ELECTROMAGNETIC WAVES IN MATTER

Imrana Ashraf Zahid

Quaid-i-Azam University Islamabad Pakistan

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Electromagnetic Wave Propagation in Linear Media

EM wave propagation inside matter - in regions with no free charges and no free currents (the medium is an insulator/non-conductor).

For this situation, Maxwell's equations become:

1) $\boxed{\vec{\nabla} \bullet \vec{D}(\vec{r},t) = 0}$

2) $\boxed{\vec{\nabla} \bullet \vec{B}(\vec{r},t) = 0}$

3) $\boxed{\vec{\nabla} \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}}$

4) $\boxed{\vec{\nabla} \times \vec{H}(\vec{r},t) = \frac{\partial \vec{D}(\vec{r},t)}{\partial t}}$

Electromagnetic Wave Propagation in Linear Media

The medium is assumed to be linear, homogeneous and isotropic- thus the following relations are valid in this medium:

$$\boxed{\vec{D}(\vec{r},t) = \varepsilon \vec{E}(\vec{r},t)} \quad \text{and} \quad \boxed{\vec{H}(\vec{r},t) = \frac{1}{\mu}\vec{B}(\vec{r},t)}$$

- ε = electric permittivity of the medium.
- $\varepsilon = \varepsilon_o(1 + \chi_e)$, χ_e = electric susceptibility of the medium.
- μ = magnetic permeability of the medium.
- $\mu = \mu_o(1 + \chi_m)$, χ_m = magnetic susceptibility of the medium.
- ε_o = electric permittivity of free space = 8.85×10^{-12} Farads/m.
- μ_o = magnetic permeability of free space = $4\pi \times 10^{-7}$ Henrys/m.

Electromagnetic Wave Propagation in Linear Media

Maxwell's equations inside the linear, homogeneous and isotropic non-conducting medium become:

1) $\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r},t)=0}$

2) $\boxed{\vec{\nabla}\cdot\vec{B}(\vec{r},t)=0}$

3) $\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t)=-\frac{\partial\vec{B}(\vec{r},t)}{\partial t}}$

4) $\boxed{\vec{\nabla}\times\vec{B}(\vec{r},t)=\mu\varepsilon\frac{\partial\vec{E}(\vec{r},t)}{\partial t}}$

In a linear /homogeneous/isotropic medium, the speed of propagation of EM waves is:

$$\boxed{v'_{prop}=\frac{1}{\sqrt{\varepsilon\mu}}}$$

Electromagnetic Wave Propagation in Linear Media

The ***E*** and ***B*** fields in the medium obey the following wave equation:

$$\boxed{\nabla^2 \vec{E}(\vec{r},t) = \varepsilon\mu \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} = \frac{1}{v'^2_{prop}} \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2}}$$

$$\boxed{\nabla^2 \vec{B}(\vec{r},t) = \varepsilon\mu \frac{\partial \vec{B}(\vec{r},t)}{\partial t} = \frac{1}{v'^2_{prop}} \frac{\partial^2 \vec{B}(\vec{r},t)}{\partial t^2}}$$

Electromagnetic Wave Propagation in Linear Media

For linear / homogeneous / isotropic media:

$$\varepsilon = K_e \varepsilon_o = (1+\chi_e)\varepsilon_o \qquad K_e = \frac{\varepsilon}{\varepsilon_o} = (1+\chi_e) = \text{relative electric permittivity}$$

$$\mu = K_m \mu_0 = (1+\chi_m)\mu_o \qquad K_m = \frac{\mu}{\mu_o} = (1+\chi_m) = \text{relative magnetic permeability}$$

$$v'_{prop} = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{K_e \varepsilon_o K_m \mu_o}} = \frac{1}{\sqrt{K_e K_m}} \frac{1}{\sqrt{\varepsilon_o \mu_o}} = \frac{1}{\sqrt{K_e K_m}} c$$

If $\boxed{K_e K_m \geq 1}$ thus $\boxed{\frac{1}{\sqrt{K_e K_m}} \leq 1} \Rightarrow \boxed{v'_{prop} = \frac{1}{\sqrt{K_e K_m}} c \leq c}$

Electromagnetic Wave Propagation in Linear Media

Note also that since $\boxed{K_e = \frac{\varepsilon}{\varepsilon_o}}$ and $\boxed{K_m = \frac{\mu}{\mu_o}}$ are dimensionless

quantities, then so is $\boxed{\frac{1}{\sqrt{K_e K_m}}}$

Define the index of refraction - *a dimensionless quantity-* of the linear / homogeneous / isotropic medium as:

$$\boxed{n \equiv \sqrt{K_e K_m} = \sqrt{\frac{\varepsilon\mu}{\varepsilon_o \mu_o}}}$$

Electromagnetic Wave Propagation in Linear Media

For linear / homogeneous / isotropic media:

$\boxed{v'_{prop} = c/n \ (\leq c)}$ because $\boxed{n \geq 1}$

For many (but not all) linear/homogeneous/isotropic materials:

$$\boxed{\mu = \mu_o(1+\chi_m) \simeq \mu_o}$$

(*True for many paramagnetic and diamagnetic-type materials*)

$$\boxed{|\chi_m| \sim \vartheta(10^{-8}) \sim 0}$$

Thus $\boxed{K_m = \dfrac{\mu}{\mu_o} = (1+\chi_m) \simeq 1} \Rightarrow \boxed{n \simeq \sqrt{K_e}}$ and $\boxed{v'_{prop} = \dfrac{c}{n} \simeq \dfrac{c}{\sqrt{K_e}}}$.

Electromagnetic Wave Propagation in Linear Media

The instantaneous EM energy density associated with a linear/homogeneous/isotropic material

$$\boxed{u_{EM}(\vec{r},t) = \frac{1}{2}\left(\varepsilon E^2(\vec{r},t) + \frac{1}{\mu}B^2(\vec{r},t)\right) = \frac{1}{2}\left(\vec{E}(\vec{r},t)\bullet\vec{D}(\vec{r},t) + \vec{B}(\vec{r},t)\bullet\vec{H}(\vec{r},t)\right)} \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

with $\boxed{\vec{D}(\vec{r},t) = \varepsilon\vec{E}(\vec{r},t)}$ and $\boxed{\vec{H}(\vec{r},t) = \frac{1}{\mu}\vec{B}(\vec{r},t)}$

Electromagnetic Wave Propagation in Linear Media

The instantaneous Poynting's vector associated with a linear/homogeneous/isotropic material

$$\boxed{\vec{S}(\vec{r},t) = \frac{1}{\mu}\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right) = \left(\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)\right)} \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

The intensity of an EM wave propagating in this medium is:

$$\boxed{I(\vec{r}) \equiv \left\langle \left|\vec{S}(\vec{r},t)\right| \right\rangle = v'_{prop}\left\langle u_{EM}(\vec{r},t)\right\rangle = \frac{1}{2}v'_{prop}\varepsilon E_o^2(\vec{r}) = \frac{1}{2}\left(\frac{c}{n}\right)\varepsilon E_o^2(\vec{r})} \left(\frac{\text{Watts}}{\text{m}^2}\right)$$

Electromagnetic Wave Propagation in Linear Media

The instantaneous linear momentum density associated with an EM wave propagating in a linear/homogeneous/isotropic medium is:

$$\boxed{\vec{\wp}_{EM}(\vec{r},t) = \varepsilon\mu\vec{S}(\vec{r},t) = \frac{1}{v'^2_{prop}}\vec{S}(\vec{r},t) = \varepsilon\cancel{\mu}\frac{1}{\cancel{\mu}}\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right) = \varepsilon\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)}\left(\frac{\text{kg}}{\text{m}^2\text{-sec}}\right)$$

The instantaneous angular momentum density associated with an EM wave propagating in this medium is:

$$\boxed{\vec{\ell}_{EM}(\vec{r},t) = \vec{r}\times\vec{\wp}_{EM}(\vec{r},t) = \varepsilon\,\vec{r}\times\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)}\left(\frac{\text{kg}}{\text{m-sec}}\right)$$

Electromagnetic Wave Propagation in Linear Media

Total instantaneous EM energy: $\boxed{U_{EM}(t) = \int_v u_{EM}(\vec{r},t)\,d\tau}$ (Joules)

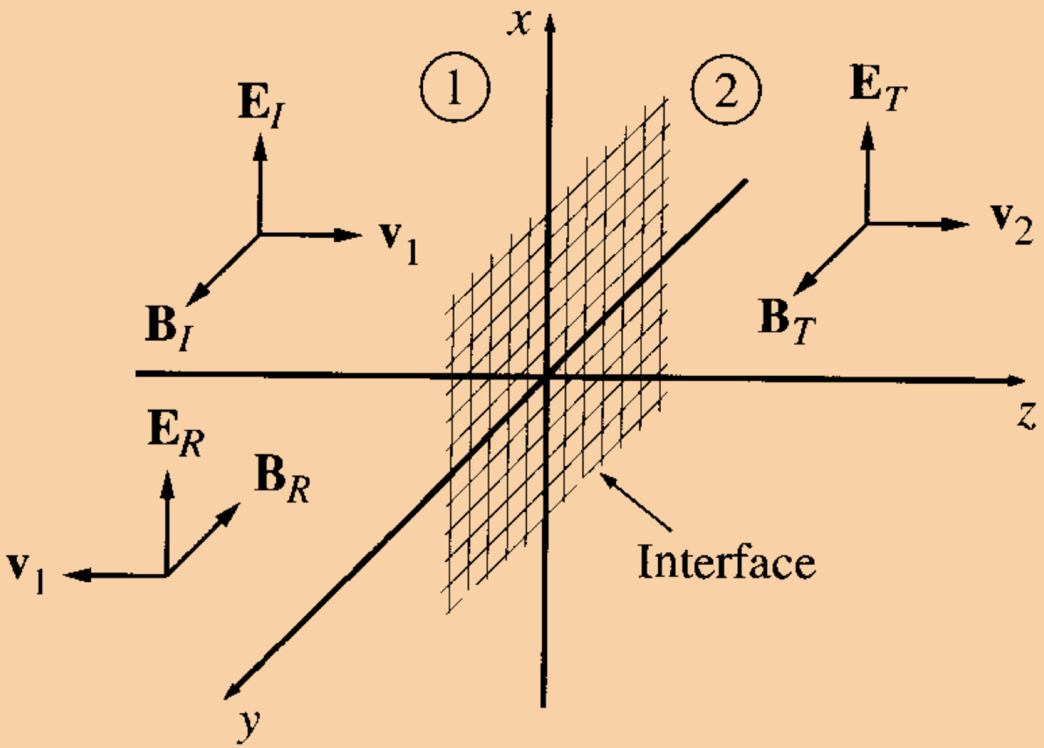
Total instantaneous linear momentum: $\boxed{\vec{p}_{EM}(t) = \int_v \vec{\wp}_{EM}(\vec{r},t)\,d\tau}$ $\left(\frac{\text{kg-m}}{\text{sec}}\right)$

Instantaneous *EM* Power: $\boxed{P_{EM}(t) = \frac{\partial U_{EM}(t)}{\partial t} = -\oint_S \vec{S}(\vec{r},t)\cdot d\vec{a}}$ (Watts)

Total instantaneous angular momentum: $\boxed{\vec{\mathcal{L}}_{EM}(t) = \int_v \vec{\ell}_{EM}(\vec{r},t)\,d\tau}$ $\left(\frac{\text{kg-m}^2}{\text{sec}}\right)$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Suppose the x-y plane forms the boundary between two linear media. A plane wave of frequency ω- travelling in the z- direction and polarized in the x- direction- approaches the interface from the left



Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Incident EM plane wave (in medium 1):

Propagates in the $+\hat{z}$-direction (*i.e.* $\hat{k}_{inc} = +\hat{k}_1 = +\hat{z}$), with polarization $\boxed{\hat{n}_{inc} = +\hat{x}}$

$$\boxed{\tilde{\vec{E}}_{inc}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}\hat{x}} \quad \text{with:} \quad \boxed{k_{inc} = \left|\vec{k}_{inc}\right| = k_1 = \left|\vec{k}_1\right| = 2\pi/\lambda_1 = \omega/v_1}$$

$$\boxed{\tilde{\vec{B}}_{inc}(z,t) = \frac{1}{v_1}\hat{k}_{inc} \times \tilde{\vec{E}}_{inc}(z,t) = \frac{1}{v_1}\tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}\hat{y}} \quad \underline{\text{since}}: \quad \boxed{\hat{k}_{inc} \times \hat{n}_{inc} = +\hat{z} \times \hat{x} = +\hat{y}}$$

Reflected EM plane wave (in medium 1):

Propagates in the $-\hat{z}$-direction (*i.e.* $\hat{k}_{refl} = -\hat{k}_1 = -\hat{z}$), with polarization $\boxed{\hat{n}_{refl} = +\hat{x}}$

$$\boxed{\tilde{\vec{E}}_{refl}(z,t) = \tilde{E}_{o_{refl}} e^{i(-k_1 z - \omega t)}\hat{x}} \quad \text{with:} \quad \boxed{k_{refl} = \left|\vec{k}_{refl}\right| = k_1 = \left|\vec{k}_1\right| = 2\pi/\lambda_1 = \omega/v_1}$$

$$\boxed{\tilde{B}_{refl}(z,t) = \frac{1}{v_1}\hat{k}_{refl} \times \tilde{\vec{E}}_{refl}(z,t) = -\frac{1}{v_1}\tilde{E}_{o_{refl}} e^{i(-k_1 z - \omega t)}\hat{y}} \quad \underline{\text{since}}: \quad \boxed{\hat{k}_{refl} \times \hat{n}_{refl} = -\hat{z} \times \hat{x} = -\hat{y}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Transmitted EM plane wave (in medium 2):

Propagates in the $+\hat{z}$ -direction (*i.e.* $\hat{k}_{trans} = +\hat{k}_2 = +\hat{z}$), with polarization $\boxed{\hat{n}_{trans} = +\hat{x}}$

$\boxed{\tilde{\vec{E}}_{trans}(z,t) = \tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)} \hat{x}}$ with: $\boxed{k_{trans} = \left|\vec{k}_{trans}\right| = k_2 = \left|\vec{k}_2\right| = 2\pi/\lambda_2 = \omega/v_2}$

$\boxed{\tilde{\vec{B}}_{trans}(z,t) = \frac{1}{v_2}\hat{k}_{trans} \times \tilde{\vec{E}}_{trans}(z,t) = \frac{1}{v_2}\tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)} \hat{y}}$ since: $\boxed{\hat{k}_{trans} \times \hat{n}_{trans} = +\hat{z} \times \hat{x} = +\hat{y}}$

In this situation the E -field - polarization vectors are all oriented in the same direction

$\boxed{\hat{n}_{inc} = \hat{n}_{refl} = \hat{n}_{trans} = +\hat{x}}$ or equivalently:

$$\boxed{\vec{E}_{inc}(\vec{r},t) \parallel \vec{E}_{refl}(\vec{r},t) \parallel \vec{E}_{trans}(\vec{r},t)}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

At the interface between the two linear / homogeneous / isotropic media -at z = 0 in the x-y plane- the boundary conditions 1 – 4 must be satisfied for the total E and B -fields immediately present on either side of the interface:

BC 1) Normal $\vec{D}$ continuous: $\boxed{\varepsilon_1 E^{\perp}_{1_{Tot}} = \varepsilon_2 E^{\perp}_{2_{Tot}}}$

(*n.b.* $\perp$ refers to the *x*-*y* boundary, *i.e.* in the $+\hat{z}$ direction)

BC 2) Tangential $\vec{E}$ continuous: $\boxed{E^{\parallel}_{1_{Tot}} = E^{\parallel}_{2_{Tot}}}$

(*n.b.* || refers to the *x*-*y* boundary, *i.e.* in the *x*-*y* plane)

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

BC 3) Normal $\vec{B}$ continuous: $\boxed{B^{\perp}_{1_{Tot}} = B^{\perp}_{2_{Tot}}}$

($\perp$ to x-y boundary, i.e. in the +z^ direction)

BC 4) Tangential $\vec{H}$ continuous: $\boxed{\frac{1}{\mu_1} B^{\parallel}_{1_{Tot}} = \frac{1}{\mu_2} B^{\parallel}_{2_{Tot}}}$

($\parallel$ to x-y boundary, i.e. in x-y plane)

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

For plane EM waves at normal incidence on the boundary at z = 0- lying in the x-y plane- no components of **E or B** (incident, reflected or transmitted waves) - allowed to be along the $\pm\hat{z}$ propagation direction (s) - the E and *B*-field are transverse fields -constraints imposed by Maxwell's equations.

BC -1) and BC- 3) impose no restrictions on such *EM* waves since:

$$\{ E^{\perp}_{1_{Tot}} = \mathrm{E}^{z}_{1_{Tot}} = 0 \, ; \; E^{\perp}_{2_{Tot}} = E^{z}_{2_{Tot}} = 0 \} \text{ and } \{ B^{\perp}_{1_{Tot}} = B^{z}_{1_{Tot}} = 0 \, ; \; B^{\perp}_{2_{Tot}} = B^{z}_{2_{Tot}} = 0 \}$$

⇒ The only restrictions on plane EM waves propagating with normal incidence on the boundary at z = 0 are imposed by BC-2) and BC- 4).

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

At z = 0 in medium 1) (i.e. z ≤ 0) we must have:

$$\boxed{\vec{\tilde{E}}^{\parallel}_{1_{Tot}}(z=0,t)=\vec{\tilde{E}}_{inc}(z=0,t)+\vec{\tilde{E}}_{refl}(z=0,t)} \text{ and}$$

$$\boxed{\frac{1}{\mu_1}\vec{\tilde{B}}^{\parallel}_{1_{Tot}}(z=0,t)=\frac{1}{\mu_1}\vec{\tilde{B}}_{inc}(z=0,t)+\frac{1}{\mu_1}\vec{\tilde{B}}_{refl}(z=0,t)}$$

While at z = 0 in medium 2) (i.e. z ≥ 0) we must have:

$$\boxed{\vec{\tilde{E}}^{\parallel}_{2_{Tot}}(z=0,t)=\vec{\tilde{E}}_{trans}(z=0,t)} \text{ and}$$

$$\boxed{\frac{1}{\mu_2}\vec{\tilde{B}}^{\parallel}_{2_{Tot}}(z=0,t)=\frac{1}{\mu_2}\vec{\tilde{B}}_{trans}(z=0,t)}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

BC 2) -Tangential *E* is continuous @ z = 0) requires that:

$$\boxed{\vec{\tilde{E}}^{\parallel}_{1_{Tot}}\Big|_{z=0} = \vec{\tilde{E}}^{\parallel}_{2_{Tot}}\Big|_{z=0}} \quad \text{or:} \quad \boxed{\vec{\tilde{E}}_{inc}(z=0,t)+\vec{\tilde{E}}_{refl}(z=0,t)=\vec{\tilde{E}}_{trans}(z=0,t)}.$$

BC 4) -Tangential *H* is continuous @ z = 0) requires that:

$$\boxed{\frac{1}{\mu_1}\vec{\tilde{B}}^{\parallel}_{1_{Tot}}\Big|_{z=0} = \frac{1}{\mu_2}\vec{\tilde{B}}^{\parallel}_{2_{Tot}}\Big|_{z=0}}$$

$$\text{or:} \quad \boxed{\frac{1}{\mu_1}\vec{\tilde{B}}_{inc}(z=0,t)+\frac{1}{\mu_1}\vec{\tilde{B}}_{refl}(z=0,t)=\frac{1}{\mu_2}\vec{\tilde{B}}_{trans}(z=0,t)}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Using explicit expressions for the complex **E** and **B** fields

$$\boxed{\vec{\tilde{E}}_{inc}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)}\hat{x}}$$

$$\boxed{\vec{\tilde{B}}_{inc}(z,t) = \frac{1}{v_1}\hat{k}_{inc} \times \vec{\tilde{E}}_{inc}(z,t) = \frac{1}{v_1}\vec{\tilde{E}}_{o_{inc}} e^{i(k_1 z - \omega t)}\hat{y}}$$

$$\boxed{\vec{\tilde{E}}_{refl}(z,t) = \tilde{E}_{o_{refl}} e^{i(-k_1 z - \omega t)}\hat{x}}$$

$$\boxed{\tilde{B}_{refl}(z,t) = \frac{1}{v_1}\hat{k}_{refl} \times \vec{\tilde{E}}_{refl}(z,t) = -\frac{1}{v_1}\tilde{E}_{o_{refl}} e^{i(-k_1 z - \omega t)}\hat{y}}$$

$$\boxed{\vec{\tilde{E}}_{trans}(z,t) = \tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)}\hat{x}}$$

$$\boxed{\tilde{B}_{trans}(z,t) = \frac{1}{v_2}\hat{k}_{trans} \times \vec{\tilde{E}}_{trans}(z,t) = \frac{1}{v_2}\tilde{E}_{o_{trans}} e^{i(k_2 z - \omega t)}\hat{y}}$$

The above boundary condition relations become

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

BC 2) (Tangential $\vec{E}$ continuous @ $z = 0$): $\boxed{\tilde{E}_{0_{inc}} \cancel{e^{-i\omega t}} + \tilde{E}_{0_{refl}} \cancel{e^{-i\omega t}} = \tilde{E}_{0_{trans}} \cancel{e^{-i\omega t}}}$

BC 4) (Tangential $\vec{H}$ continuous @ $z = 0$): $\boxed{\frac{1}{\mu_1 v_1} \tilde{E}_{0_{inc}} \cancel{e^{-i\omega t}} - \frac{1}{\mu_1 v_1} \tilde{E}_{0_{refl}} \cancel{e^{-i\omega t}} = \frac{1}{\mu_2 v_2} \tilde{E}_{0_{trans}} \cancel{e^{-i\omega t}}}$

Cancelling the common $e^{-i\omega t}$ factors on the LHS & RHS of above equations - we have at z = 0 (everywhere in the x-y plane- must be independent of any time t):

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

BC 2) (Tangential $\vec{E}$ continuous @ $z = 0$): $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$

BC 4) (Tangential $\vec{H}$ continuous @ $z = 0$): $\boxed{\frac{1}{\mu_1 v_1}\tilde{E}_{o_{inc}} - \frac{1}{\mu_1 v_1}\tilde{E}_{o_{refl}} = \frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}}$

Assuming that {μ_1 and μ_2} and {v_1 and v_2} are known / given for the two media, we have <u>two</u> equations {from BC 2) and BC 4)} and <u>*three*</u> unknowns {$\tilde{E}_{o_{inc}}$, $\tilde{E}_{o_{refl}}$, $\tilde{E}_{o_{trans}}$}

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

→ Solve above equations simultaneously for

$\{\tilde{E}_{o_{refl}}$ and $\tilde{E}_{o_{trans}}\}$ in terms of / scaled to $\tilde{E}_{o_{inc}}$.

Define: $$\boxed{\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}}$$

BC 4) -Tangential ***H*** continuous @ z = 0- relation becomes:

$$\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \beta\,\tilde{E}_{o_{trans}}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

BC 2) -Tangential **E** continuous @ z = 0 - gives:

$$\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$$

BC 4) -Tangential ***H*** continuous @ z = 0- reduces to

$$\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \beta\,\tilde{E}_{o_{trans}}} \quad \text{with} \quad \boxed{\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Add and Subtract BC -2) and BC- 4) relations:

$$\boxed{2\tilde{E}_{o_{inc}} = (1+\beta)\,\tilde{E}_{o_{trans}}} \Rightarrow \boxed{\tilde{E}_{o_{trans}} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{o_{inc}}} \quad (2+4)$$

$$\boxed{2\tilde{E}_{o_{refl}} = (1-\beta)\,\tilde{E}_{o_{trans}}} \Rightarrow \boxed{\tilde{E}_{o_{refl}} = \left(\frac{1-\beta}{2}\right)\tilde{E}_{o_{trans}}} \quad (2-4)$$

Insert the result of eqn. (2+4) into eqn. (2—4):

$$\boxed{\tilde{E}_{o_{refl}} = \left(\frac{1-\beta}{\cancel{2}}\right)\left(\frac{\cancel{2}}{1+\beta}\right)\tilde{E}_{o_{inc}} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{o_{inc}}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

$$\boxed{\tilde{E}_{o_{refl}} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{o_{inc}}} \quad \text{and} \quad \boxed{\tilde{E}_{o_{trans}} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{o_{inc}}}$$

<u>Now</u>: $\boxed{\beta \equiv \dfrac{\mu_1 v_1}{\mu_2 v_2}}$ and: $\boxed{v_1 = \dfrac{c}{n_1}}$, $\boxed{v_2 = \dfrac{c}{n_2}}$ where: $\boxed{n_1 = \sqrt{\dfrac{\varepsilon_1\mu_1}{\varepsilon_o\mu_o}}}$ and $\boxed{n_2 = \sqrt{\dfrac{\varepsilon_2\mu_2}{\varepsilon_o\mu_o}}}$

$$\boxed{\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 (c/n_1)}{\mu_2 (c/n_2)} = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\mu_1\sqrt{\varepsilon_2\mu_2/\varepsilon_o\mu_o}}{\mu_2\sqrt{\varepsilon_1\mu_1/\varepsilon_o\mu_o}} = \frac{\mu_1}{\mu_2}\frac{\sqrt{\varepsilon_2\mu_2}}{\sqrt{\varepsilon_1\mu_1}} = \sqrt{\left(\frac{\varepsilon_2}{\mu_2}\right)\Big/\left(\frac{\varepsilon_1}{\mu_1}\right)} = \sqrt{\frac{\varepsilon_2\mu_1}{\varepsilon_1\mu_2}}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Now if the two media are both paramagnetic or diamagnetic-such that

$$\boxed{\left|\chi_{m_{1,2}}\right| \ll 1}$$

$$i.e.\ \boxed{\mu_1 = \mu_o\left(1+\chi_{m_1}\right) \approx \mu_o} \text{ and: } \boxed{\mu_2 = \mu_o\left(1+\chi_{m_2}\right) \approx \mu_o}$$

Common for many (but not all) non-conducting linear/homogeneous/isotropic media

Then $$\boxed{\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} \simeq \left(\frac{v_1}{v_2}\right) = \left(\frac{n_2}{n_1}\right)} \text{ for } \boxed{\mu_1 \approx \mu_2 \approx \mu_o} \text{ or } \boxed{\left|\chi_{m_{1,2}}\right| \ll 1}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Then

$$\tilde{E}_{o_{refl}} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{o_{inc}} \simeq \left(\frac{1-(v_1/v_2)}{1+(v_1/v_2)}\right)\tilde{E}_{o_{inc}} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right)\tilde{E}_{o_{inc}}$$

$$\tilde{E}_{o_{trans}} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{o_{inc}} \simeq \left(\frac{2}{1+(v_1/v_2)}\right)\tilde{E}_{o_{nc}} = \left(\frac{2v_2}{v_2 + v_1}\right)\tilde{E}_{o_{inc}}$$

We can alternatively express these relations in terms of the indices of refraction n_1 and n_2:

$$\tilde{E}_{o_{refl}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)\tilde{E}_{o_{inc}} \quad \text{and} \quad \tilde{E}_{o_{trans}} = \left(\frac{2n_1}{n_1 + n_2}\right)\tilde{E}_{o_{inc}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Now since:

$$\boxed{\tilde{E}_{o_{inc}} = E_{o_{inc}} e^{i\delta}}$$

$$\boxed{\tilde{E}_{o_{refl}} = E_{o_{refl}} e^{i\delta}}$$

$$\boxed{\tilde{E}_{o_{trans}} = E_{o_{trans}} e^{i\delta}}$$

δ = phase angle (in radians) defined at the zero of time - $t = 0$

Then for the purely real amplitudes $(E_{o_{inc}}, E_{o_{refl}}, E_{o_{trans}})$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

The relations between real amplitudes become:

for $\boxed{\mu_1 \simeq \mu_2 \simeq \mu_o}$

$$\boxed{E_{o_{refl}} = \left(\frac{1-\beta}{1+\beta}\right) E_{o_{inc}} \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right) E_{o_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{o_{inc}}} \qquad \boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)}$$

$$\boxed{E_{o_{trans}} = \left(\frac{2}{1+\beta}\right) E_{o_{inc}} \simeq \left(\frac{2v_2}{v_1 + v_1}\right) E_{o_{inc}} = \left(\frac{2n_1}{n_1 + n_2}\right) E_{o_{inc}}}$$

for $\boxed{\mu_1 \simeq \mu_2 \simeq \mu_o}$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Monochromatic plane EM wave at normal incidence on a boundary between two linear / homogeneous / isotropic media for $\boxed{\mu_1 \simeq \mu_2 \simeq \mu_o}$ for the following cases:

If $v_2 > v_1$ $(i.e.\ n_2 < n_1)$ $\{e.g.$ medium 1) = glass $\Rightarrow$ medium 2) = air$\}$:

$$\boxed{E_{o_{refl}} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) E_{o_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{o_{inc}}} \Rightarrow$$

$E_{o_{refl}}$ is precisely in-phase with $E_{o_{inc}}$ because $(v_2 - v_1) > 0$.

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

If $v_2 < v_1$ $(i.e.\ n_2 > n_1)$ $\{e.g.$ medium 1) = air $\Rightarrow$ medium 2) = glass$\}$:

$$\boxed{E_{0_{refl}} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) E_{0_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{0_{inc}}} \Rightarrow$$

$E_{0_{refl}}$ is 180^{o} out-of-phase with $E_{0_{inc}}$ because $(v_2 - v_1) < 0$.

$$i.e. \quad \boxed{E_{0_{refl}} = -\left|\frac{v_2 - v_1}{v_2 + v_1}\right| E_{0_{inc}} = -\left|\frac{n_1 - n_2}{n_1 + n_2}\right| E_{0_{inc}}} \Rightarrow$$

The minus sign indicates a 180° phase shift occurs upon reflection for $v_2 < v_1$ (i.e. $n_2 > n_1$) !!!

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

$E_{o_{trans}}$ is <u>always</u> in-phase with $E_{o_{inc}}$ for all possible v_1 & v_2 $(n_1$ & $n_2)$ because:

$$\boxed{E_{o_{trans}} = \left(\frac{2}{1+\beta}\right)E_{o_{inc}} \simeq \left(\frac{2v_2}{v_1+v_1}\right)E_{o_{inc}} = \left(\frac{2n_1}{n_1+n_2}\right)E_{o_{inc}}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

What fraction of the incident *EM* wave energy is reflected ?

What fraction of the incident *EM* wave energy is transmitted?

In a given linear/homogeneous/isotropic medium with

$$\boxed{v = \sqrt{\frac{\varepsilon_o \mu_o}{\varepsilon \mu}}\, c = c/n}$$

The time-averaged energy density in the EM wave is:

$$\boxed{\langle u_{EM}(\vec{r},t)\rangle = \frac{1}{2}\varepsilon E_o^2(\vec{r}) = \varepsilon E_{o_{rms}}^2(\vec{r})} \left(\frac{\text{Joules}}{\text{m}^3}\right)$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

The time-averaged Poynting's vector is:

$$\boxed{\left\langle \vec{S}(\vec{r},t) \right\rangle = \frac{1}{\mu}\left\langle \vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t) \right\rangle} \left(\frac{\text{Watts}}{\text{m}^2} \right)$$

The intensity of the EM wave is:

$$\boxed{I(\vec{r}) \equiv \left\langle \left| \vec{S}(\vec{r},t) \right| \right\rangle = v\left\langle u_{EM}(\vec{r},t) \right\rangle = v\left(\frac{1}{2}\varepsilon E_o^2(\vec{r}) \right) = \frac{1}{2}\varepsilon v E_o^2(\vec{r}) = \varepsilon v E_{o_{rms}}^2(\vec{r})} \left(\frac{\text{Watts}}{\text{m}^2} \right)$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

The three Poynting's vectors associated with this problem are such that

$$\vec{S}_{inc} \parallel (+\hat{z}), \quad \vec{S}_{refl} \parallel (-\hat{z}) \text{ and } \vec{S}_{trans} \parallel (+\hat{z})$$

For a monochromatic plane EM wave at normal incidence with

$$\mu_1 \simeq \mu_2 \simeq \mu_o$$

$$\boxed{E_{o_{refl}} = \left(\frac{1-\beta}{1+\beta}\right) E_{o_{inc}} \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right) E_{o_{inc}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{o_{inc}}} \qquad \boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)}$$

$$\boxed{E_{o_{trans}} = \left(\frac{2}{1+\beta}\right) E_{o_{inc}} \simeq \left(\frac{2v_2}{v_1 + v_1}\right) E_{o_{inc}} = \left(\frac{2n_1}{n_1 + n_2}\right) E_{o_{inc}}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Take the ratios $\left(E_{o_{refl}} / E_{o_{inc}}\right)$ and $\left(E_{o_{trans}} / E_{o_{inc}}\right)$ - then square them:

$$\boxed{\left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2 = \left(\frac{1-\beta}{1+\beta}\right)^2 \approx \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2}$$

and

$$\boxed{\left(\frac{E_{o_{trans}}}{E_{o_{inc}}}\right)^2 = \left(\frac{2}{1+\beta}\right)^2 \approx \left(\frac{2v_2}{v_2 + v_1}\right)^2 = \left(\frac{2n_1}{n_1 + n_2}\right)^2}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Define the reflection coefficient as:

$$R(\vec{r}) \equiv \left(\frac{I_{refl}(\vec{r})}{I_{inc}(\vec{r})} \right) = \frac{\left\langle \left| \vec{S}_{refl}(\vec{r},t) \right| \right\rangle}{\left\langle \left| \vec{S}_{inc}(\vec{r},t) \right| \right\rangle} = \frac{v_1 \left\langle u_{EM}^{refl}(\vec{r},t) \right\rangle}{v_1 \left\langle u_{EM}^{inc}(\vec{r},t) \right\rangle} = \frac{\left\langle u_{EM}^{refl}(\vec{r},t) \right\rangle}{\left\langle u_{EM}^{inc}(\vec{r},t) \right\rangle} = \frac{\frac{1}{2}\varepsilon_1 v_1 E_{o_{refl}}^2(\vec{r})}{\frac{1}{2}\varepsilon_1 v_1 E_{o_{inc}}^2(\vec{r})} = \frac{E_{o_{refl}}^2(\vec{r})}{E_{o_{inc}}^2(\vec{r})}$$

Define the transmission coefficient as:

$$T(\vec{r}) \equiv \left(\frac{I_{trans}(\vec{r})}{I_{inc}(\vec{r})} \right) = \frac{\left\langle \left| \vec{S}_{trans}(\vec{r},t) \right| \right\rangle}{\left\langle \left| \vec{S}_{inc}(\vec{r},t) \right| \right\rangle} = \frac{v_2 \left\langle u_{EM}^{trans}(\vec{r},t) \right\rangle}{v_1 \left\langle u_{EM}^{inc}(\vec{r},t) \right\rangle} = \frac{\left(\frac{1}{2}\varepsilon_2 v_2 E_{o_{trans}}^2(\vec{r})\right)}{\left(\frac{1}{2}\varepsilon_1 v_1 E_{o_{inc}}^2(\vec{r})\right)} = \frac{\varepsilon_2 v_2 E_{o_{trans}}^2(\vec{r})}{\varepsilon_1 v_1 E_{o_{inc}}^2(\vec{r})}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

For a linearly-polarized monochromatic plane EM wave at normal incidence on a boundary between two linear / homogeneous / isotropic media, with $\mu_1 \simeq \mu_2 \simeq \mu_o$

Reflection coefficient:

$$R(\vec{r}) \equiv \left(\frac{I_{refl}(\vec{r})}{I_{inc}(\vec{r})} \right) = \left(\frac{E_{o_{refl}}(\vec{r})}{E_{o\,inc}(\vec{r})} \right)^2$$

Transmission coefficient:

$$T(\vec{r}) \equiv \left(\frac{I_{trans}(\vec{r})}{I_{inc}(\vec{r})} \right) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} \right) \left(\frac{E_{o_{trans}}(\vec{r})}{E_{o_{inc}}(\vec{r})} \right)^2$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

But:

$$\boxed{\left(\frac{E_{o_{refl}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2 = \left(\frac{1-\beta}{1+\beta}\right)^2 \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2} \quad \&$$

$$\boxed{\left(\frac{E_{o_{trans}}(\vec{r})}{E_{o_{inc}}(\vec{r})}\right)^2 = \left(\frac{2}{1+\beta}\right)^2 \simeq \left(\frac{2v_2}{v_2 + v_1}\right)^2 = \left(\frac{2n_1}{n_1 + n_2}\right)^2}$$

Thus Reflection and Transmission coefficient:

$$\boxed{R(\vec{r}) \equiv \left(\frac{1-\beta}{1+\beta}\right)^2 \simeq \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2} \qquad \boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)}$$

$$\boxed{T(\vec{r}) \equiv \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{2}{1+\beta}\right)^2 \simeq \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{2v_2}{v_2 + v_1}\right)^2 = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{2n_1}{n_1 + n_2}\right)^2}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

Now: $\boxed{\dfrac{\varepsilon_2 v_2}{\varepsilon_1 v_1} = \dfrac{\dfrac{\varepsilon_2\mu_2 v_2}{\mu_2}}{\dfrac{\varepsilon_1\mu_1 v_1}{\mu_1}}}$ but: $\boxed{\begin{array}{l} v_2^2 = \dfrac{1}{\varepsilon_2\mu_2} \;\Rightarrow\; \varepsilon_2\mu_2 = \dfrac{1}{v_2^2} \\ v_1^2 = \dfrac{1}{\varepsilon_1\mu_1} \;\Rightarrow\; \varepsilon_1\mu_1 = \dfrac{1}{v_1^2} \end{array}}$

$$\boxed{\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} = \frac{\left(\dfrac{1}{v_2{}^2}\cdot v_2\right)\Big/\mu_2}{\left(\dfrac{1}{v_1{}^2}\cdot v_1\right)\Big/\mu_2} = \frac{1\big/\mu_2 v_2}{1\big/\mu_1 v_1} = \frac{\mu_1 v_1}{\mu_2 v_2} \equiv \beta} \;!!! \quad i.e. \quad \boxed{\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}}$$

Reflection & Transmission of Linear Polarized Plane EM Waves at Normal Incidence

for $\boxed{\mu_1 \approx \mu_2 \approx \mu_o}$

$$\boxed{T(\vec{r}) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{2}{1+\beta}\right)^2 = \beta\left(\frac{2}{1+\beta}\right)^2 = \frac{4\beta}{(1+\beta)^2} \simeq \frac{4v_2 v_1}{(v_2+v_1)^2} = \frac{4n_1 n_2}{(n_1+n_2)^2}}$$

Thus:

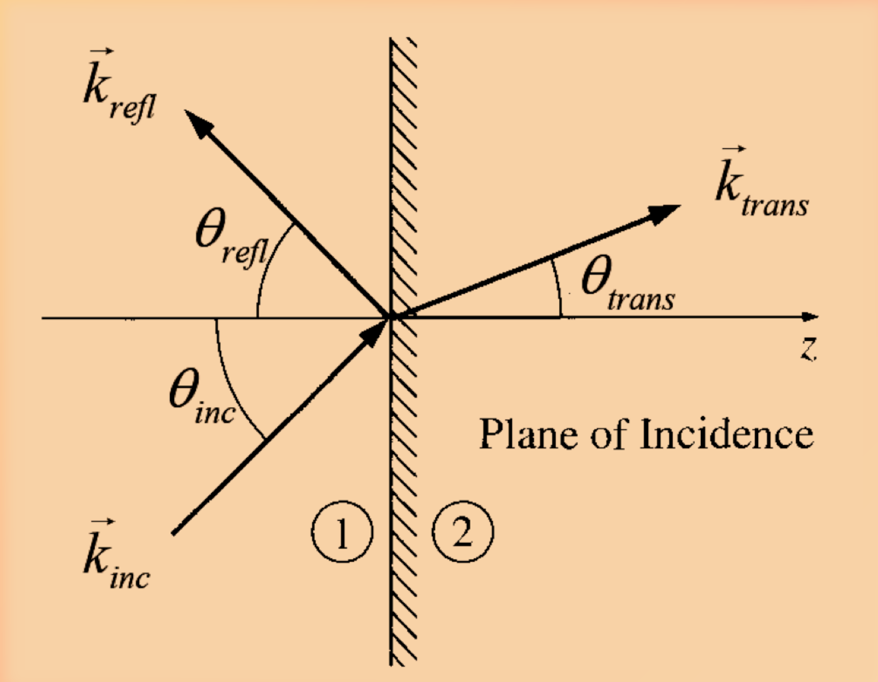
$$\boxed{R(\vec{r})+T(\vec{r}) = \frac{(1-\beta)^2}{(1+\beta)^2}+\frac{4\beta}{(1+\beta)^2} = \frac{(1-\beta)^2+4\beta}{(1+\beta)^2} = \frac{1-2\beta+\beta^2+4\beta}{(1+\beta)^2} = \frac{1+2\beta+\beta^2}{(1+\beta)^2} = \frac{(1+\beta)^2}{(1+\beta)^2} = 1}$$

$$\boxed{R(\vec{r})+T(\vec{r}) = 1}$$

$\Rightarrow$EM energy is conserved at the interface/boundary between two L/H/I media

Reflection & Transmission of Monochromatic Plane EM Waves at Oblique Incidence

A monochromatic plane EM wave incident at an oblique angle θ_{inc} on a boundary between two linear/ homogeneous/isotropic media, defined with respect to the normal to the interface- as shown



Reflection & Transmission of Monochromatic Plane EM Waves at Oblique Incidence

The incident EM wave is:

$$\boxed{\tilde{\vec{E}}_{inc}(\vec{r},t)=\tilde{\vec{E}}_{o_{inc}}e^{i\left(\vec{k}_{inc}\cdot\vec{r}-\omega t\right)}} \quad \text{and} \quad \boxed{\tilde{\vec{B}}_{inc}(\vec{r},t)=\frac{1}{v_1}\hat{k}_{inc}\times\tilde{\vec{E}}_{inc}(\vec{r},t)}$$

The reflected EM wave is:

$$\boxed{\tilde{\vec{E}}_{refl}(\vec{r},t)=\tilde{\vec{E}}_{o_{refl}}e^{i\left(\vec{k}_{refl}\cdot\vec{r}-\omega t\right)}} \quad \text{and} \quad \boxed{\tilde{\vec{B}}_{refl}(\vec{r},t)=\frac{1}{v_1}\hat{k}_{refl}\times\tilde{\vec{E}}_{refl}(\vec{r},t)}$$

The transmitted EM wave is:

$$\boxed{\tilde{\vec{E}}_{trans}(\vec{r},t)=\tilde{\vec{E}}_{o_{trans}}e^{i\left(\vec{k}_{trans}\cdot\vec{r}-\omega t\right)}} \quad \text{and} \quad \boxed{\tilde{\vec{B}}_{trans}(\vec{r},t)=\frac{1}{v_2}\hat{k}_{trans}\times\tilde{\vec{E}}_{trans}(\vec{r},t)}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

All three EM waves have the same frequency- $f = \omega/2\pi$

$$\omega = k_{inc} v_1 = k_{refl} v_1 = k_{trans} v_2$$

$$\boxed{k_{inc} = k_{refl} = k_1 = \left(\frac{v_2}{v_1}\right) k_{trans} = \left(\frac{v_2}{v_1}\right) k_2 = \left(\frac{n_1}{n_2}\right) k_{trans} = \left(\frac{n_1}{n_2}\right) k_2}$$

$$\boxed{v_i = c/n_i \quad i = 1,2}$$

The total EM fields in medium 1

$$\boxed{\vec{\tilde{E}}_{Tot_1}(\vec{r},t) = \vec{\tilde{E}}_{inc}(\vec{r},t) + \vec{\tilde{E}}_{refl}(\vec{r},t)} \quad \text{and} \quad \boxed{\vec{\tilde{B}}_{Tot_1}(\vec{r},t) = \vec{\tilde{B}}_{inc}(\vec{r},t) + \vec{\tilde{B}}_{refl}(\vec{r},t)}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Must match to the total EM fields in medium 2:

$$\boxed{\vec{\tilde{E}}_{Tot_2}(\vec{r},t) = \vec{\tilde{E}}_{trans}(\vec{r},t)} \quad \text{and} \quad \boxed{\vec{\tilde{B}}_{Tot_2}(\vec{r},t) = \vec{\tilde{B}}_{trans}(\vec{r},t)}$$

Using the boundary conditions BC1) → BC4) at z = 0.

At $z = 0$- four boundary conditions are of the form:

$$\boxed{(\sim)\, e^{i(\vec{k}_{inc}\cdot\vec{r}-\omega t)} + (\sim)\, e^{i(\vec{k}_{refl}\cdot\vec{r}-\omega t)} = (\sim)\, e^{i(\vec{k}_{trans}\cdot\vec{r}-\omega t)}}$$

They must hold for all (x,y) on the interface at z = 0 - and also must hold for all times t. The above relation is already satisfied for arbitrary time, t - the factor $e^{-i\omega t}$ is common to all terms.

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The following relation must hold for all (x,y) on interface at at z = 0:

$$\boxed{(\sim\!\sim)\, e^{i\left(\vec{k}_{inc}\cdot\vec{r}\right)} + (\sim\!\sim)\ e^{i\left(\vec{k}_{refl}\cdot\vec{r}\right)} = (\sim\!\sim)\ e^{i\left(\vec{k}_{trans}\cdot\vec{r}\right)}}$$

When $z = 0$ - at interface we must have:

$$\vec{k}_{inc} \bullet \vec{r} = \vec{k}_{refl} \bullet \vec{r} = \vec{k}_{trans} \bullet \vec{r}$$

$$k_{inc_x} x + k_{inc_y} y = k_{refl_x} x + k_{refl_y} y = k_{trans_x} x + k_{trans_y} y \qquad @\, z = 0$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The above relation can only hold for arbitrary (x, y, z = 0) **iff (= if and only if):**

$$k_{inc_x} x = k_{refl_x} x = k_{trans_x} x \quad \Rightarrow \quad k_{inc_x} = k_{refl_x} = k_{trans_x}$$

$$k_{inc_y} y = k_{refl_y} y = k_{trans_y} y \quad \Rightarrow \quad k_{inc_y} = k_{refl_y} = k_{trans_y}$$

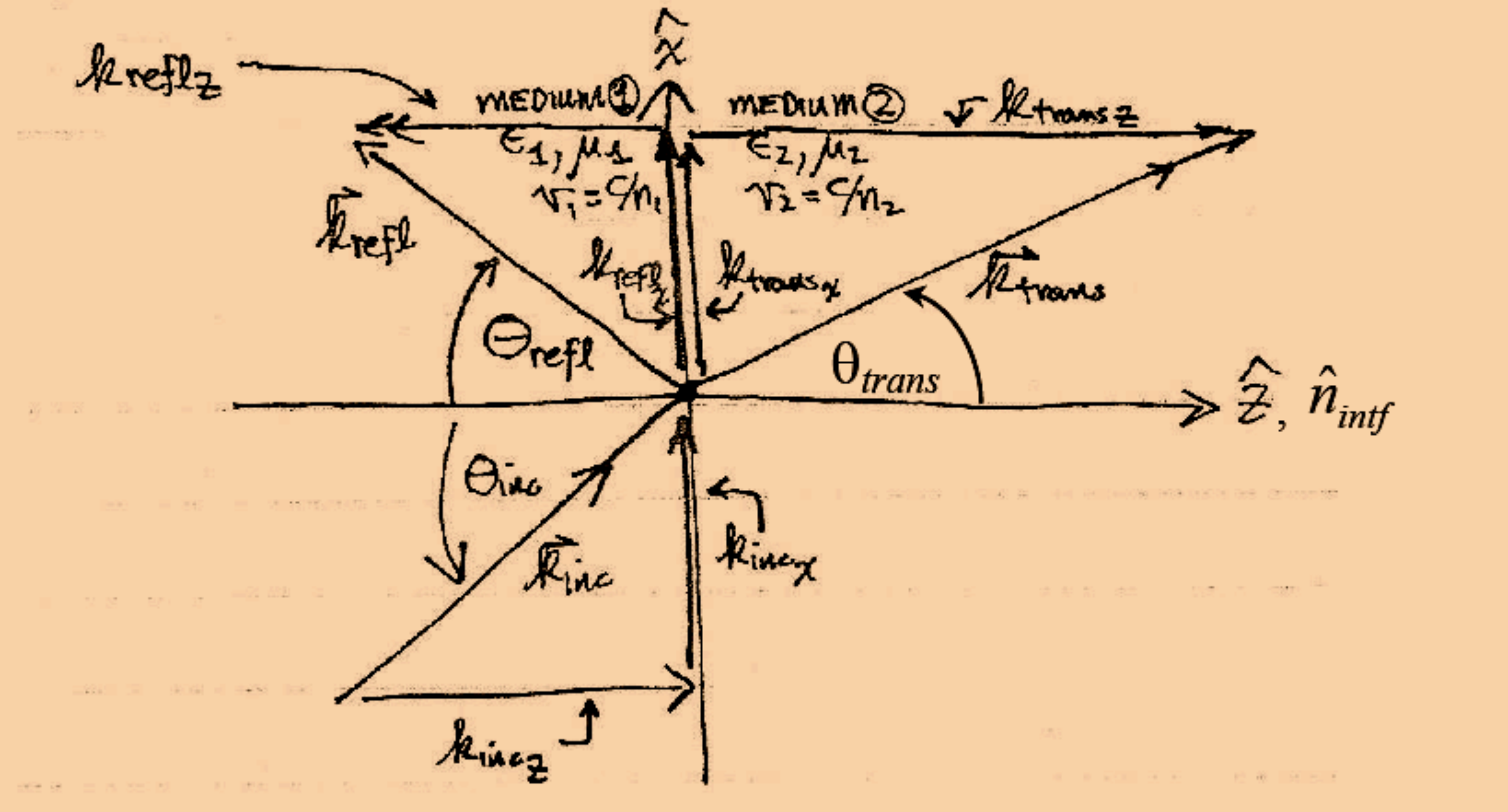
Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The problem has rotational symmetry about the z –axis- without any loss of generality - choose k to lie entirely within the x-z plane- that is no component of k in y-direction as shown in the figure on next slide

$$k_{inc_y} = k_{refl_y} = k_{trans_y} = 0 \quad \text{and thus:} \quad k_{inc_x} = k_{refl_x} = k_{trans_x}$$

The transverse components of $\vec{k}_{inc}, \vec{k}_{refl}, \vec{k}_{trans}$ re all equal and point in the +x^ direction.

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*



Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The First Law of Geometrical Optics:

The incident, reflected, and transmitted wave vectors form a plane - called the plane of incidence- which also includes the normal to the surface -here the z axis.

The Second Law of Geometrical Optics (Law of Reflection):

From the figure- we see that:

$$\boxed{k_{inc_x} = k_{inc} \sin\theta_{inc}} = \boxed{k_{refl_x} = k_{refl} \sin\theta_{refl}} = \boxed{k_{trans_x} = k_{trans} \sin\theta_{trans}}$$

$$\boxed{k_{inc} = k_{refl} = k_1} \Rightarrow \boxed{\sin\theta_{inc} = \sin\theta_{refl}}$$

Angle of Incidence = Angle of Reflection

$$\boxed{\theta_{inc} = \theta_{refl}}$$

Law of Reflection!

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The Third Law of Geometrical Optics (Law of Refraction – Snell's Law):

For the transmitted angle - θ_{trans} we see that:

$$\boxed{k_{inc} \sin\theta_{inc} = k_{trans} \sin\theta_{trans}}$$

In medium 1): $\boxed{k_{inc} = k_1 = \omega/v_1 = n_1\omega/c = n_1 k_o}$

where $\boxed{k_o = \text{ vacuum wave number } = 2\pi/\lambda_o}$

and $\boxed{\lambda_o = \text{ vacuum wave length}}$

In medium 2): $\boxed{k_{trans} = k_2 = \omega/v_2 = n_2\omega/c = n_2 k_o}$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

$$\boxed{k_{inc}\sin\theta_{inc} = k_{trans}\sin\theta_{trans}} \Rightarrow \boxed{k_1\sin\theta_{inc} = k_2\sin\theta_{trans}}$$

$\boxed{k_{inc} = k_1 = n_1 k_o}$ and $\boxed{k_{trans} = k_2 = n_2 k_o}$

$$\boxed{k_1\sin\theta_{inc} = k_2\sin\theta_{trans}} \Rightarrow \boxed{n_1\sin\theta_{inc} = n_2\sin\theta_{trans}}$$

Law of Refraction
(Snell's Law)

Which can also be written as: $\boxed{\dfrac{\sin\theta_{trans}}{\sin\theta_{inc}} = \dfrac{n_1}{n_2}}$

Since θ_{trans} refers to medium 2) and θ_{inc} refers to medium 1)

$$\boxed{n_1\sin\theta_1 = n_2\sin\theta_2} \quad \text{or:} \quad \boxed{\frac{\sin\theta_2}{\sin\theta_1} = \frac{n_1}{n_2}}$$

(incident) (transmitted)

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Using three laws of geometrical optics we can see that :

$$\boxed{\vec{k}_{inc} \bullet \vec{r}\,\Big|_{z=0} = \vec{k}_{refl} \bullet \vec{r}\,\Big|_{z=0} = \vec{k}_{trans} \bullet \vec{r}\,\Big|_{z=0}}$$

everywhere on the interface at $z = 0$ -in the x-y plane

Thus $$\boxed{e^{i\left(\vec{k}_{inc} \bullet \vec{r} - \omega t\right)}\Big|_{z=0} = e^{i\left(\vec{k}_{refl} \bullet \vec{r} - \omega t\right)}\Big|_{z=0} = e^{i\left(\vec{k}_{trans} \bullet \vec{r} - \omega t\right)}\Big|_{z=0}}$$

everywhere on the interface at z = 0 -in the x-y plane and valid also for all time(s) t, since ω is the same in either medium (1 or 2).

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The BC 1) → BC 4) for a monochromatic plane *EM* wave incident on an interface at an oblique angle between two linear/ homogeneous/isotropic media become:

BC 1): Normal *(* z-) component of ***D*** continuous at z = 0 (no free surface charges)*:*

$$\boxed{\varepsilon_1\left(\tilde{E}_{o_{inc_z}}+\tilde{E}_{o_{refl_z}}\right)=\varepsilon_2\tilde{E}_{o_{trans_z}}} \qquad \left\{\text{using } \vec{D}=\varepsilon\vec{E}\right\}$$

BC 2): Tangential (x-, y-) components of ***E*** continuous at z = 0*:*

$$\boxed{\left(\tilde{E}_{o_{inc_{x,y}}}+\tilde{E}_{o_{refl_{x,y}}}\right)=\tilde{E}_{o_{trans_{x,y}}}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

BC 3): Normal (z-) component of $\boldsymbol{B}$ continuous at $z = 0$:

$$\boxed{\left(\tilde{B}_{o_{inc_z}} + \tilde{B}_{o_{refl_z}}\right) = \tilde{B}_{o_{trans_z}}}$$

BC 4): Tangential (x-, y-) components of **H** continuous at z = 0 (no free surface currents):

$$\boxed{\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_{x,y}}} + \tilde{B}_{o_{refl_{x,y}}}\right) = \frac{1}{\mu_2}\tilde{B}_{o_{trans_{x,y}}}}$$

Note that in each of the above, we also have the relation

$$\boxed{\vec{\tilde{B}}_o = \frac{1}{v}\hat{k} \times \vec{\tilde{E}}_o}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

For a monochromatic plane EM wave incident on a boundary between two L / H/ I media at an oblique angle of incidence - three possible polarization cases to consider:

Case I): $\vec{E}_{inc} \perp$ plane of incidence

$\{\vec{B}_{inc} \parallel$ plane of incidence$\}$

Transverse Electric (TE) Polarization

Case II): $\vec{E}_{inc} \parallel$ plane of incidence

$\{\vec{B}_{inc} \perp$ plane of incidence$\}$

Transverse Magnetic (TM) Polarization

Case III): The most general case: $\vec{E}_{inc}$ is neither $\perp$ nor $\parallel$ to the plane of incidence.

$\{\Rightarrow \vec{B}_{inc}$ is neither $\parallel$ nor $\perp$ to the plane of incidence$\}$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Case I): Electric Field Vectors Perpendicular to the Plane of Incidence: Transverse Electric (*TE) Polarization*

- A monochromatic plane EM wave is incident on a boundary at z = 0 -in the x-y plane between two L/H/I media - at an oblique angle of incidence.
- The polarization of the incident EM wave is transverse ($\perp$) to the plane of incidence (containing the three wave-vectors and the unit normal to the boundary $\hat{n} = +\hat{z}$).
- The three B-field vectors are related to their respective E -field vectors by the right hand rule - all three B-field vectors lie in the x-z plane (the plane of incidence)

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The four boundary conditions on the complex *E* and *B* fields on the boundary at z = 0 are:

BC 1) Normal (*z-*) component of D continuous at z = 0 (no free surface charges)

$$\boxed{\varepsilon_1\left(\cancelto{=0}{\tilde{E}_{o_{inc_z}}} + \cancelto{=0}{\tilde{E}_{o_{refl_z}}}\right) = \varepsilon_2\, \cancelto{=0}{\tilde{E}_{o_{trans_z}}}} \Rightarrow \boxed{0+0=0}$$

BC 2) Tangential (*x-*, *y-*) components of E continuous at *z = 0:*

$$\boxed{\left(\tilde{E}_{o_{inc_y}} + \tilde{E}_{o_{refl_y}}\right) = \tilde{E}_{o_{trans_y}}} \Rightarrow \boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

BC 3) Normal (*z-)* component of B continuous at z = 0:

$$\boxed{\left(\tilde{B}_{o_{inc_z}} + \tilde{B}_{o_{refl_z}}\right) = \tilde{B}_{o_{trans_z}}}$$

$$\hat{k}_{inc} = \hat{k}_{inc_x} + \hat{k}_{inc_z} = \sin\theta_{inc}\hat{x} + \cos\theta_{inc}\hat{z}$$

$$\hat{k}_{refl} = \hat{k}_{refl_x} + \hat{k}_{refl_z} = \sin\theta_{refl}\hat{x} - \cos\theta_{refl}\hat{z}$$

$$\hat{k}_{trans} = \hat{k}_{trans_x} + \hat{k}_{trans_z} = \sin\theta_{trans}\hat{x} + \cos\theta_{trans}\hat{z}$$

$$\boxed{\left(\tilde{B}_{o_{inc_z}}\hat{z} + \tilde{B}_{o_{refl_z}}\hat{z}\right) = \tilde{B}_{o_{trans_z}}\hat{z}} = \boxed{\frac{1}{v_1}\left(\tilde{E}_{o_{inc}}\sin\theta_{inc} + \tilde{E}_{o_{refl}}\sin\theta_{refl}\right)\hat{z} = \frac{1}{v_2}\tilde{E}_{o_{trans}}\sin\theta_{trans}\hat{z}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Using the Law of Reflection on the BC 3) result:

$$\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \left(\frac{v_1}{v_2} \cdot \frac{\sin\theta_{trans}}{\sin\theta_{inc}}\right)\tilde{E}_{o_{trans}}}$$

Using Snell's Law / Law of Refraction:

$$\boxed{n_1 \sin\theta_{inc} = n_2 \sin\theta_{trans}} \Rightarrow \boxed{\frac{n_1}{c}\sin\theta_{inc} = \frac{n_2}{c}\sin\theta_{trans}} \Rightarrow \boxed{\frac{1}{v_1}\sin\theta_{inc} = \frac{1}{v_2}\sin\theta_{trans}}$$

or: $\boxed{v_2 \sin\theta_{inc} = v_1 \sin\theta_{trans}}$ or: $\boxed{\left(\frac{v_1}{v_2} \cdot \frac{\sin\theta_{trans}}{\sin\theta_{inc}}\right) = 1}$

Reduces to BC2) $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

BC 4) Tangential (*x*-, *y*-) components of H continuous at z = 0 (no free surface currents):

$$\boxed{\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_x}}\hat{x}+\tilde{B}_{o_{refl_x}}\hat{x}\right)=\frac{1}{\mu_2}\tilde{B}_{o_{trans_x}}\hat{x}}$$

$$=\boxed{\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}}\left(-\cos\theta_{inc}\right)+\tilde{E}_{o_{refl}}\cos\theta_{refl}\right)\hat{x}=\frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}\left(-\cos\theta_{trans}\right)\hat{x}}$$

Using the Law of Reflection on the BC 4) result:

$$\boxed{\left(\tilde{E}_{o_{inc}}-\tilde{E}_{o_{refl}}\right)=\left(\frac{\mu_1 v_1}{\mu_2 v_2}\cdot\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\tilde{E}_{o_{trans}}}$$

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From BC 1) → BC 4) actually have only two independent relations for the case of transverse electric (TE) polarization:

1) $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$

2) $\boxed{\left(\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}}\right) = \left(\dfrac{\mu_1 v_1}{\mu_2 v_2} \cdot \dfrac{\cos\theta_{trans}}{\cos\theta_{inc}}\right) \tilde{E}_{o_{trans}}}$

Define: $\boxed{\beta \equiv \left(\dfrac{\mu_1 v_1}{\mu_2 v_2}\right)}$ $\boxed{\alpha \equiv \left(\dfrac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)}$

Then eqn. 2) becomes:

$$\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \alpha\beta\, \tilde{E}_{o_{trans}}}$$

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Adding and subtracting Eqn's 1 &2 to get:

$$\boxed{\tilde{E}_{o_{trans}} = \left(\frac{2}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}}} \quad \text{eqn. (1+2)} \qquad \boxed{\tilde{E}_{o_{refl}} = \left(\frac{1-\alpha\beta}{2}\right)\tilde{E}_{o_{trans}}} \quad \text{eqn. (2−1)}$$

Plug eqn. (2+1) into eqn. (2−1) to obtain:

$$\boxed{\tilde{E}_{o_{refl}} = \left(\frac{1-\alpha\beta}{2}\right)\left(\frac{2}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)\tilde{E}_{o_{inc}}}$$

$$\boxed{\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)} \quad \text{and} \quad \boxed{\frac{\tilde{E}_{o_{trans}}}{\tilde{E}_{o_{inc}}} = \left(\frac{2}{1+\alpha\beta}\right)}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The Fresnel Equations for $\vec{E}\,||$ to Interface

$=\vec{E}\perp$ Plane of Incidence = Transverse Electric (TE) Polarization

$$\boxed{E_{o_{refl}}^{TE}=\left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)E_{o_{inc}}^{TE}} \text{ and } \boxed{E_{o_{trans}}^{TE}=\left(\frac{2}{1+\alpha\beta}\right)E_{o_{inc}}^{TE}}$$

with $$\boxed{\alpha\equiv\left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)} \text{ and } \boxed{\beta\equiv\left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

For *TE polarization:*

Incident Intensity

$$I_{inc}^{TE} = \left|\left\langle \vec{S}_{inc}^{TE}(t)\right\rangle \bullet \hat{z}\right| = \left(\frac{1}{2}v_1\varepsilon_1\left(E_{o_{inc}}^{TE}\right)^2\right)\left|\hat{k}_{inc}\bullet\hat{z}\right| = \left(\frac{1}{2}v_1\varepsilon_1\left(E_{o_{inc}}^{TE}\right)^2\right)\cos\theta_{inc} = \frac{1}{2}\varepsilon_1 v_1\left(E_{o_{inc}}^{TE}\right)^2\cos\theta_{inc}$$

Reflection Intensity

$$I_{refl}^{TE} = \left|\left\langle \vec{S}_{refl}^{TE}(t)\right\rangle \bullet \hat{z}\right| = \left(\frac{1}{2}v_1\varepsilon_1\left(E_{o_{refl}}^{TE}\right)^2\right)\cos\theta_{refl} = \frac{1}{2}\varepsilon_1 v_1\left(E_{o_{refl}}^{TE}\right)^2\cos\theta_{inc}$$

Transmission Intensity

$$I_{trans}^{TE} = \left|\left\langle \vec{S}_{trans}^{TE}(t)\right\rangle \bullet \hat{z}\right| = \left(\frac{1}{2}v_2\varepsilon_2\left(E_{o_{trans}}^{TE}\right)^2\right)\cos\theta_{trans} = \frac{1}{2}\varepsilon_2 v_2\left(E_{o_{trans}}^{TE}\right)^2\cos\theta_{trans}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Reflection and Transmission coefficients for transverse electric (*TE) polarization*

$$\boxed{R_{TE} \equiv \frac{I_{refl}^{TE}}{I_{inc}^{TE}} = \frac{\frac{1}{2}\varepsilon_1 v_1 \left(E_{o_{refl}}^{TE}\right)^2 \cos \overbrace{\theta_{inc}}^{=\theta_{refl}}}{\frac{1}{2}\varepsilon_1 v_1 \left(E_{o_{inc}}^{TE}\right)^2 \cos\theta_{inc}} = \left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2}$$

$$\boxed{T_{TE} \equiv \frac{I_{trans}^{TE}}{I_{inc}^{TE}} = \frac{\frac{1}{2}\varepsilon_2 v_2 \left(E_{o_{trans}}^{TE}\right)^2 \cos\theta_{trans}}{\frac{1}{2}\varepsilon_1 v_1 \left(E_{o_{inc}}^{TE}\right)^2 \cos\theta_{inc}} = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The reflection and transmission coefficients for transverse electric (*TE*) polarization

$$R_{TE} = \left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2$$

$$T_{TE} = \alpha\beta\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2 = \frac{4\alpha\beta}{(1+\alpha\beta)^2}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Case II): Electric Field Vectors Parallel to the Plane of Incidence:

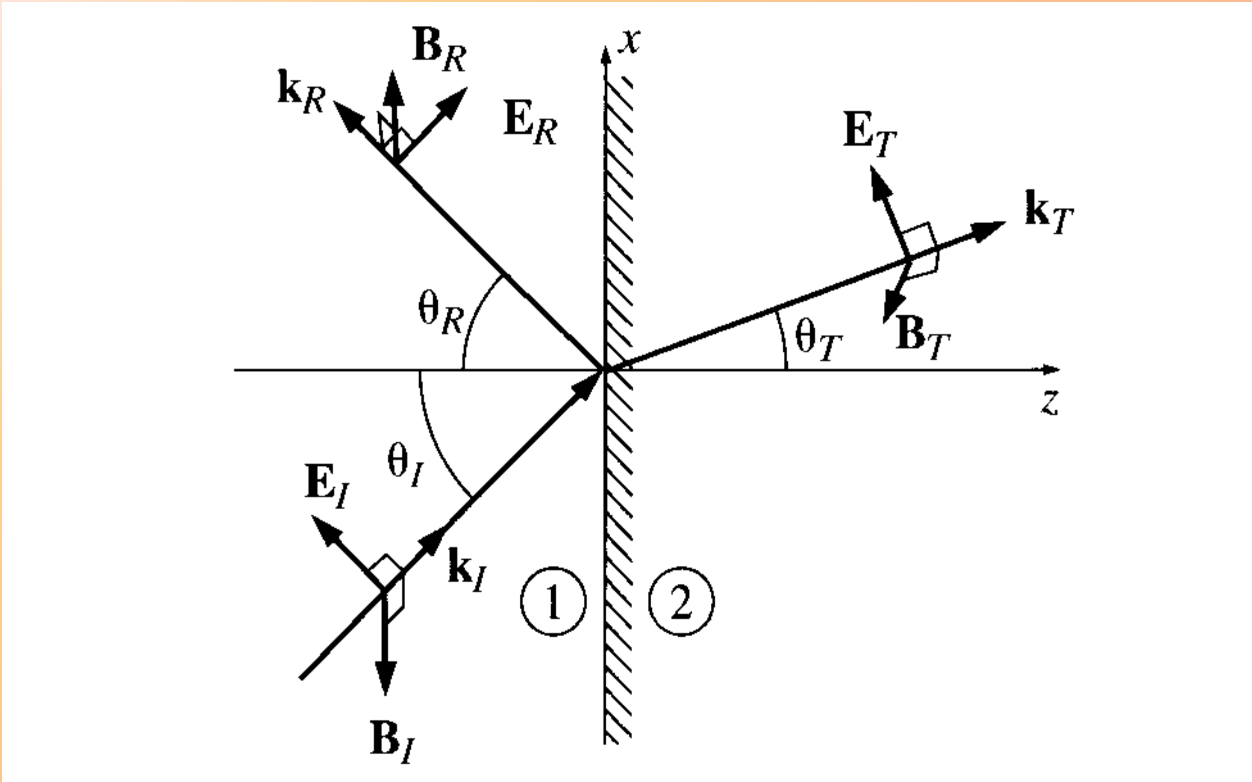
Transverse Magnetic (TM) Polarization

•A monochromatic plane EM wave is incident on a boundary at z = 0 in the x-y plane between two L / H/ I media at an oblique angle of incidence.

•The polarization of the incident EM wave is now parallel to the plane of incidence –(containing the three wave-vectors and the unit normal to the boundary nˆ = +zˆ).

• The three B -field vectors are related to E -field vectors by the right hand rule –then all three B-field vectors are ⊥ to the plane of incidence {hence the origin of the name transverse magnetic polarization}.

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*



Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The four boundary conditions on the complex E and B-fields on the boundary at z = 0 are:

BC 1) Normal (z-) component of D continuous at z = 0 (no free surface charges)

$$\boxed{\varepsilon_1\left(\tilde{E}_{o_{inc_z}}+\tilde{E}_{o_{refl_z}}\right)=\varepsilon_2\tilde{E}_{o_{trans_z}}}$$

$$\boxed{\varepsilon_1\left(-\tilde{E}_{o_{inc}}\sin\theta_{inc}+\tilde{E}_{o_{refl}}\sin\theta_{refl}\right)=\varepsilon_2\left(-\tilde{E}_{o_{trans}}\sin\theta_{trans}\right)}$$

BC 2) Tangential (x-, y-) components of E continuous at *z = 0:*

$$\boxed{\left(\tilde{E}_{o_{inc_x}}+\tilde{E}_{o_{refl_x}}\right)=\tilde{E}_{o_{trans_x}}}$$

$$\boxed{\left(\tilde{E}_{o_{inc}}\cos\theta_{inc}+\tilde{E}_{o_{refl}}\cos\theta_{refl}\right)=\tilde{E}_{o_{trans}}\cos\theta_{trans}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

BC 3) Normal (**z-**) component of *B* continuous at *z = 0:*

$$\boxed{\left(\cancelto{=0}{\tilde{B}_{o_{inc_z}}} + \cancelto{=0}{\tilde{B}_{o_{refl_z}}}\right) = \cancelto{=0}{\tilde{B}_{o_{trans_z}}}} \Rightarrow \boxed{0+0=0}$$

BC 4) Tangential (x-, y-) components of H continuous at z = 0 (no free surface currents):

$$\boxed{\frac{1}{\mu_1}\left(\tilde{B}_{o_{inc_y}} + \tilde{B}_{o_{refl_y}}\right) = \frac{1}{\mu_2}\left(\tilde{B}_{o_{trans_y}}\right)} \Rightarrow \boxed{\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}}\right) = \frac{1}{\mu_2 v_2}\tilde{E}_{o_{trans}}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

From BC 1) at $z = 0$:

$$\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \left(\frac{\varepsilon_2}{\varepsilon_1}\frac{n_1}{n_2}\right)\tilde{E}_{o_{trans}} = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)\tilde{E}_{o_{trans}} = \beta\,\tilde{E}_{o_{trans}}}$$

From BC 4) at $z = 0$:

$$\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right)\tilde{E}_{o_{trans}} = \beta\tilde{E}_{o_{trans}}}$$

where:

$$\boxed{\beta \equiv \left(\frac{\mu_1 v_1}{\mu_2 v_2}\right) = \left(\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\right)}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

From BC 2) at $z = 0$:

$$\boxed{\left(\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}}\right) = \left(\frac{\cos\theta_{trans}}{\cos\theta_{inc}}\right)\tilde{E}_{o_{trans}} = \alpha\tilde{E}_{o_{trans}}}$$

where:

$$\boxed{\alpha \equiv \frac{\cos\theta_{trans}}{\cos\theta_{inc}}}$$

Thus for the case of transverse magnetic (TM) polarization:

$$\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \beta\tilde{E}_{o_{trans}}} \quad \text{and} \quad \boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \alpha\tilde{E}_{o_{trans}}}$$

Solving these two above equations simultaneously, we obtain:

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

$$\tilde{E}_{o_{trans}} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{o_{inc}} \qquad \tilde{E}_{o_{refl}} = \left(\frac{\alpha-\beta}{2}\right)\tilde{E}_{o_{trans}} \qquad \tilde{E}_{o_{refl}} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{o_{inc}}$$

The Fresnel Equations for $\vec{B}\,||$ to Interface

$=\vec{B}\perp$ Plane of Incidence = Transverse Magnetic (*TM*) Polarization

$$\boxed{\left(\frac{E^{TM}_{o_{refl}}}{E^{TM}_{o_{inc}}}\right) = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)} \quad \text{and} \quad \boxed{\left(\frac{E^{TM}_{o_{trans}}}{E^{TM}_{o_{inc}}}\right) = \left(\frac{2}{\alpha+\beta}\right)}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Reflected & transmitted intensities at oblique incidence for the *TM case*

$$I_{inc}^{TM} = v_1 \left| \left\langle \vec{S}_{inc}^{TM}(t) \right\rangle \bullet \hat{z} \right| = \left(\frac{1}{2} v_1 \varepsilon_1 \left(E_{o_{inc}}^{TM} \right)^2 \right) \cos\theta_{inc} = \frac{1}{2} \varepsilon_1 v_1 \left(E_{o_{inc}}^{TM} \right)^2 \cos\theta_{inc}$$

$$I_{refl}^{TM} = v_1 \left| \left\langle \vec{S}_{refl}^{TM}(t) \right\rangle \bullet \hat{z} \right| = \left(\frac{1}{2} v_1 \varepsilon_1 \left(E_{o_{refl}}^{TM} \right)^2 \right) \cos\theta_{refl} = \frac{1}{2} \varepsilon_1 v_1 \left(E_{o_{refl}}^{TM} \right)^2 \cos\theta_{inc}$$

$$I_{trans}^{TM} = v_2 \left| \left\langle \vec{S}_{trans}^{TM}(t) \right\rangle \bullet \hat{z} \right| = \left(\frac{1}{2} v_2 \varepsilon_2 \left(E_{o_{trans}}^{TM} \right)^2 \right) \cos\theta_{trans} = \frac{1}{2} \varepsilon_2 v_2 \left(E_{o_{trans}}^{TM} \right)^2 \cos\theta_{trans}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Reflection and Transmission coefficients

$$\boxed{R_{TM} = \left(\frac{E^{TM}_{o_{refl}}}{E^{TM}_{o_{inc}}} \right)^2 = \left(\frac{\alpha - \beta}{\alpha + \beta} \right)^2}$$

$$\boxed{T_{TM} = \alpha\beta \left(\frac{E^{TM}_{o_{trans}}}{E^{TM}_{o_{inc}}} \right)^2 = \frac{4\alpha\beta}{(\alpha + \beta)^2}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

The Fresnel Equations

TE Polarization

$$\boxed{\left(\frac{E^{TE}_{o_{refl}}}{E^{TE}_{o_{inc}}}\right)=\left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)}$$

$$\boxed{\left(\frac{E^{TE}_{o_{trans}}}{E^{TE}_{o_{inc}}}\right)=\frac{2}{(1+\alpha\beta)}}$$

$$\boxed{\alpha\equiv\frac{\cos\theta_{trans}}{\cos\theta_{inc}}}$$

$$\boxed{\beta\equiv\frac{\mu_1 v_1}{\mu_2 v_2}=\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}=\frac{\mu_1 n_2}{\mu_2 n_1}=\frac{\varepsilon_2 n_1}{\varepsilon_1 n_2}}$$

TM Polarization

$$\boxed{\left(\frac{E^{TM}_{o_{refl}}}{E^{TM}_{o_{inc}}}\right)=\left(\frac{\alpha-\beta}{\alpha+\beta}\right)}$$

$$\boxed{\left(\frac{E^{TM}_{o_{trans}}}{E^{TM}_{o_{inc}}}\right)=\frac{2}{(\alpha+\beta)}}$$

$$\boxed{v_1={}^{c}/_{n_1}={}^{1}/_{\sqrt{\varepsilon_1\mu_1}}}$$

$$\boxed{v_2={}^{c}/_{n_2}={}^{1}/_{\sqrt{\varepsilon_2\mu_2}}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Reflection and Transmission Coefficients *R* & *T*

$$R + T = 1$$

TE Polarization

$$R_{TE} \equiv \frac{I_{refl}^{TE}}{I_{inc}^{TE}} = \left(\frac{E_{o_{refl}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2$$

$$T_{TE} \equiv \left(\frac{I_{trans}^{TE}}{I_{inc}^{TE}}\right) = \alpha\beta\left(\frac{E_{o_{trans}}^{TE}}{E_{o_{inc}}^{TE}}\right)^2 = \frac{4\alpha\beta}{(1+\alpha\beta)^2}$$

$$\alpha \equiv \frac{\cos\theta_{trans}}{\cos\theta_{inc}}$$

$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\varepsilon_2 n_1}{\varepsilon_1 n_2}$$

TM Polarization

$$R_{TM} \equiv \frac{I_{refl}^{TM}}{I_{inc}^{TM}} = \left(\frac{E_{o_{refl}}^{TM}}{E_{o_{inc}}^{TM}}\right)^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$$

$$T_{TM} \equiv \left(\frac{I_{trans}^{TM}}{I_{inc}^{TM}}\right) = \alpha\beta\left(\frac{E_{o_{trans}}^{TM}}{E_{o_{inc}}^{TM}}\right)^2 = \frac{4\alpha\beta}{(\alpha+\beta)^2}$$

$$v_1 = c/n_1 = 1/\sqrt{\varepsilon_1\mu_1}$$

$$v_2 = c/n_2 = 1/\sqrt{\varepsilon_2\mu_2}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

- Now explore the physics associated with the Fresnel Equations -the reflection and transmission coefficients.
- Comparing results for TE vs. TM polarization for the cases of external reflection ($n_1 < n_2$) and internal reflection $n_1 > n_2$)

Comment 1):

- When $(E_{refl} / E_{inc}) < 0$ - E_{orefl} *is* 180^o out-of-phase with E_{oinc} since the numerators of the original Fresnel Equations for TE & TM polarization are $(1 - \alpha\beta)$ and $(\alpha - \beta)$ respectively.

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Comment 2):

•For TM Polarization (only)- there exists an angle of incidence where $(E_{refl} / E_{inc}) = 0$ - no reflected wave occurs at this angle for TM polarization!

•This angle is known as Brewster's angle θ_B (also known as the polarizing angle θ_P - because an incident wave which is a linear combination of TE and TM polarizations will have a reflected wave which is 100% pure-TE polarized for an incidence angle $\theta_{inc} = \theta_B = \theta_P$!!).

•Brewster's angle θ_B exists for both external ($n_1 < n_2$) & internal reflection ($n_1 > n_2$) for TM polarization (only).

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Brewster's Angle θ_B */ the Polarizing Angle* θ_P *for Transverse Magnetic (TM) Polarization*

From the numerator of $\left(E^{TM}_{o_{refl}} / E^{TM}_{o_{inc}}\right) = \left(\dfrac{\alpha - \beta}{\alpha + \beta}\right)$ this ratio = 0 at Brewster's angle θ_B when $(\alpha - \beta) = 0$, i.e. when $\alpha = \beta$.

<u>But</u>: $\boxed{\alpha \equiv \dfrac{\cos\theta_{trans}}{\cos\theta_{inc}}}$ and $\boxed{\beta \equiv \dfrac{\mu_1 n_2}{\mu_2 n_1} \simeq \dfrac{n_2}{n_1}}$ for $\boxed{\mu_1 \simeq \mu_2 \simeq \mu_o}$

$\cos\theta_{trans} = \sqrt{1 - \sin^2\theta_{trans}}$ and Snell's Law: $\sin\theta_{trans} = \left(\dfrac{n_1}{n_2}\right)\sin\theta_{inc}$

$$\boxed{\alpha = \frac{\sqrt{1 - \left(\dfrac{n_1}{n_2}\right)^2 \sin^2\theta_{inc}}}{\cos\theta_{inc}} \simeq \left(\frac{n_2}{n_1}\right) = \beta}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Brewster's Angle θ_B / *the Polarizing Angle* θ_P *for Transverse Magnetic (TM) Polarization*

$$\boxed{1-\frac{1}{\beta^2}\sin^2\theta_{inc}=\beta^2\cos^2\theta_{inc}=\beta^2\left(1-\sin^2\theta_{inc}\right)} \leftarrow \text{Solve for } \sin^2\theta_{inc}$$

$$\boxed{1-\beta^2=\left(\frac{1}{\beta^2}-\beta^2\right)\sin^2\theta_{inc}} \Rightarrow \boxed{\sin^2\theta_{inc}=\frac{1-\beta^2}{1/\beta^2-\beta^2}=\frac{\left(1-\beta^2\right)\beta^2}{\left(1-\beta^4\right)}}$$

$$\boxed{1-\beta^4=\left(1-\beta^2\right)\left(1+\beta^2\right)}$$

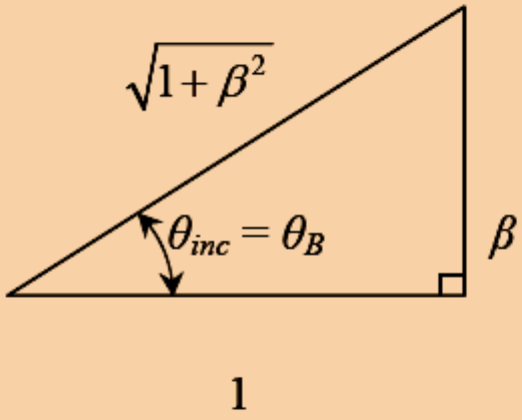
$$\boxed{\sin^2\theta_{inc}=\frac{\left(1-\beta^2\right)\beta^2}{\left(1-\beta^2\right)\left(1+\beta^2\right)}=\frac{\beta^2}{1+\beta^2}} \Rightarrow \boxed{\sin\theta_{inc}=\frac{\beta}{\sqrt{1+\beta^2}}}$$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

Geometrically: $\boxed{\sin\theta_{inc} = \frac{\beta}{\sqrt{1+\beta^2}}} = \boxed{\frac{\text{opp. side}}{\text{hypotenuse}}}$

$\boxed{\cos\theta_{inc} = \frac{1}{\sqrt{1+\beta^2}}} = \boxed{\frac{\text{adjacent}}{\text{hypotenuse}}}$

$\boxed{\tan\theta_{inc} = \beta} = \boxed{\frac{\text{opp. side}}{\text{adjacent}}} \simeq \left(\frac{n_2}{n_1}\right)$



Thus, at an angle of incidence $\theta_{inc} = \theta_B^{inc} \equiv \theta_P^{inc}$ = Brewster's angle / the polarizing angle for a *TM* polarized incident wave, where *no* *reflected* wave exists, we have:

$$\boxed{\tan\theta_B^{inc} \equiv \tan\theta_P^{inc} \simeq \left(\frac{n_2}{n_1}\right)} \quad \text{for} \quad \mu_1 \simeq \mu_2 \simeq \mu_o$$

From Snell's Law: $n_1 \sin\theta_{inc} = n_2 \sin\theta_{trans}$ we also see that: $\tan\theta_B^{inc} = \dfrac{\sin\theta_B^{inc}}{\cos\theta_B^{inc}} \simeq \dfrac{n_2}{n_1}$

or: $n_1 \sin\theta_B^{inc} \simeq n_2 \cos\theta_B^{inc}$ for $\mu_1 \simeq \mu_2 \simeq \mu_o$.

Thus, from Snell's Law we see that: $\cos\theta_B^{inc} = \sin\theta_{trans}$ when $\theta_{inc} = \theta_B^{inc} \equiv \theta_P^{inc}$.

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

So what's so interesting about this???

Well: $\boxed{\cos\theta_B^{inc} = \sin\left(\tfrac{\pi}{2}-\theta_B^{inc}\right) = \sin\left(\tfrac{\pi}{2}\right)\cos\theta_B^{inc} - \cancel{\overset{=0}{\cos\left(\tfrac{\pi}{2}\right)}}\sin\theta_B^{inc} = \sin\theta_{trans}}$ *i.e.* $\boxed{\sin\left(\frac{\pi}{2}-\theta_B^{inc}\right) = \sin\theta_{trans}}$

$\therefore$ When $\theta_{inc} = \theta_B^{inc} \equiv \theta_P^{inc}$ for an incident *TM*-polarized *EM* wave, we see that $\theta_{trans} = \pi/2 - \theta_B^{inc}$

Thus: $\theta_B^{inc} + \theta_{trans} = \pi/2$, *i.e.* $\theta_B^{inc} \equiv \theta_P^{inc}$ and θ_{trans} are *complimentary* angles !!!

Comment 3):

For internal reflection ($n_1 > n_2$) there exists a critical angle of incidence past which no transmitted beam exists for either TE or TM polarization. The critical angle does not depend on polarization – it is actually defined by Snell's Law:

$\boxed{n_1\sin\theta_{critical}^{inc} = n_2\sin\theta_{trans}^{\max} = n_2\sin\left(\frac{\pi}{2}\right) = n_2}$ or: $\boxed{\sin\theta_{critical}^{inc} = \left(\frac{n_2}{n_1}\right)}$ or: $\boxed{\theta_{critical}^{inc} = \sin^{-1}\left(\frac{n_2}{n_1}\right)}$

Reflection & Transmission of Monochromatic Plane *EM Waves at Oblique Incidence*

For $\theta_{inc} \geq \theta_{critical}^{inc}$, no transmitted beam exists → incident beam is totally internally reflected.

For $\theta_{inc} > \theta_{critical}^{inc}$, the transmitted wave is actually exponentially damped – becomes a so-called:

Evanescent Wave:

$$\boxed{\vec{\tilde{E}}_{trans}(\vec{r},t) = \vec{\tilde{E}}_{o_{trans}} \underbrace{e^{-\alpha z}}_{\text{Exp. damping in } z} \underbrace{e^{i\left(k_2 x \sin\theta_{inc}\left(\frac{n_1}{n_2}\right) - \omega t\right)}}_{\text{Oscillatory along interface in } x\text{-direction}}}$$

$$\boxed{\alpha = k_2\sqrt{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_{inc} - 1}}$$

Exp. damping in z Oscillatory along <u>interface</u> in x-direction

ELECTROMAGNETIC WAVES IN CONDUCTORS

Imrana Ashraf Zahid

Quaid-i-Azam University Islamabad Pakistan

Preparatory School to Winter College on Optics: Light : A Bridge between Earth and Space. 2nd February - 6th February 2015

ELECTROMAGNETIC WAVES IN CONDUCTORS

- Free charge and free currents are zero for propagation through a vacuum or insulating materials such as glass or pure water.
- Inside a conductor- free charges can move around in response to *EM* fields contained therein- free current is not zero.
- Assume that the conductor is linear/homogeneous/ isotropic media.
- From Ohm's Law

where σ_C = conductivity $\vec{J}_{free}(\vec{r},t) = \sigma_C \vec{E}(\vec{r},t)$ r (*Ohm*$^{-1}$*/m) and* σ_C =*1/* ρ_C where ρ_C = resistivity of the metal conductor (*Ohm-m*).

ELECTROMAGNETIC WAVES IN CONDUCTORS

Assume that the linear/ homogeneous/isotropic conducting medium has electric permittivity ε and magnetic permeability μ. Maxwell's equations inside such a conductor are thus:

1) $\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r},t)=\rho_{free}(\vec{r},t)/\varepsilon}$

2) $\boxed{\vec{\nabla}\cdot\vec{B}(\vec{r},t)=0}$

3) $\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t)=-\dfrac{\partial\vec{B}(\vec{r},t)}{\partial t}}$

Using Ohm's Law:
$\vec{J}_{free}(\vec{r},t)=\sigma_C\vec{E}(\vec{r},t)$

4) $\boxed{\vec{\nabla}\times\vec{B}(\vec{r},t)=\mu\vec{J}_{free}(\vec{r},t)+\mu\varepsilon\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}=\mu\sigma_C\vec{E}(\vec{r},t)+\mu\varepsilon\dfrac{\partial\vec{E}(\vec{r},t)}{\partial t}}$

ELECTROMAGNETIC WAVES IN CONDUCTORS

Electric charge is conserved- thus the continuity equation inside the conductor is:

$$\boxed{\vec{\nabla}\bullet\vec{J}_{free}(\vec{r},t) = -\frac{\partial\rho_{free}(\vec{r},t)}{\partial t}}$$

<u>but</u>: $\boxed{\vec{J}_{free}(\vec{r},t) = \sigma_C\vec{E}(\vec{r},t)}$

$$\boxed{\sigma_C\left(\vec{\nabla}\bullet\vec{E}(\vec{r},t)\right) = -\frac{\partial\rho_{free}(\vec{r},t)}{\partial t}}$$

<u>but</u>: $\boxed{\vec{\nabla}\bullet\vec{E}(\vec{r},t) = \rho_{free}(\vec{r},t)/\varepsilon}$

thus:

$$\boxed{\frac{\sigma_C\rho_{free}(\vec{r},t)}{\varepsilon} = -\frac{\partial\rho_{free}(\vec{r},t)}{\partial t}}$$

<u>or</u>: $\boxed{\frac{\partial\rho_{free}(\vec{r},t)}{\partial t} + \left(\frac{\sigma_C}{\varepsilon}\right)\rho_{free}(\vec{r},t) = 0}$

1st order linear, homogeneous differential equation

ELECTROMAGNETIC WAVES IN CONDUCTORS

The general solution of this differential equation for the free charge density is of the form:

$$\rho_{free}\left(\vec{r},t\right)=\rho_{free}\left(\vec{r},t=0\right)e^{-\sigma_C t/\varepsilon}=\rho_{free}\left(\vec{r},t=0\right)e^{-t/\tau_{relax}}$$

A damped exponential!!!

The continuity equation inside a conductor tells us that any free charge density initially present at time t = 0 is exponentially damped in a characteristic time $= \tau_{relax} \equiv \varepsilon/\sigma_C$ ation time.

ELECTROMAGNETIC WAVES IN CONDUCTORS

Maxwell's equations for a ***charge-equilibrated conductor***

1) $\boxed{\vec{\nabla}\bullet\vec{E}(\vec{r},t)=0}$

2) $\boxed{\vec{\nabla}\bullet\vec{B}(\vec{r},t)=0}$

3) $$\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t)=-\frac{\partial\vec{B}(\vec{r},t)}{\partial t}}$$

4) $$\boxed{\vec{\nabla}\times\vec{B}(\vec{r},t)=\mu\sigma_C\vec{E}(\vec{r},t)+\mu\varepsilon\frac{\partial\vec{E}(\vec{r},t)}{\partial t}=\mu\left(\sigma_C\vec{E}(\vec{r},t)+\varepsilon\frac{\partial\vec{E}(\vec{r},t)}{\partial t}\right)}$$

ELECTROMAGNETIC WAVES IN CONDUCTORS

These equations are different from the previous derivation(s) of monochromatic plane EM waves propagating in free space and in linear/homogeneous/ isotropic non-conducting materials. Re-derive the wave equations for ***E and B.*** Apply **∇** × ( ) to equations 3) and 4):

We get $$\boxed{\nabla^2 \vec{E}(\vec{r},t) = \mu\varepsilon \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} + \mu\sigma_C \frac{\partial \vec{E}(\vec{r},t)}{\partial t}}$$

and $$\boxed{\nabla^2 \vec{B}(\vec{r},t) = \mu\varepsilon \frac{\partial^2 \vec{B}(\vec{r},t)}{\partial t^2} + \mu\sigma_C \frac{\partial \vec{B}(\vec{r},t)}{\partial t}}$$

ELECTROMAGNETIC WAVES IN CONDUCTORS

General solution(s) - are usually in the form of an oscillatory function times a damping term (*a decaying exponential)* – in the direction of the propagation of the EM wave. A complex plane-wave type solutions for E and B associated with the above wave equation(s) are of the general form:

$$\boxed{\vec{\tilde{E}}(z,t) = \vec{\tilde{E}}_o e^{i(\tilde{k}z - \omega t)}}$$

$$\boxed{\vec{\tilde{B}}(z,t) = \vec{\tilde{B}}_o e^{i(\tilde{k}z - \omega t)} = \left(\frac{\tilde{k}}{\omega}\right)\hat{k} \times \vec{\tilde{E}}(z,t) = \frac{1}{\omega}\vec{\tilde{k}} \times \vec{\tilde{E}}(z,t)}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

With (frequency-dependent) complex wave number:

$$\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$$

$$\boxed{k(\omega) = \Re e\left(\tilde{k}(\omega)\right) = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}+1\right]^{1/2}}$$

$$\boxed{\kappa(\omega) = \Im m\left(\tilde{k}(\omega)\right) = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}-1\right]^{1/2}}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

The imaginary part of k that is - $\kappa = \Im m(k)$ results in an exponential damping of the monochromatic plane *EM* wave with increasing z*:*

$$\boxed{\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_o e^{-\kappa z} e^{i(kz-\omega t)}}$$

$$\boxed{\tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_o e^{-\kappa z} e^{i(kz-\omega t)} = \frac{1}{\omega}\tilde{\vec{k}} \times \tilde{\vec{E}}(z,t) = \frac{1}{\omega}\tilde{\vec{k}} \times \tilde{\vec{E}}_o e^{-\kappa z} e^{i(kz-\omega t)}}$$

These solutions satisfy the above wave equations for any choice

$\tilde{\vec{E}}_o$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

- The above plane wave solutions satisfy the above wave equations(s).
- Maxwell's equations rule out the presence of any longitudinal i.e, z- component of E and B.
- E and B are purely transverse waves (as before)- even in a conductor.
- Consider a linearly polarized monochromatic plane EM wave propagating in the +zˆ -direction in a conducting medium.

$$\tilde{\vec{E}}(z,t) = \tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{x}$$

then

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

$$\boxed{\tilde{\vec{B}}(z,t) = \frac{1}{\omega}\tilde{\vec{k}} \times \tilde{\vec{E}}(z,t) = \left(\frac{\tilde{k}}{\omega}\right)\tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{y} = \left(\frac{k+i\kappa}{\omega}\right)\tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{y}}$$

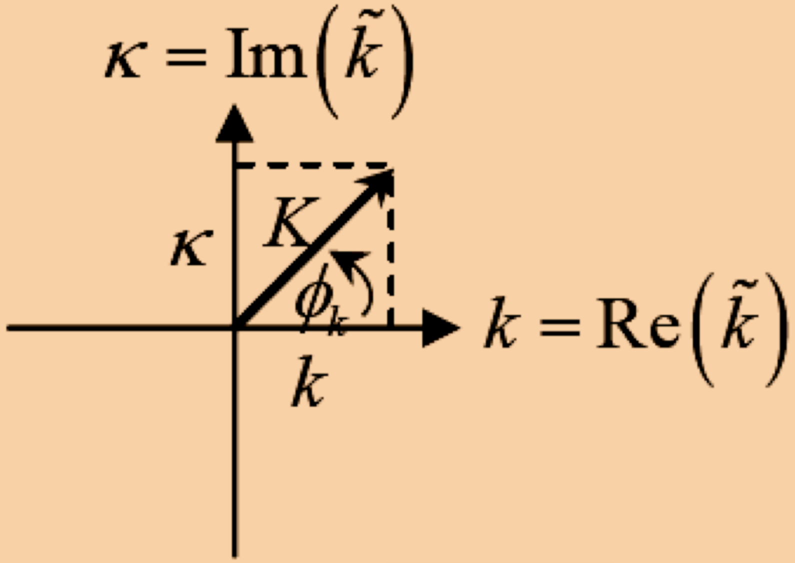
$$\Rightarrow \quad \tilde{\vec{E}}(z,t) \perp \tilde{\vec{B}}(z,t) \perp \hat{z} \quad (+\hat{z} = \text{propagation direction})$$

The complex wave-number $\tilde{k} = k + ik = Ke^{i\phi}$

$$\text{where: } K \equiv \left|\tilde{k}\right| = \sqrt{k^2 + \kappa^2} \;\; \text{and} \;\; \phi_k \equiv \tan^{-1}\left(\kappa/k\right)$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

In the complex $\tilde{k}$-plane:



$$K = \left|\tilde{k}\right| = \sqrt{k^2 + \kappa^2}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

Then we see that:

$$\boxed{\tilde{\vec{E}}(z,t) = \tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{x}}$$

has

$$\boxed{\tilde{E}_o = E_o e^{i\delta_E}}$$

$$\boxed{\tilde{\vec{B}}(z,t) = \tilde{B}_0 e^{-\kappa z} e^{i(kz-\omega t)}\hat{y} = \frac{\tilde{k}}{\omega}\tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}\hat{y}}$$

$$\boxed{\tilde{k} = K e^{i\phi_k}}$$

has

$$\boxed{\tilde{B}_o = B_o e^{i\delta_B} = \frac{\tilde{k}}{\omega}\tilde{E}_o = \frac{K e^{i\phi_k}}{\omega} E_o e^{i\delta_E}}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

$$\boxed{B_o e^{i\delta_B} = \frac{K e^{i\phi_k}}{\omega} E_o e^{i\delta_B} = \frac{K}{\omega} E_o e^{i(\delta_E + \phi_k)} = \frac{\sqrt{k^2 + \kappa^2}}{\omega} E_o e^{i(\delta_E + \phi_k)}}$$

inside a conductor, **E** and **B** are no longer in phase with each other!!!

Phases of ***E** and **B***

$$\boxed{\delta_B = \delta_E + \phi_k}$$

With phase difference:

$$\boxed{\Delta\varphi_{B-E} \equiv \delta_B - \delta_E = \phi_k}$$

Magnetic field lags behind electric field

We also see that:

$$\boxed{\frac{B_o}{E_o} = \frac{K}{\omega} = \left[\varepsilon\mu\sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]^{1/2} \neq \frac{1}{c}}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

The real/physical ***E*** and ***B*** fields associated with linearly polarized monochromatic plane *EM* waves propagating in a conducting medium are exponentially damped:

$$\boxed{\vec{E}(z,t) = \Re e\left(\tilde{\vec{E}}(z,t)\right) = E_o e^{-\kappa z}\cos\left(kz - \omega t + \delta_E\right)\hat{x}} \qquad \boxed{\delta_B = \delta_E + \phi_k}$$

$$\boxed{\vec{B}(z,t) = \Re e\left(\vec{B}(z,t)\right) = B_o e^{-\kappa z}\cos\left(kz - \omega t + \delta_B\right)\hat{y} = B_o e^{-\kappa z}\cos\left(kz - \omega t + \{\delta_E + \phi_k\}\right)\hat{y}}$$

$$\boxed{\frac{B_o}{E_o} = \frac{K(\omega)}{\omega} = \left[\varepsilon\mu\sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]^{\frac{1}{2}}}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

where

$$\boxed{K(\omega) \equiv \left|\tilde{k}(\omega)\right| = \sqrt{k^2(\omega) + \kappa^2(\omega)} = \omega\left[\varepsilon\mu\sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\,\right]^{1/2}}$$

$$\boxed{\delta_B = \delta_E + \phi_k},\quad \boxed{\phi_k(\omega) \equiv \tan^{-1}\left(\frac{\kappa(\omega)}{k(\omega)}\right)}$$

and

$$\boxed{\tilde{k}(\omega) = \left|\tilde{\vec{k}}(\omega)\right| = k(\omega) + i\kappa(\omega)}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

The real part of *k*- determines the spatial wavelength λ (ω)-the propagation speed v(ω) and also the index of refraction

$$\boxed{\lambda(\omega) = \frac{2\pi}{k(\omega)} = \frac{2\pi}{\Re e\left(\tilde{k}(\omega)\right)}}$$

$$\boxed{v(\omega) = \frac{\omega}{k(\omega)} = \frac{\omega}{\Re e\left(\tilde{k}(\omega)\right)}}$$

$$\boxed{n(\omega) = \frac{c}{v(\omega)} = \frac{ck(\omega)}{\omega} = \frac{c\Re e\left(\tilde{k}(\omega)\right)}{\omega}}$$

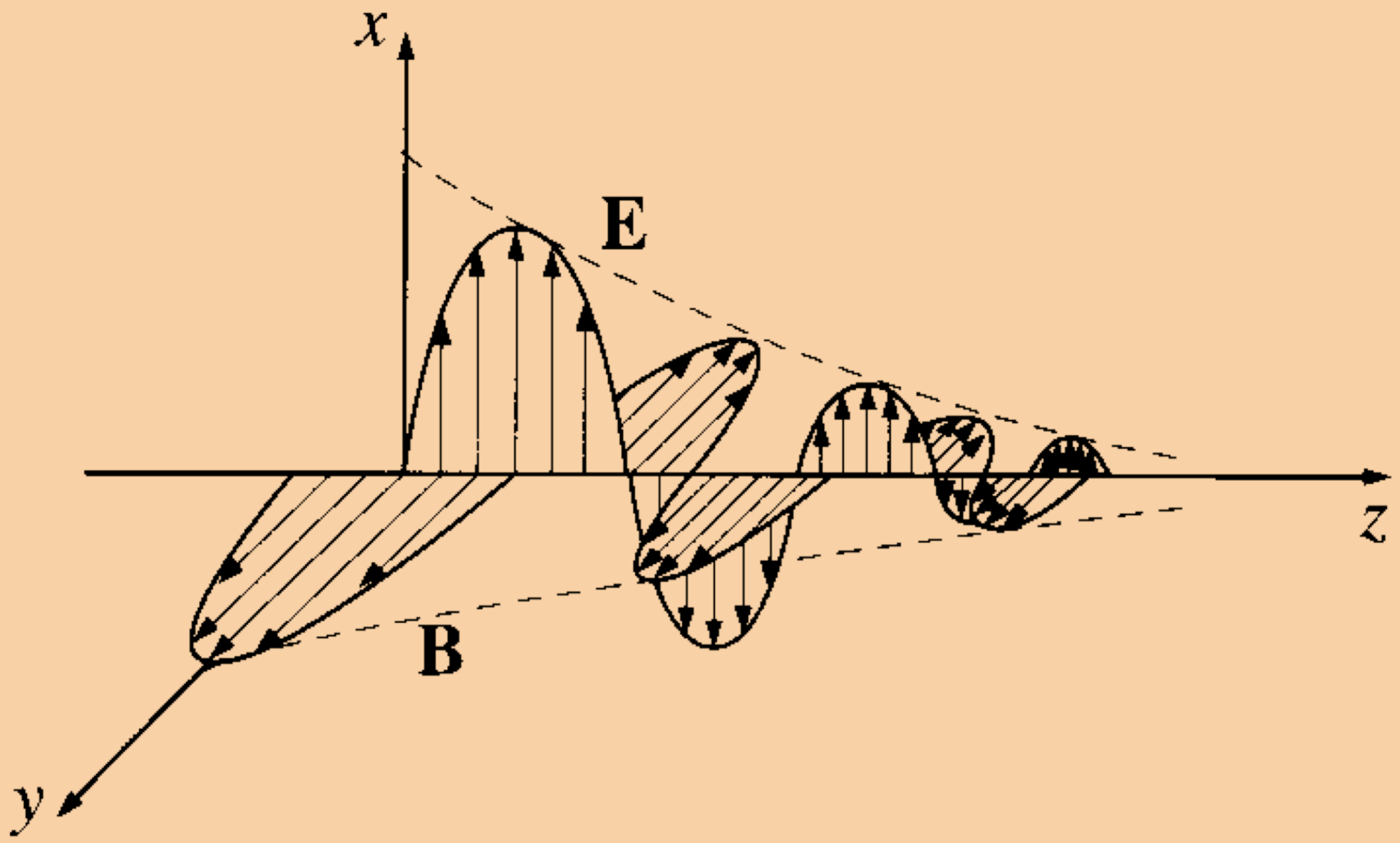
MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA

The characteristic distance over which E and B are reduced to 1/e=0.3679- of their initial values (at *z = 0)* is known as the skin depth

$$\boxed{\delta_{sc}(\omega) \equiv 1/\kappa(\omega)}$$

$$\boxed{\delta_{sc}(\omega) = \frac{1}{\kappa(\omega)} = \frac{1}{\omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}-1\right]^{1/2}}} \Rightarrow \boxed{\begin{array}{l} \tilde{\vec{E}}(z=\delta_{sc},t) = \tilde{\vec{E}}_o e^{-1} e^{i(kz-\omega t)} \\ \tilde{\vec{B}}(z=\delta_{sc},t) = \tilde{\vec{B}}_o e^{-1} e^{i(kz-\omega t)} \end{array}}$$

MONOCHROMATIC PLANE WAVES IN CONDUCTING MEDIA



Reflection of EM Waves at Normal Incidence from a Conducting Surface

In the presence of free surface charges σ and free surface currents- the BC's for reflection and refraction at *e.g.* a dielectric-conductor interface become:

BC 1): (normal ***D*** at interface):

$$\boxed{\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_{free}}$$

BC 2): (tangential ***E*** at interface):

$$\boxed{E_1^{\parallel} - E_2^{\parallel} = 0} \Rightarrow \boxed{E_1^{\parallel} = E_2^{\parallel}}$$

BC 3): (normal ***B*** at interface):

$$\boxed{B_1^{\perp} - B_2^{\perp} = 0} \Rightarrow \boxed{B_1^{\perp} = B_2^{\perp}}$$

BC 4): (tangential ***H*** at interface):

$$\boxed{\frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = \vec{K}_{free} \times \hat{n}_{\underset{21}{\rightarrow}}}$$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

$\perp$ = normal to plane of interface
$||$ = parallel to plane of interface

Where $n_{21} \rightarrow$ is a unit vector $\perp$ to the interface - pointing from medium (2) into medium (1).

Incident *EM* wave [medium (1)]:

$$\tilde{\vec{E}}_{inc}(z,t) = \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)} \hat{x}$$

and

$$\tilde{\vec{B}}_{inc}(z,t) = \frac{1}{v_1} \tilde{E}_{o_{inc}} e^{i(k_1 z - \omega t)} \hat{y}$$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

Reflected *EM* wave [medium (1)]:

$$\boxed{\tilde{\vec{E}}_{refl}(z,t) = \tilde{E}_{o_{refl}}\, e^{i(-k_1 z = \omega t)}\,\hat{x}}$$

and

$$\boxed{\tilde{\vec{B}}_{refl}(z,t) = -\frac{1}{v_1}\tilde{E}_{o_{refl}}\, e^{i(-k_1 z - \omega t)}\,\hat{y}}$$

Transmitted *EM* wave [medium (2)]:

$$\boxed{\tilde{\vec{E}}_{trans}(z,t) = \tilde{E}_{o_{trans}}\, e^{i(\tilde{k}_2 z - \omega t)}\,\hat{x}}$$

and

$$\boxed{\tilde{\vec{B}}_{trans}(z,t) = \frac{\tilde{k}_2}{\omega}\tilde{E}_{o_{trans}}\, e^{i(\tilde{k}_2 z - \omega t)}\,\hat{y}}$$

complex wave-number in (conducting) medium (2) is:

$$\boxed{\tilde{k}_2 = k_2 + i\kappa_2}$$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

In medium (1) EM fields are:

$$\boxed{\tilde{\vec{E}}_{Tot_1}(z,t) = \tilde{\vec{E}}_{inc}(z,t) + \tilde{\vec{E}}_{refl}(z,t)} \quad \boxed{\tilde{\vec{B}}_{Tot_1}(z,t) = \tilde{\vec{B}}_{inc}(z,t) + \tilde{\vec{B}}_{refl}(z,t)}$$

In medium (2) EM fields are:

$$\boxed{\tilde{\vec{E}}_{Tot_2}(z,t) = \tilde{\vec{E}}_{trans}(z,t)} \quad \underline{\text{and}}: \quad \boxed{\tilde{\vec{B}}_{Tot_2}(z,t) = \tilde{\vec{B}}_{trans}(z,t)}$$

Reflection of EM Waves at Normal Incidence from a Conducting Surface

Apply BC's at the z = 0 interface in the x-y plane:

BC 1): $\boxed{\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_{free}}$ but $\boxed{E_1^{\perp} = \tilde{E}_{1_z} = 0}$ **and**: $\boxed{E_2^{\perp} = \tilde{E}_{2_z} = 0}$

$$\boxed{0 - 0 = \sigma_{free}} \Rightarrow \boxed{\sigma_{free} = 0}$$

BC 2): $\boxed{E_1^{\parallel} = E_2^{\parallel}}$ $\therefore$ $\boxed{\tilde{E}_{o_{inc}} + \tilde{E}_{o_{refl}} = \tilde{E}_{o_{trans}}}$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

BC 3): $\boxed{B_1^{\perp} = B_2^{\perp}}$ but: $\boxed{B_1^{\perp} = B_{1_z} = 0}$ and: $\boxed{B_2^{\perp} = B_{2_z} = 0} \Rightarrow \boxed{0 = 0}$

BC 4): $\boxed{\dfrac{1}{\mu_1} B_1^{\parallel} - \dfrac{1}{\mu_2} B_2^{\parallel} = \vec{K}_{free} \times \hat{n}_{\underset{21}{\rightarrow}}}$ but: $\boxed{\vec{K}_{free} = 0}$ $\therefore$ $\boxed{\dfrac{1}{\mu_1 v_1}\left(\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}}\right) - \dfrac{\tilde{k}_2}{\mu_2 \omega} \tilde{E}_{o_{trans}} = 0}$

or: $\boxed{\tilde{E}_{o_{inc}} - \tilde{E}_{o_{refl}} = \tilde{\beta} \tilde{E}_{o_{trans}}}$ with: $\boxed{\tilde{\beta} \equiv \left(\dfrac{\mu_1 v_1 \tilde{k}_2}{\mu_2 \omega}\right) = \left(\dfrac{\mu_1 v_1}{\mu_2 \omega}\right) \tilde{k}_2}$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

Thus we obtain:

$$\boxed{\left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)=\left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)} \quad \underline{\text{and}}\text{:} \quad \boxed{\left(\frac{\tilde{E}_{o_{trans}}}{\tilde{E}_{o_{inc}}}\right)=\frac{2}{\left(1+\tilde{\beta}\right)}}$$

with

$$\boxed{\tilde{\beta}\equiv\left(\frac{\mu_1 v_1 \tilde{k}_2}{\mu_2\omega}\right)=\left(\frac{\mu_1 v_1}{\mu_2\omega}\right)\tilde{k}_2}$$

The relations for reflection/transmission of EMW at normal incidence on a non-conductor/conductor boundary are identical to those obtained for reflection/transmission of EMW at normal incidence on a boundary between two non-conductors- except for the replacement of β with a complex β.

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

For the case of a perfect conductor- the conductivity

$$\sigma_C = \infty \quad \{\text{thus resistivity}, \rho_C = 1/\sigma_C = 0\}$$

$\Rightarrow$ <u>*both*</u> $\boxed{k_2 \simeq \kappa_2 \simeq \sqrt{\frac{\omega\mu_2\sigma_C}{2}} = \infty}$ and since: $\boxed{\tilde{k}_2 = k_2 + i\kappa_2}$ then: $\boxed{\tilde{k}_2 = \infty + i\infty = \infty(1+i)}$

and since: $\boxed{\tilde{\beta} \equiv \left(\frac{\mu_1 v_1 \tilde{k}_2}{\mu_2\omega}\right) = \left(\frac{\mu_1 v_1}{\mu_2\omega}\right)\tilde{k}_2} \Rightarrow \underline{\underline{\tilde{\beta} = \infty}}$

Thus, for a perfect conductor, we see that:

$$\boxed{\tilde{E}_{o_{refl}} = -\tilde{E}_{o_{inc}}} \text{ and } \boxed{\tilde{E}_{trans} = 0}$$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

For a perfect conductor the reflection and transmission coefficients are:

$$\boxed{R \equiv \left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2 = \left|\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right|^2 = \left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)\left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)^* = 1}$$

<u>and</u>: $\boxed{T = 1 - R = 0}$

In case of a perfect conductor - for normal incidence- the reflected wave undergoes a 180 degree phase shift with respect to the incident wave at the interface at $z = 0$ in the x-y plane. A perfect conductor screens out all *EM* waves from propagating in its interior.

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

For a good conductor- the conductivity is large- but finite. The reflection coefficient R for monochromatic plane EM waves at normal incidence on a good conductor is not unity- but close to it. {*This is why good conductors make good mirrors!*}.

$$R \equiv \left(\frac{E_{o_{refl}}}{E_{o_{inc}}}\right)^2 = \left|\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right|^2 = \left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)\left(\frac{\tilde{E}_{o_{refl}}}{\tilde{E}_{o_{inc}}}\right)^* = \left|\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right|^2 = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)^*$$

Where $$\tilde{\beta} = \left(\frac{\mu_1 v_1}{\mu_2 \omega}\right)\tilde{k}_2 = \left(\frac{\mu_1 v_1}{\mu_2 \omega}\right)\sqrt{\frac{\omega \mu_2 \sigma_C}{2}}(1+i) = \mu_1 v_1 \sqrt{\frac{\sigma_C}{2\mu_2 \omega}}(1+i)$$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

Define $$\boxed{\gamma \equiv \mu_1 v_1 \sqrt{\frac{\sigma_C}{2\mu_2\omega}}}$$ Then: $$\boxed{\tilde{\beta} = \gamma(1+i)}$$

Thus, the reflection coefficient R for monochromatic plane EM waves at normal incidence on a good conductor is:

$$\boxed{R = \left|\frac{\tilde{E}_{0_{refl}}}{\tilde{E}_{0_{inc}}}\right|^2 = \left|\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right|^2 = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)^* = \left(\frac{1-\gamma-i\gamma}{1+\gamma+i\gamma}\right)\left(\frac{1-\gamma+i\gamma}{1+\gamma-i\gamma}\right) = \left[\frac{(1-\gamma)^2+\gamma^2}{(1+\gamma)^2+\gamma^2}\right]}$$

with $$\boxed{\gamma \equiv \mu_1 v_1 \sqrt{\frac{\sigma_C}{2\mu_2\omega}}}$$

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

Obviously, only a small fraction of the normally-incident monochromatic plane EM wave *is* transmitted into the good conductor- since $R < 1$ and $\boxed{T = 1 - R}$, *i.e.*:

$$\boxed{T = 1 - R = 1 - \left[\frac{(1-\gamma)^2 + \gamma^2}{(1+\gamma)^2 + \gamma^2}\right] \quad (\ll 1)}$$

Note that the transmitted wave is exponentially attenuated in the z-direction- the E and *B* fields in the good conductor fall to 1/e of their initial {z = 0} values at the interface- the monochromatic plane *EM* wave propagates a distance of one skin depth in z-direction into the conductor:

Reflection of *EM Waves at Normal Incidence from a Conducting Surface*

$$\boxed{\delta_{sc}(\omega) \equiv \frac{1}{\kappa_2(\omega)} \simeq \sqrt{\frac{2}{\omega\mu_2\sigma_C}}}$$

Note also that the energy associated with the transmitted monochromatic plane *EM* wave is ultimately dissipated in the conducting medium as heat.

In metals - the transmitted wave is absorbed in the metal- we can only study the reflection coefficient *R*.

Full Maxwell Equations in Matter

The electromagnetic state of matter at a given observation point ***r*** at a given time t is described by four macroscopic quantities:

1.) The volume density of free charge: $\boxed{\rho_{free}(\vec{r},t)}$

2.) The volume density of electric dipoles: $\boxed{\vec{\mathrm{P}}(\vec{r},t)}$ ⇐ electric polarization

3.) The volume density of magnetic dipoles: $\boxed{\vec{\mathrm{M}}(\vec{r},t)}$ ⇐ magnetization

4.) The free electric current / unit area: $\boxed{\vec{J}_{free}(\vec{r},t)}$ ⇐ {free} current density

Full Maxwell Equations in Matter

Then Maxwell's equations in matter, for $\rho_{free} = 0$ and $\vec{\mathrm{M}} = 0$

1) Gauss' Law: $\boxed{\vec{\nabla} \bullet \vec{D} = 0}$ or: $\boxed{\vec{\nabla} \bullet \vec{E} = -\frac{1}{\varepsilon_o} \vec{\nabla} \bullet \vec{\mathrm{P}} = \rho_{free} / \varepsilon_o}$

2) No magnetic charges: $\boxed{\vec{\nabla} \bullet \vec{B} = 0}$

3) Faraday's Law: $\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$

4) Ampere's Law: $\boxed{\vec{\nabla} \times \vec{B} = \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t} + \mu_o \frac{\partial \vec{\mathrm{P}}}{\partial t} + \mu_o \vec{J}_{free}}$

Full Maxwell Equations in Matter

Then applying the curl operator to Faraday's Law:

We thus obtain the inhomogeneous wave equation:

$$\boxed{\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \underbrace{\frac{1}{\varepsilon_o}\nabla \rho_{bound} + \mu_o \frac{\partial^2 \vec{\mathrm{P}}}{\partial t^2} + \mu_o \frac{\partial \vec{J}_{free}}{\partial t}}_{\text{source terms}}}$$

{and a similar one for ***B*** }

Full Maxwell Equations in Matter

For non-conducting or poorly-conducting media, i.e. insulators/ dielectrics- the first two terms on the RHS are important – they explain many optical effects such as dispersion (frequency-dependence of the index of refraction), absorption . . .

Note that the $\vec{\nabla}\rho_{bound} = -\vec{\nabla}\left(\vec{\nabla}\bullet\vec{\mathrm{P}}\right)$ term is often zero- P uniform

$$\vec{\nabla}\bullet\vec{\mathrm{P}} = \frac{\partial \mathrm{P}_x}{\partial x} + \frac{\partial \mathrm{P}_y}{\partial y} + \frac{\partial \mathrm{P}_z}{\partial z} \text{ and } \vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

$e.g.$ for $\vec{\mathrm{P}} \propto \vec{E}$ (i.e. $\vec{\mathrm{P}}$ proportional to $\vec{E}$) where: $\vec{E}(z,t) = E_o \cos(kz - \omega t + \delta)\hat{x}$

Full Maxwell Equations in Matter

For good conductors (e.g. metals), the conduction term

$$\boxed{\mu_o \frac{\partial \vec{J}_{free}}{\partial t} = \mu_o \sigma_C \frac{\partial \vec{E}}{\partial t}}$$

is the most important, because it explains the opacity of metals (e.g. in the visible light region) and also explains the high reflectance of metals.

THANK YOU