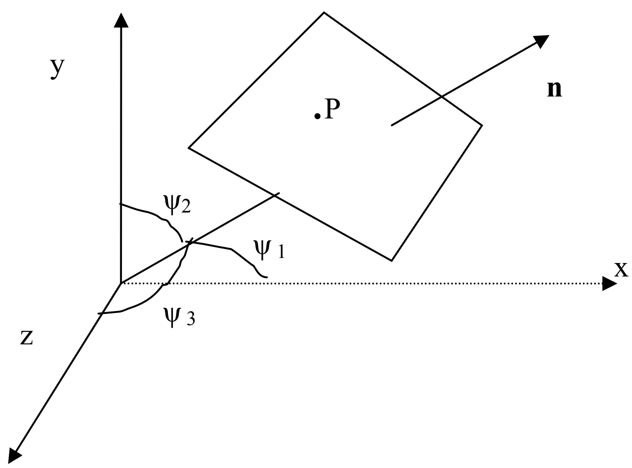

Fig. 1

At time t_1 the wavefront Φ_1 is at distance p_1 from origin:

$$\Phi_1 = k\, p_1 - \omega\, t_1$$

after a time dt position of Φ_1 is

$$\Phi_1 = k\,(p_1 + dp) - \omega\,(t_1 + dt)$$

from which

$$k\, dp - \omega\, dt = 0; \qquad dp = (\omega/k)\, dt$$

p increases. Linear motion. Velocity

12) $$V_f = \frac{\omega}{k}$$

Wavefront moves with velocity V_f **"phase velocity"**, in our case $V_f = V$.
Wavelength: distance between two subsequent equiphase planes. At time t_1 :

$$\left[k\,(p_1 + \lambda) - \omega\, t_1\right] - (kp_1 - \omega\, t_1) = 2\pi$$

Therefore

13) $$\lambda = \frac{2\pi}{k} = 2\pi \frac{V_f}{\omega} = \frac{V_f}{\nu} = T V_f$$ useful relations

Important fact: **Frequency ν is from source and does not change** in linear media; the effect of a medium is to **change propagation velocity and wavelength**.

Amplitude, A, can be complex $A = A_0 e^{i\varphi_0}$ and constant φ_0 represents initial phase. A real plane wave solution, e.g. the real part of the complex solution, of the wave equation, written in complete explicit, form is

14) $$v_1(P,t) = A_0 \cos\left[k(\alpha x + \beta y + \gamma z) + \varphi_0 - \omega t\right]$$

Quantity φ_0 represents the initial phase in the origin (t=0, p=0). Generally one has to deal with phase differences where it does not play a role, often here we assume φ_0=0. Quantity A_0^2 is the **Intensity** (proportional to the power density flux) on a surface normal to propagation direction **n.**

IMPORTANCE OF PLANE WAVES:

- plane wave is an approximation to describe a wave in limited regions,

- e.g. the beam from a lens due to a point source in the focus

- the field from a distant source; distance much larger than wavelength and limited region

- basically plane waves are the elements for representing any e.m. field in terms of Fourier Series or Fourier Integrals.

SPHERICAL WAVES

By taking the Laplacian in polar coordinates (r, θ, φ) one can write the wave equation in these coordinates. In general solution u = u(r, θ, φ). First consider a solution depending on r, u = u(r). In this case the wave Equation gives

$$\frac{\partial^2 (r u)}{\partial r^2} + k^2 (r u) = 0$$

whose solution is

$$r u = A e^{\pm ikr} \quad ;$$

where A is, generally, complex constant. One obtains two solutions:

15) $$u_1 = A\frac{e^{ikr}}{r} \quad ; \quad u_2 = A\frac{e^{-ikr}}{r}$$

Equiphase surfaces are spherical surfaces:

$$\pm kr + \varphi_0 = \text{Constant} \qquad \varphi_0 \text{ initial phase}$$

At a given instant the total phase

$$\phi = -\omega t \pm kr + \varphi_0$$

that is

16) $$\begin{aligned} kr_1 &= \phi + \omega t - \varphi_0 \\ kr_2 &= -\phi - \omega t + \varphi_0 \end{aligned}$$

which represent the phase of a **diverging spherical wave** , u_1, and a **converging spherical wave** , u_2. respectively.

Wavelength and velocity are equal to those of plane waves.

Dependence of Amplitude on 1/r represents **conservation of energy**. An element of spherical surface is:

$$d\Sigma = r^2 \sin\vartheta \, d\vartheta \, d\varphi$$

Power across the element is

$$dP = u u^* \, d\Sigma = \frac{AA^*}{r^2} r^2 \sin\vartheta \, d\vartheta \, d\varphi$$

Power across an entire sphere is

$$P = \iint_{\text{sphere}} dP = 4\pi A A^*$$

Note: spherical waves have singularity for $r = 0$.
Physical significance: **diverging wave u_1** represents radiation emitted by a point source, valid everywhere apart from a small volume around $r = 0$ where the source is. **Converging wave u_2** represents focussing of a wave, for instance by a lens, and is valid everywhere apart from a small region near the focus. The effect of a converging lens can be described by a converging spherical wave before the focus and a diverging one after it.

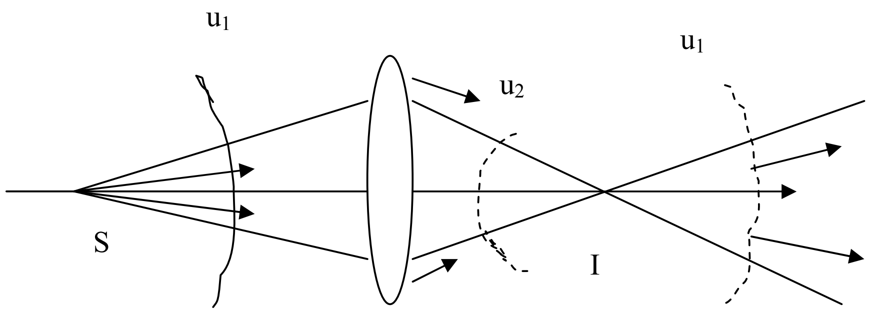

Figure 2. Spherical diverging wave, from a source, focussed by a lens at point image I, then diverging again.

INTERFERENCE

TIME INTERFERENCE

Two time coherent waves

Two coherent waves (of the same frequency) at point P:

$$v_1 = A\, e^{-i\omega t} \qquad v_2 = A\, e^{-(i\omega t - \varphi)}$$

Without loss of generality the amplitudes are assumed equal and real.
Total field $v = v_1 + v_2$. Intensity I:

$$I = |v|^2 = (v_1 + v_2)(v_1^* + v_2^*) = 2A^2 + 2A^2 \cos\varphi$$

Therefore the intensity value depends on the phase difference, and can be higher or lower than the total intensity of the sum of the two waves, as expected. If $\varphi = \pi/2$ I=0, and if $\varphi=0$ $I = 4\ A^2$. Of course total energy in space is conserved due to the phase of other points.

Two waves of different frequency (time incoherent)

$$\omega \neq \omega_1 \qquad \text{for simplicity } \omega_1 > \omega$$

$$v_1 = A\, e^{-i\omega t} \qquad v_2 = A\, e^{-(i\omega_1 t - i\varphi)}$$

$$I = 2A^2 + 2A^2 \left[Re\left(e^{-i\omega t + i\omega_1 t - i\varphi}\right)\right]^2 = 2A^2 + 2A^2 cos^2[(\omega_1 - \omega)t - \varphi]$$

Intensity is oscillating function, called "beating", with frequency $\omega_1 - \omega$ a high frequency in optics. Our eyes (or conventional instruments) cannot follow it, and one sees the mean value $\bar{I}$ of I, averaged over a characteristic time τ of the eye or the instrument:

$$\bar{I} = \frac{1}{\tau}\int_0^{\tau} I(t)\,dt$$

If τ is large with respect to the period of the beating $T_b = 2\pi/(\omega_1-\omega)$ the average oscillating term vanishes and the total intensity of the interference is the sum of the intensity of the two waves. Sum of energies means incoherent waves.

On the contrary, if τ is of the order of or smaller than the period T_b , then the time dependence can be revealed. Modern instrumentation can do this.
Each instrument has its characteristic time τ and one has:

if $\tau \leq \dfrac{2\pi}{\omega_1 - \omega}$ the instrument "sees" **coherent waves**

if $\tau > \dfrac{2\pi}{\omega_1 - \omega}$ the instrument "sees" **incoherent waves**

Therefore interference from two time incoherent waves can be seen as coherent or incoherent depending on the characteristic time of the instrument with respect to the period of the beating.

SPACE INTERFERENCE of COHERENT WAVES

Two plane Waves

Let us consider two plane waves, of same frequency and amplitude propagating in two directions (+ θ and - θ) symmetric with respect to z axis, in plane (x,z). ($\beta = 0$).
The first wave u_1 has cosine directors $\alpha = sin\,\vartheta$, $\beta = 0$, $\gamma = cos\,\vartheta$

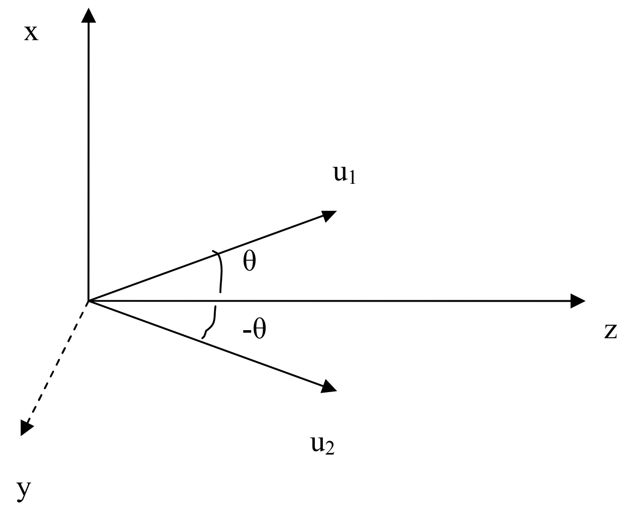

Fig. 3. Scheme of interference of two plane waves

$$u_1 = A\,e^{ik(\alpha x + \gamma z)} \quad \text{and} \quad u_2 = A\,e^{ik(-\alpha x + \gamma z)}$$

$$u = u_1 + u_2 = Ae^{ik\gamma z}\left(e^{ik\alpha x} + e^{-ik\alpha x}\right) = 2Ae^{ik\gamma z}\, cos(kx\, sin\,\vartheta)$$

Intensity

$$I = u\,u^* = 4A^2 \cos^2(kx \sin\vartheta)$$

is periodic function of x, with period p:

$$k\,p \sin\vartheta = \pi \quad ; \quad p = \frac{\lambda}{2\sin\vartheta}$$

On a screen normal to z axis: interference fringes parallel to y axis.
On those planes parallel to plane (y, z) where one has

$$kx \sin\vartheta = (2n+1)\frac{\pi}{2} \quad ,$$

field u(P) vanishes. One can place metallic plane surfaces on these planes, without disturbing the field in between: this is the basis of **metallic guiding** of waves, used not only for microwaves but also in optics, for instance in applications of high power lasers.

Michelson interferometer, **Fabry-Perot interferometer** and many others such as **Mach-Zehnder interferometer** are based on simple or multiple interference between plane waves.

Two spherical waves

Two spherical waves (of the same frequency and equal amplitude coefficient A_1, are centred on $-z_0$ and z_0, respectively.

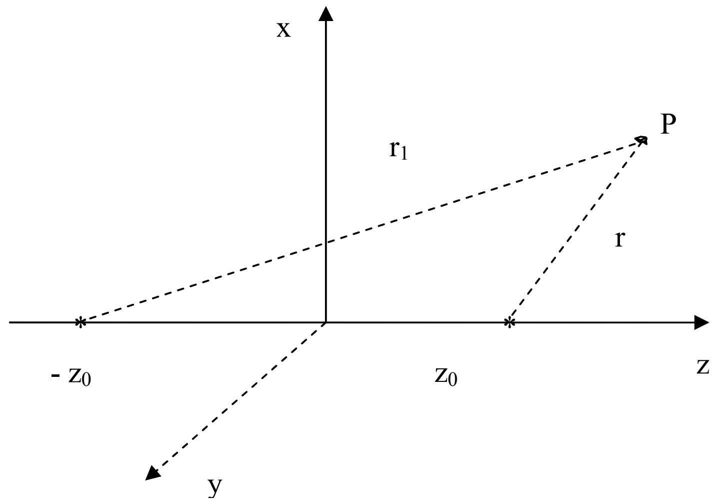

Fig. 4 Two spherical waves with centres on the z axis

$$u_1 = \frac{A_1 \; e^{ikr}}{r} \qquad \text{and} \qquad u_2 = \frac{A_1 \, e^{ikr_1}}{r_1}$$

For small changes of r the phase of the first wave varies strongly, while its denominator does not change appreciably. We can neglect the dependence of denominator on r and call A the resulting amplitude. The same can be said for the second wave. When the two distances are not much different from one another, which is true when the distance between the two sources is small with respect to the distance of point P, one can assume $r = r_1$.

Therefore

$$u = u_1 + u_2 = A\,e^{ikr} + A\,e^{ikr_1} = A\,e^{ik\left(\frac{r+r_1}{2}\right)} \left\{ e^{ik\left(\frac{r-r_1}{2}\right)} + e^{-ik\left(\frac{r-r_1}{2}\right)} \right\} =$$

$$= 2A\,e^{ik\left(\frac{r+r_1}{2}\right)} \cos\left[\frac{k(r-r_1)}{2}\right]$$

Intensity I is:

$$I = 4A^2 \cos^2\left[\frac{k(r-r_1)}{2}\right]$$

Surfaces where

$$\left[\frac{k(r-r_1)}{2}\right] = n\,\pi \qquad \text{that is} \quad r - r_1 = n\,\lambda$$

are lines of maximum intensity: they are hyperboloids of rotation, with foci in the two source points. Analogous relation is found for the lines of zero intensity.

Intersections in a plane $x = d$, are hyperbolas. If d is large with respect to all other distances, by means of a series development, one can see that the fringes are linear in the central region.

Superposition of spherical waves is the basis for the description of many interferometers, such as **Young interferometer** and **Ronchi test**. When one source goes to infinity, one obtains **Newton's rings**.

INTERFEROMETERS

Interferometers played, and still play, a basic role to establish a number of fundamental discoveries of physics. At present interferometers are tools which utilize interference to measure small "quantities", for example in metrology, as well as to evaluate the quality of optics systems.
Below some examples are described of interest here, with no sake of completeness.

a) Young interferometer

Young interferometer was invented by Young to demonstrate the wave nature of the light in opposition to the corpuscular theory by Newton. Young first measured the wavelength of light. A sketch of the experiment is shown in Fig.5. It is simpler than Young experiment as it is based on the use of laser coherent radiation.

In the observation plane, at very large distance $z=L$, and in the central region (θ very small) the fringes are parallel lines normal to the figure plane.
Generally, as the holes are not small, the fringes appear inside a diffraction pattern.
More generally, fringes appear in the intersection between whatever observation plane and the rotation hyperboloids produced by the interference of the two spherical waves, as described in previous theoretical sections.

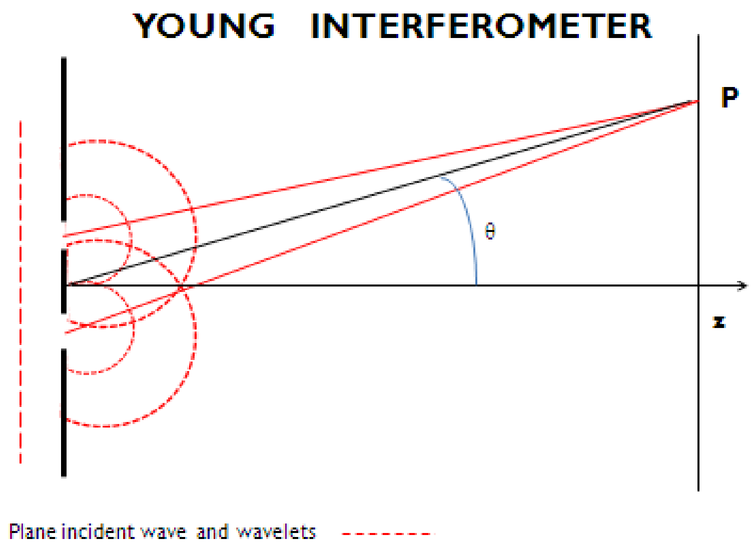

Fig. 5. Scheme of a Young Interferometer.

A plane wave impinges on a screen where two small holes are present. The holes are small enough to neglect their dimensions.
The output of each hole becomes a source of a spherical wave; on a far screen the interference fringes appear.

To be precise the experiment by Young was different, he used two parallel slit, instead of holes, and located another slit between source and slits to produce a coherent cylindrical wave. The slit was needed due to the lack of coherent sources. Now, a laser beam and a lens can give us a field which well approaches a plane monochromatic wave in the region of our interest.

b) Michelson interferometer

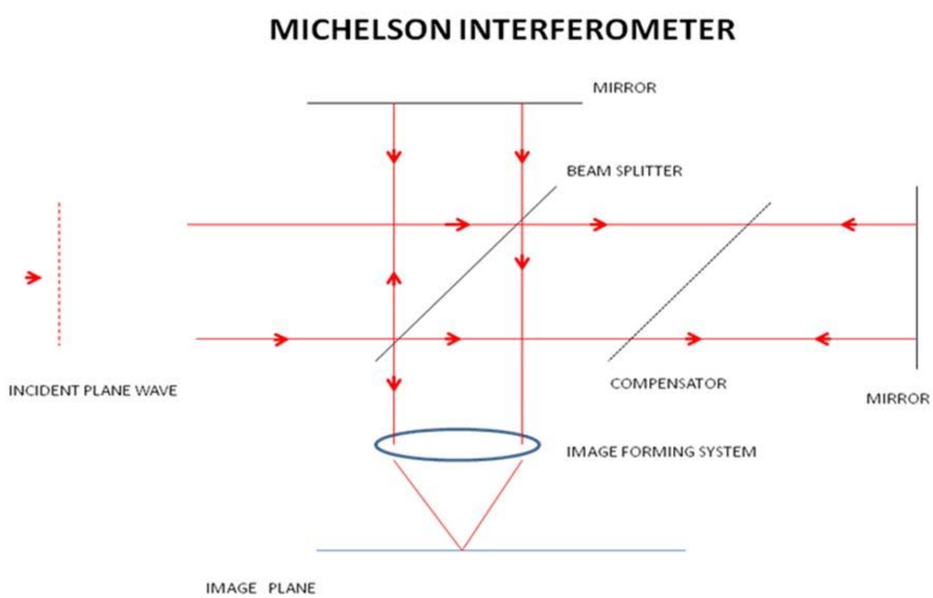

Fig.6. Scheme of a Michelson interferometer.

Michelson's interferometer, Fig 6, was used to make basic experiments in Optics and Physics in general. For instance, the so called Michelson-Morley experiment allowed one to "eliminate" the hypothesis of the existence of the "aether" for light propagation in the empty space, and was the basis for Einstein's theory of relativity.

In this scheme, an impinging plane wave is divided in two equal waves by the beam splitter; the two waves are reflected by the two corresponding mirrors, and are sent to the image plane one by reflection at the beam splitter and the other by crossing it.

In Michelson interferometer the fringe shapes depend on source and configuration. If the two mirrors are not perfectly orthogonal, and there is a small angle α of difference from orthogonality, the fringes are lines as shown in the previous section. When the mirrors are exactly orthogonal, the fringes are circular; this configuration was (and still is) used to measure small lengths.

Michelson made another important application of the interferometer to measure the diameter of the stars. The configuration used is sketched in Fig.7.

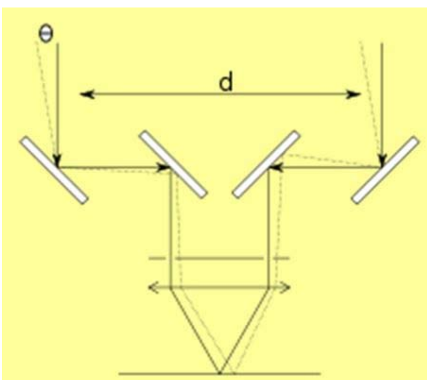

Fig 7. Michelson stellar interferometer to measure star diameter (from Wikipedia)

c) Mach-Zehnder Interferometer

Mach-Zehnder interferometer, Fig.8, was an alternative to Michelson interferometer to be used over very long paths and far apart from one another.

In this interferometer the light crosses each path (arm) only once. In the figure, R is the region where the "material to be investigated" is located. Mach-Zehnder interferometer finds large applications in fluid dynamics, in particular to measure small changes of refractive index in large areas, such as in wind tunnels.

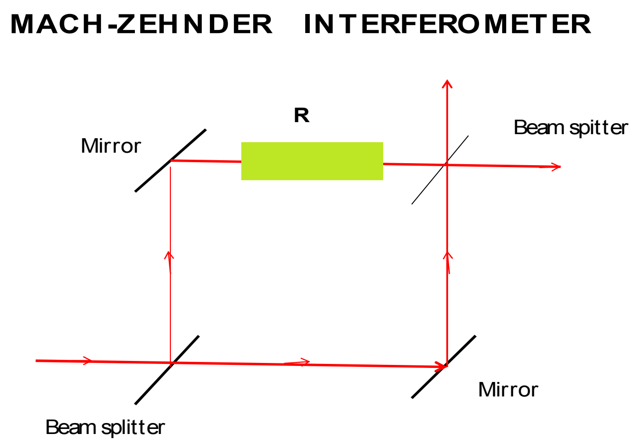

Fig. 8 Scheme of a Mach-Zehnder interferometer. R region where the material to be investigated is located.

FINAL REMARKS

Interference is present always when there are waves, not only in the coherent case, but also in the case of partial coherence or incoherence. Interferometers were also used to measure the coherence degree of fields.
Fabry-Perot interferometer was a basic configuration of laser cavities. Interference filters select monochromatic radiation from non monochromatic one, by multiple reflections at parallel surfaces and consequent multiple interference.
Note also that image formation by any system is an interference procedure, as can be immediately understood thinking of the image of a star by a perfect lens. By remembering that rays are the normals to the wavefront (geometrical optics), one obtains the image in the point where the phases along the different rays are equal. In the case of aberrations the phases are different and the image is spoiled.
It is also to be pointed out that a hologram is nothing else than an interference pattern obtained by interference of an illuminating wave and the wave "scattered" by an object. This pattern is recorded on a support and when illuminated again reproduces the field of the object: amplitude and phase.
Interference of matter waves was also demonstrated, and utilized in the electronic microscope where high resolution is reached (remember De Broglie wavelength).
Atomic interference was the last discovery obtained by using Bose-Einstein condensate, BEC.