Strongly disordered superconductors and Superconductor-Insulator Transitions (SIT)

Mikhail Feigel'man
L. D. Landau Institute for Theoretical Physics

- Granular superconductors and artificial arrays
- Homogeneously disordered thin films and similar systems

- **Superconductivity in amorphous poor conductors and SIT**

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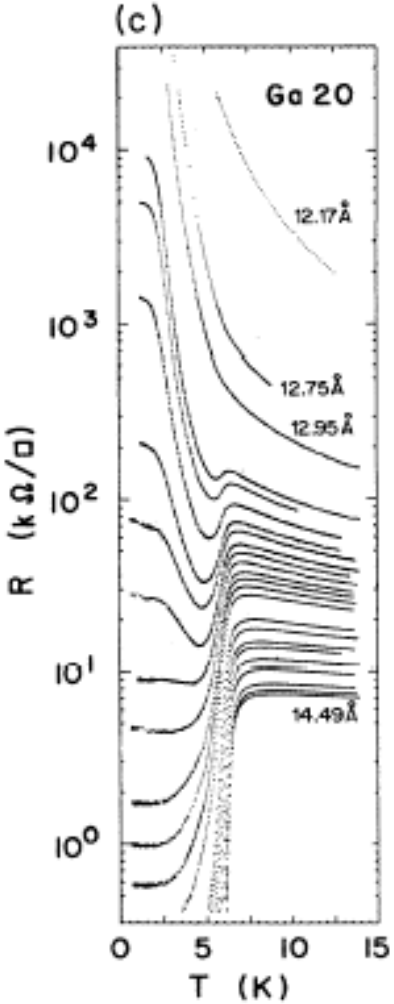
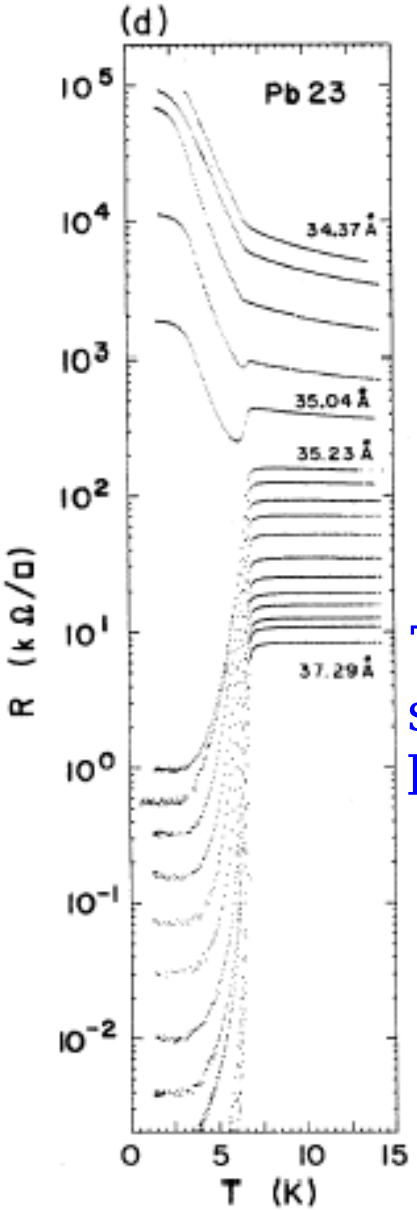
Onset of superconductivity in ultrathin granular metal films





Phys Rev B **40** 182 (1989)

H. M. Jaeger,* D. B. Haviland, B. G. Orr,† and A. M. Goldman

Why and how the transformation
From Superconducting to Insulating
behaviour occurs ?

There are many different answers;
some will be presented in the followin
lectures

Recent review:

V.F.Gantmakher, V.T.Dolgopolov
Physics-Uspekhi, **180** (2010)

1. Disordered superconductors (basic facts)
2. Role of Coulomb interaction: 3 types of materials and 3 major mechanisms of SC suppression by disorder
3. Granular superconductors and artificial arrays (bosonic mechanism)
   1) Granular superconductors: experiments
   2) Theories of SIT. Which parameter drives the transition ?
4. Homogeneously disordered metal films (fermionic mechanism)
   1) Suppression of T_c by disorder-enhanced Coulomb
   2) Mesoscopic fluctuations of T_c near the QPT
5. **Superconductivity v/s Localization (no Coulomb)**
   1) Motivation from experiments
   2) BCS-like theory for critical eigenstates
   3) Superconductivity with pseudo-gap
   4) Quantum phase transition: Bethe lattice model

Disordered superconductors (classical results)

- Potential disorder does not affect superconductive transition temperature *(for s-wave)* – *A.A. Abrikosov & L.P.Gor'kov 1958*
  P.W.Anderson 1959
- In the "dirty limit" $l << \xi_0$ coherence length decreases as $\xi \sim (l\, \xi_0)^{1/2}$ whereas London length grows as $\lambda \sim l^{-1/2}$

Accuracy limit: semi-classical approx. $k_F l >> 1$

or (the same in another form) $G = \sigma\, (h/e^2)\, \xi^{d-2} >> 1$

What happens if G ~ 1 ?

“Anderson theorem”

$$\Delta(\mathbf{r}) = \int d^d\mathbf{r}'\, K(\mathbf{r},\mathbf{r}')\, \Delta(\mathbf{r}').$$

$$K(\mathbf{r},\mathbf{r}') = \frac{g}{2} \sum_{ij} \eta_{ij}\, \psi_i(\mathbf{r})\psi_j(\mathbf{r})\psi_j(\mathbf{r}')\psi_i(\mathbf{r}'),$$

$$\eta_{ij} = \frac{\tanh(\xi_i/2T) + \tanh(\xi_j/2T)}{\xi_i + \xi_j},$$

Approx. $\Delta(\mathbf{r}) =$ const leads to BCS gap equation

$$1 = g \int N(0)\, (d\xi/\xi)\, \mathrm{th}(\xi/2T)$$

Why Coulomb repulsion does not kill phonon attraction in SC ?

Coulomb potential $\mu(\omega) = \mu_0 / [1 + \mu_0 \ln (E_F/\omega)]$

"Tolmachev Logarithm"

Normally, $\mu_0 \sim 1$ and $\ln (E_F/\omega) >> 1$

$\omega \sim \omega_D \sim 0.05$ eV whereas $E_F \sim 5$ eV

Pairing of electrons occurs in the momentum space, **(p, -p)**

At the energy scale ω_D relevant for phonon process, full amplitude in the Cooper channel can be attractive with small λ_0

$$\lambda = \lambda_0 - 1/\ln (E_F/\omega) > 0$$

What happens to Coulomb repulsion upon increase of disorder ?

1. “Microscopic” Coulomb repulsion between electrons in a pair is enhanced due to disorder, thus attraction becomes weaker (“fermionic scenario”)
2. In granular systems, “macroscopic” Coulomb repulsion (charging energy) is preventing coherence between Cooper pairs in different grains (“bosonic scenario”)
3. Special cases: Coulomb rep. is absent or strongly suppressed, Anderson localization rules

Superconductivity v/s Disorder

- Granular systems with Coulomb interaction ***"Bosonic mechanism"***
- Coulomb-induced suppression of Tc in uniform films ***"Fermionic mechanism"***
- Competition of Cooper pairing and Localization (no Coulomb)

I. Granular arrays

Reviews: I.Beloborodov *et al*, Rev. Mod.Phys.**79**, 469 (2007)
R.Fazio and H. van der Zant, Phys. Rep. **355,** 235 (2001)

Critical grain size

Mean-field theory gap equation:

$$\Delta = (g/2)\, \Sigma_i\, \Delta/[\varepsilon_i^2 + \Delta^2]^{1/2}$$

$$\delta \sum_{\xi_k} \rightarrow \int_{-\infty}^{\infty} d\xi,$$

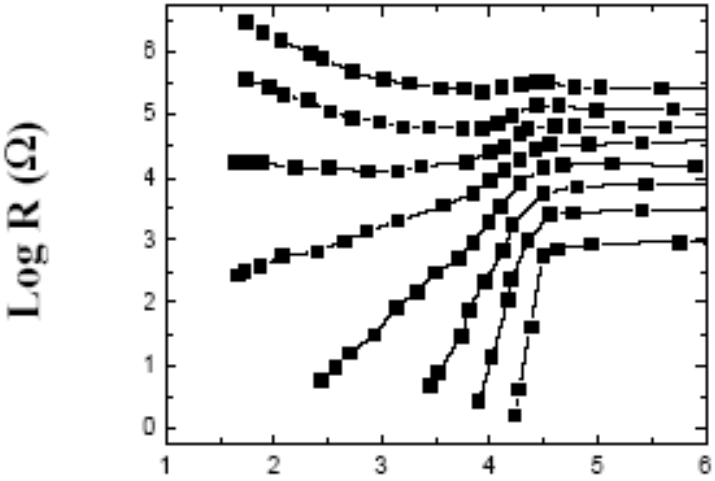
Level spacing $\delta << \Delta$ allows to replace sum by the integral and get back usual BCS equation

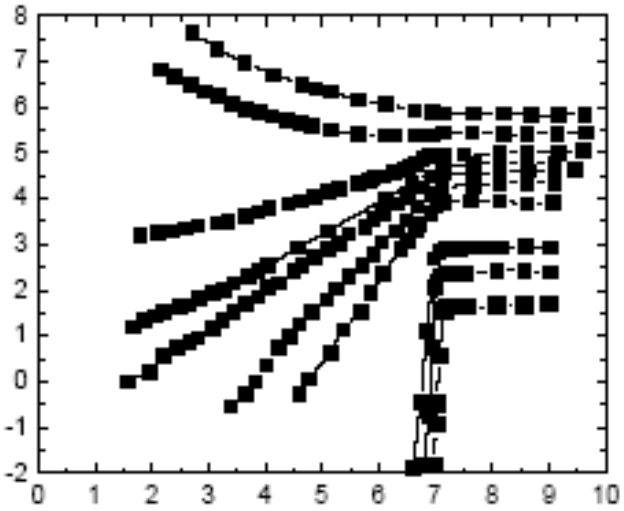
Grain radius $a >> a_c = (1/\Delta V)^{1/3}$

Note: $a_c << \xi_0$

Thin quenched-condensed films

A. Frydman, O. Naaman, R. Dynes 2002





Sn grains

Pb grains

Onset of superconductivity in ultrathin granular metal films Phys Rev B **40** 182 (1989)

H. M. Jaeger,* D. B. Haviland, B. G. Orr,† and A. M. Goldman

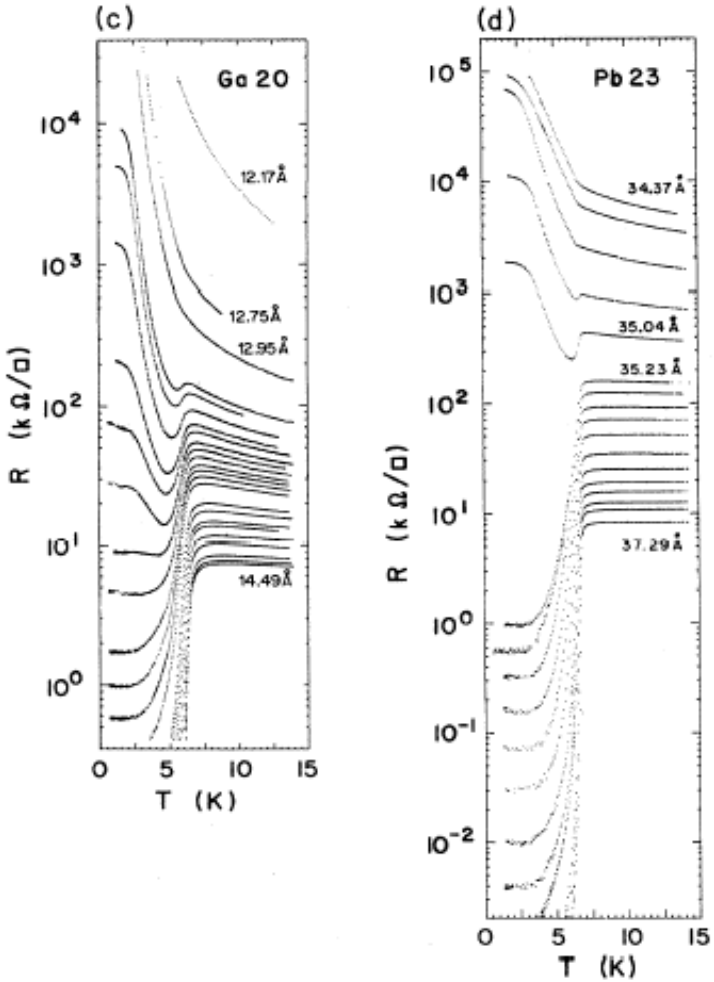


FIG. 2. The evolution of $R(T)$ curves for (a) Al, (b) In, (c) Ga, and (d) Pb, obtained *in situ* after successive increments of film thickness. Note that in all cases the entire evolution from insulating to globally superconducting spans an interval of nominal thickness of less than one monolayer.

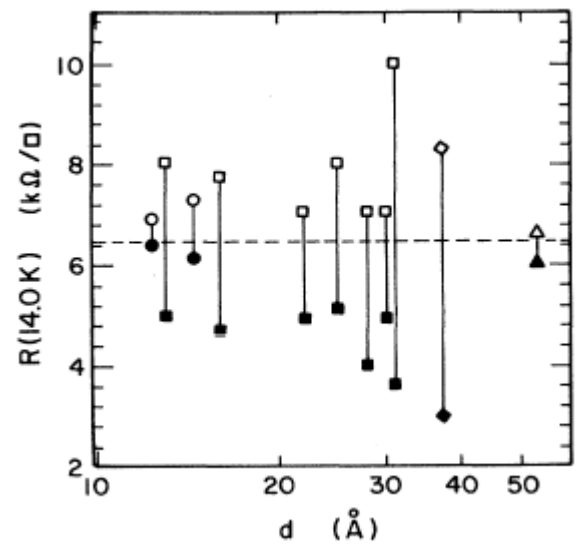


FIG. 6. The pairs of connected points represent the normal-state resistance of successive depositions in sequences of Ga (○), Sn (□), Pb (◇), and Al (△) films. The nominal thickness

<u>Conclusion in this paper</u>: control parameter is the normal resistance R. Its critical value is $R_Q = h/4e^2 = 6.5$ kOhm

Artificial regular JJ arrays

PHYSICAL REVIEW B VOLUME 50, NUMBER 6 1 AUGUST 1994-II

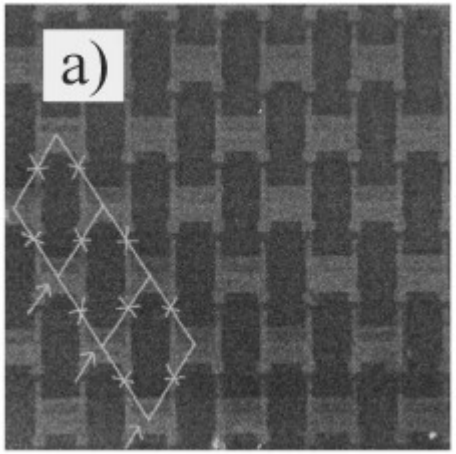
Charge solitons and quantum fluctuations in two-dimensional arrays of small Josephson junctions

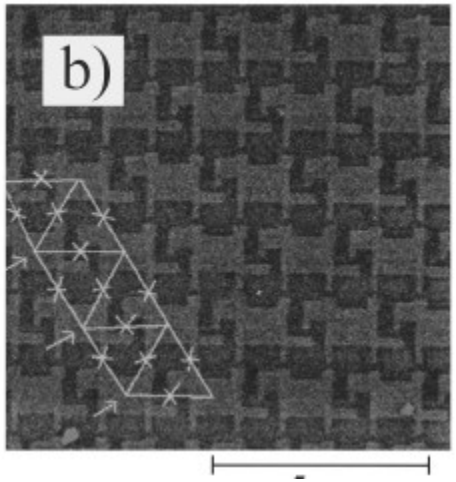
P. Delsing, C. D. Chen, D. B. Haviland, Y. Harada, and T. Claeson
Department of Physics, Chalmers University of Technology and University of Göteborg, S-412 96 Göteborg, Sweden

PHYSICAL REVIEW B VOLUME 54, NUMBER 14 1 OCTOBER 1996-II

Quantum phase transitions in two dimensions: Experiments in Josephson-junction arrays

H. S. J. van der Zant, W. J. Elion, L. J. Geerligs, and J. E. Mooij
Department of Applied Physics and Delft Institute of Microelectronics and Submicron-technology (DIMES),





Al : $T_c = 1.2$ K
$\Delta \approx 2$ K

$E_C = e^2/2C \sim 1$ K

$E_J \sim 1$K

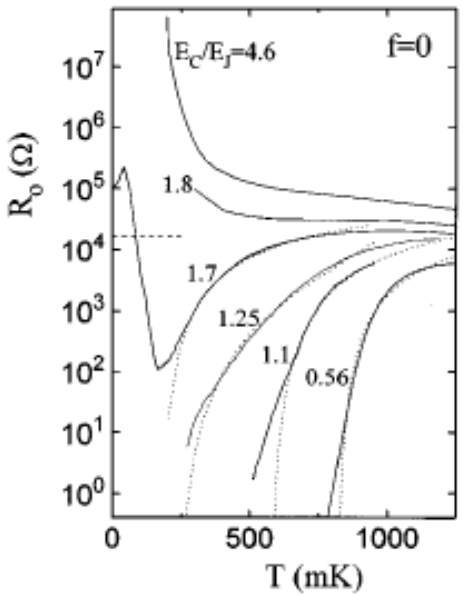


FIG. 5. The zero-field linear resistance per junction measured as a function of temperature for six different arrays. Dotted lines are fits to the vortex-KTB square-root cusp formula. The dashed horizontal line shows the zero-temperature universal resistance ($8R_q/\pi$=16.4 kΩ) of the S–I transition at f=0.

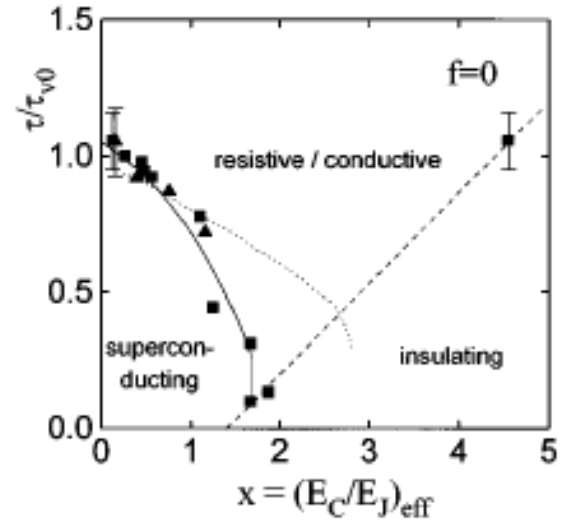


FIG. 6. Measured phase diagram of our square arrays (solid squares) and triangular (solid triangles) in zero magnetic field, showing the superconductor-to-insulator transition at $(E_C/E_J)_{eff}\approx 1.7$. The solid line is a guide to the eye connecting the data points and the dotted line at the superconducting side is the result of a recent calculation (Ref. 39).

What is the parameter that controls SIT in granular superconductors ?

- Ratio E_C/E_J ?

- Dimensionless conductance

  $g = (h/4e^2)\, R^{-1}$? (for 2D case)

2) Theoretical approaches to SIT in granular arrays

- K.Efetov ZhETF **78**, 2017 (1980) [Sov.Phys.-JETP **52**, 568 (1980)]

  Hamiltonian for charge-phase variables

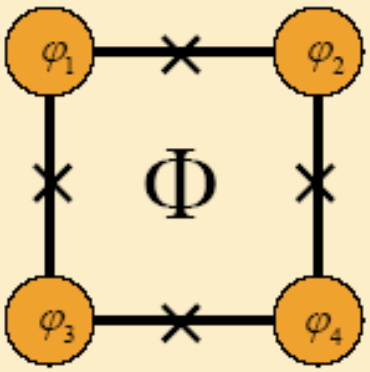
- M.P.A.Fisher, Phys.Rev.Lett. **65**, 923 (1990) General "duality" *Cooper pairs – Vortices* in 2D

- R.Fazio and G.Schön, Phys. Rev. B**43,** 5307 (1991) Effective action for 2D arrays

K.Efetov's microscopic Hamiltonian

JOSEPHSON ARRAYS

Elementary building block



Ideal Hamiltonian:

$$H = \frac{1}{2}\sum_{i,j} C_{ij}^{-1} q_i q_j + E_J \cos(\varphi_i - \varphi_j - 2\pi \frac{\Phi_{ij}}{\Phi_0}) \quad q_i = 2e\, i \frac{d}{d\varphi_i}$$

C_{ij} - capacitance matrix E_J - Josephson energy

Control parameter

$$x = E_C/E_J$$

$$Ec = e^2/2C$$

Artificial arrays: major term in capacitance matrix is n-n capacitance C

q_i and φ_i are canonically conjugated

M.P.A.Fisher's duality arguments

- Competition between Coulomb repulsion and Cooper pair hopping:

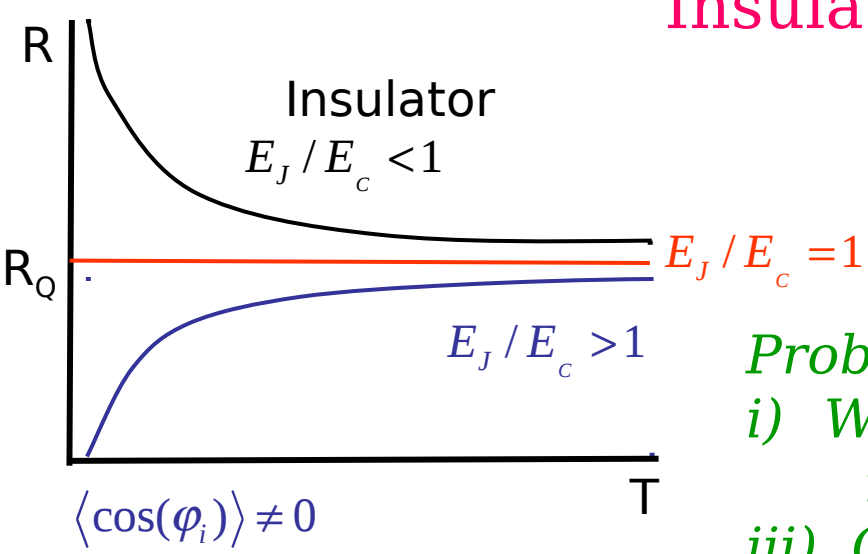
  Duality charge-vortex: both charge-charge and vortex-vortex interaction are Log(R) in 2D.

  Vortex motion generates voltage: $V=\varphi_0 j_V$

  Charge motion generates current: $I=2e\, j_c$

  At the self-dual point the currents are equal → $R_Q=V/I=h/(2e)^2=6.5k\Omega$.

Insulator is a superfluid of vortices



In favor of this idea:
usually SIT in films occurs at R near R_Q

Problems: i) how to <u>derive</u> that duality ?
i) What about capacitance matrix in granular films ?
iii) Critical R(T) is not flat usually

Efetov's theory v/s "duality approach" ?

Capacitance renormalization due to virtual electron tunneling via AES [PRB **30**, 6419 (1984)] action functional

$$S[\varphi] = S_c[\varphi] + S_{ts}[\varphi] + S_J[\varphi]$$

Virtual tunneling $\alpha_s(\tau) = \frac{1}{\pi^2}\Delta^2 K_1^2(\Delta|\tau|),$

Charging

$$S_c = \frac{1}{2e^2}\sum_{ij}\int_0^\beta d\tau C_{ij}\frac{d\varphi_i(\tau)}{d\tau}\frac{d\varphi_j(\tau)}{d\tau},$$

$$S_{ts} = g\sum_{\langle ij\rangle}\int_0^\beta d\tau d\tau' \alpha(\tau-\tau')\sin^2\left[\frac{\varphi_{ij}(\tau)-\varphi_{ij}(\tau')}{2}\right]$$

$$S_{ts} = (3g/32\,\Delta)\sum_{<ij>}\int_0^\beta \dot{\varphi}_{ij}^2(\tau)d\tau,$$

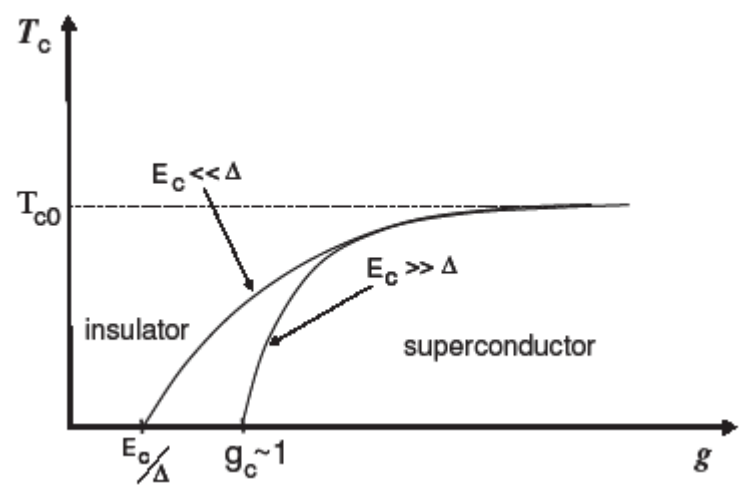
Josephson (if $d\varphi/dt << \Delta$)

$$C_{ind} = (3/16)\, ge^2/\Delta$$

$$S_J = -\frac{1}{2}\sum_{\langle ij\rangle}\int_0^\beta J_{ij}\cos\left(2\left[\varphi_i(\tau)-\varphi_j(\tau)\right]\right)d\tau$$

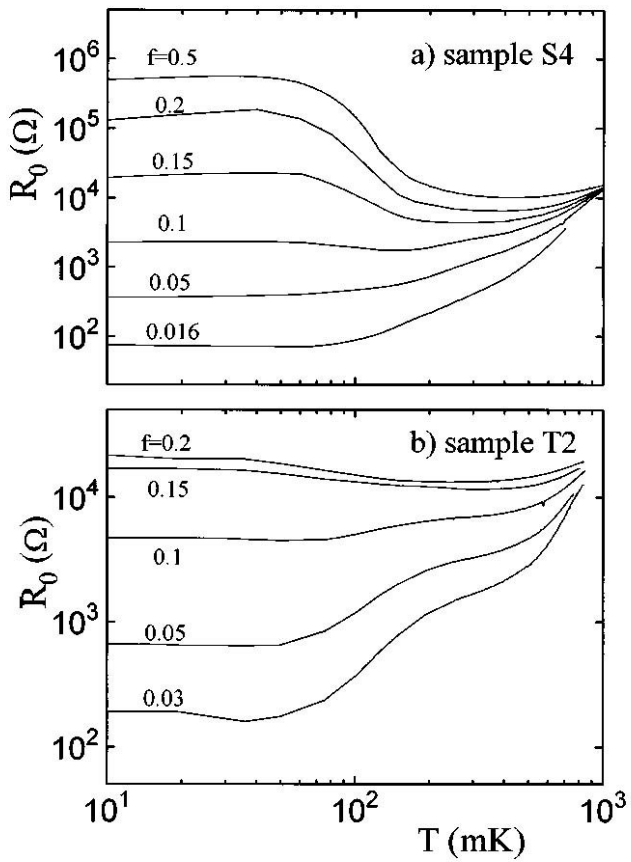
$$J = g\Delta/2$$

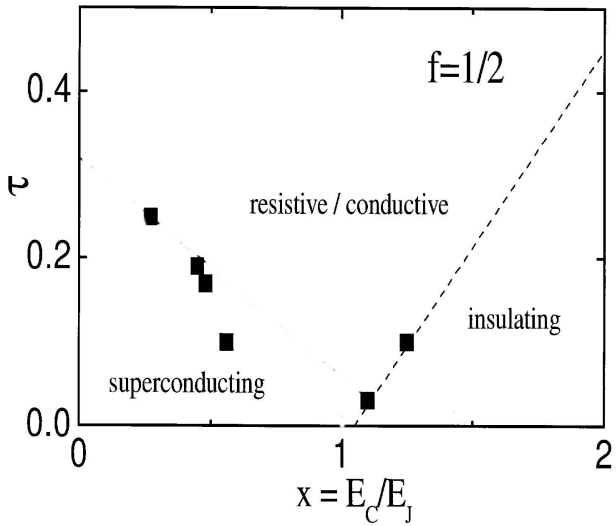
$$E_c = \Delta/[3g/8 + 2C\Delta/e^2]$$



“Bose metal” in JJ array ?

van der ZANT, ELION, GEERLIGS, AND MOOIJ





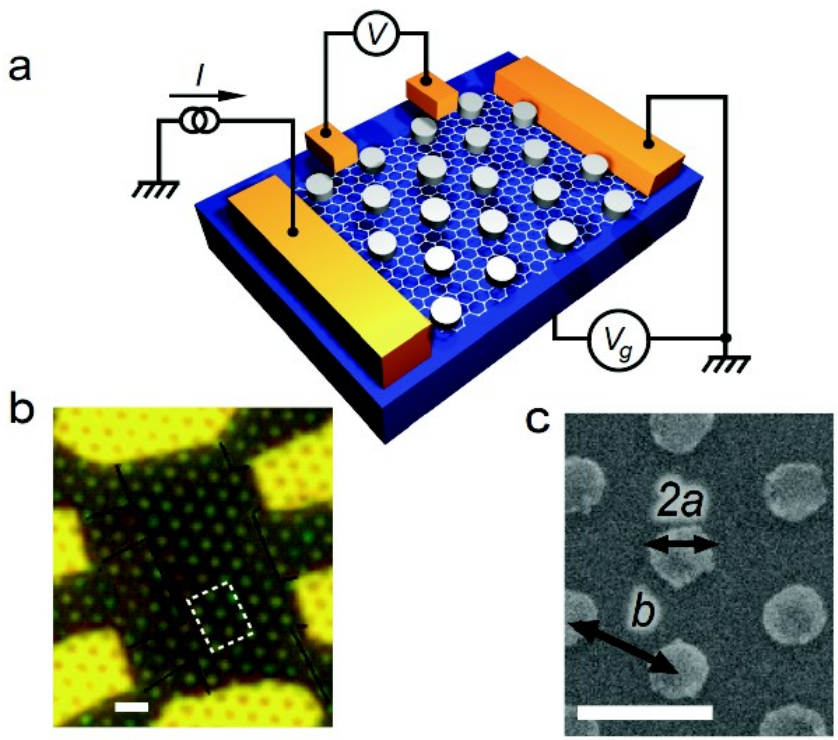
At non-zero field simple Josephson arrays show temperature-independent resistance that change by orders of magnitude with B

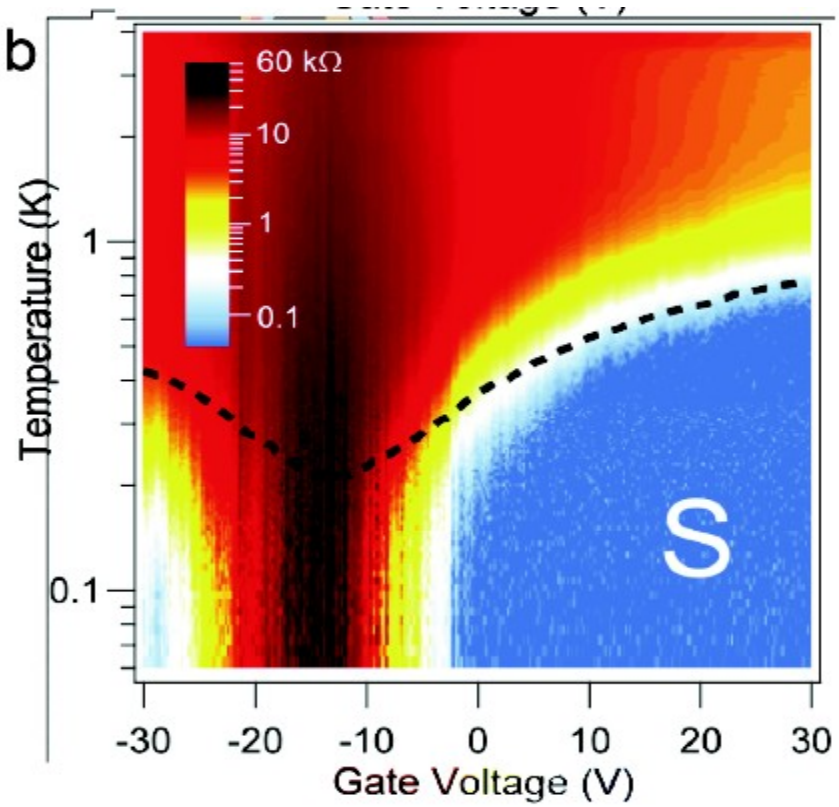
The origin of “Bose metal” state is unknown

Collapse of superconductivity in a hybrid tin-graphene Josephson junction array

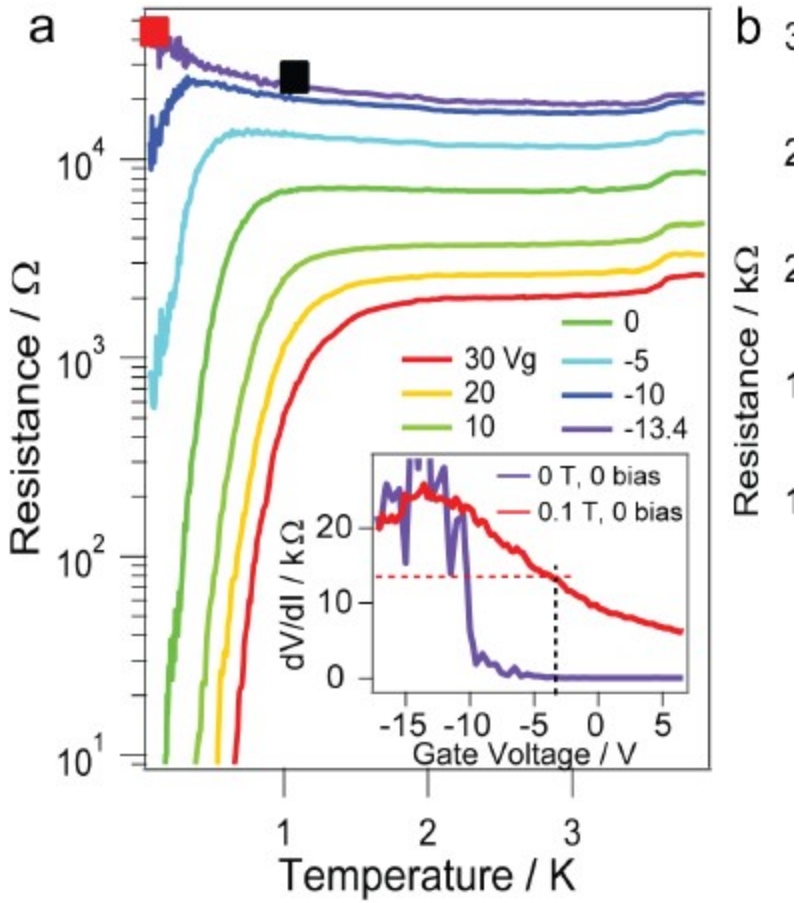
Nature Physics 2014

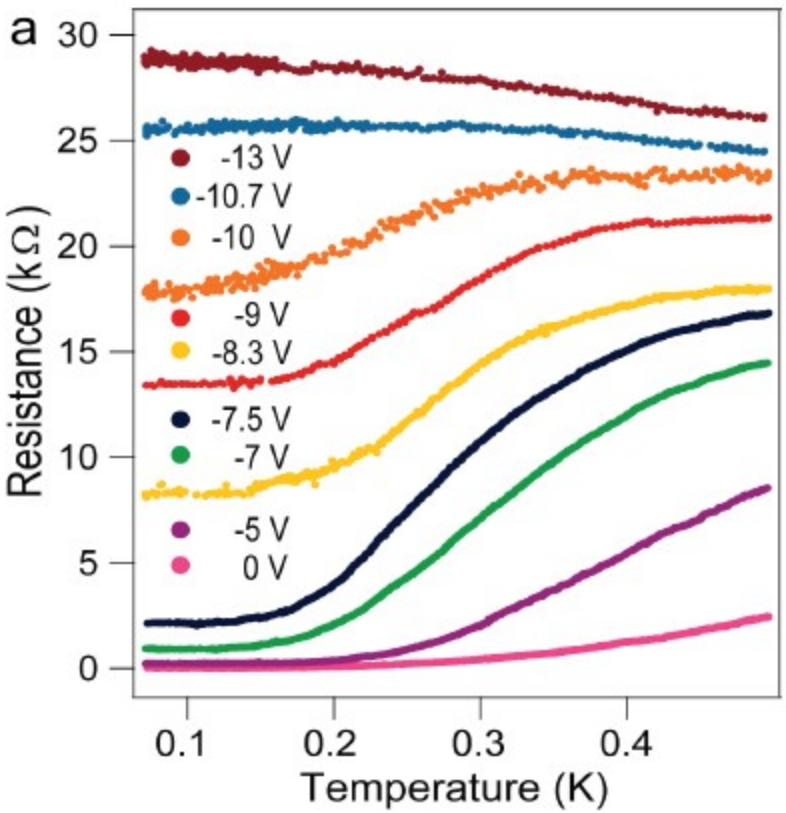
Zheng Han[1,2], Adrien Allain[1,2], Hadi Arjmandi-Tash[1,2], Konstantin Tikhonov[3,4], Mikhail Feigel'man[3,5], Benjamin Sacépé[1,2] and Vincent Bouchiat[1,2]





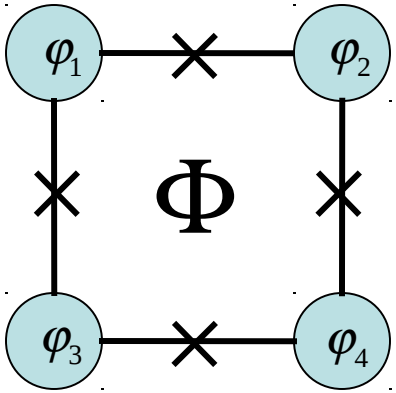
Intermediate metal state – now B=0





JOSEPHSON ARRAYS

Elementary building block



Hypothesis:
it might be related to charge offset noise

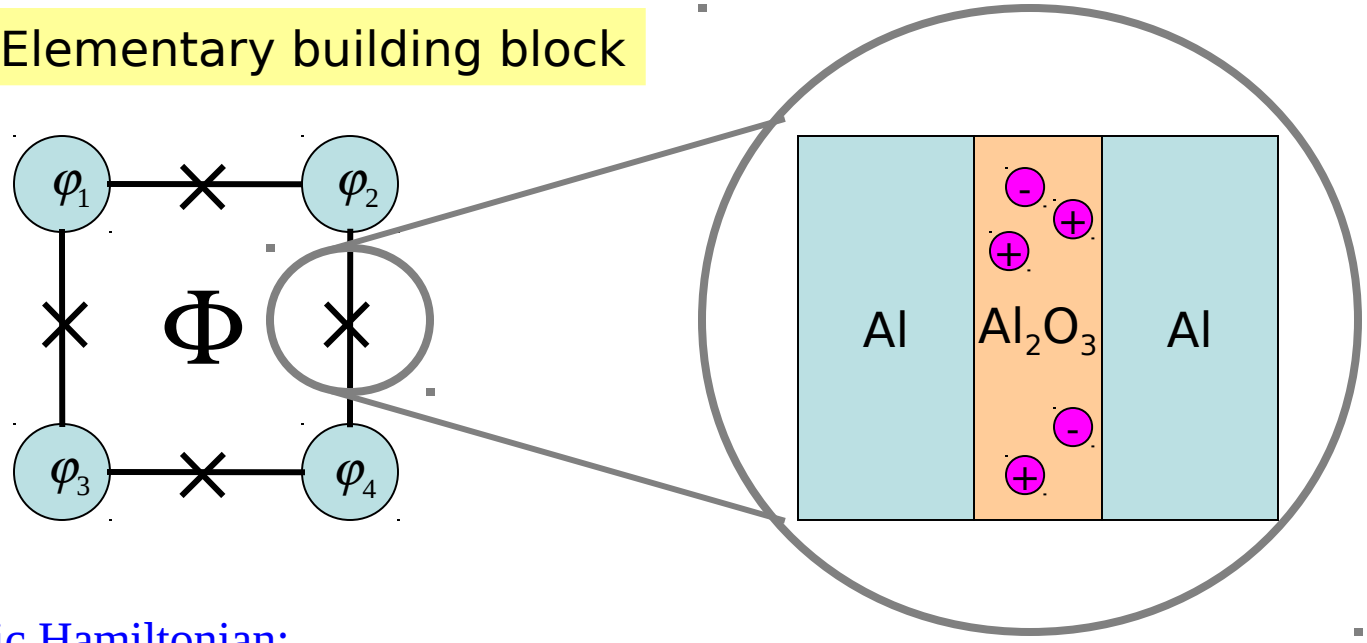
Ideal Hamiltonian:

$$H = \frac{1}{2}\sum_{i,j} C_{ij}^{-1} q_i q_j + E_J \cos(\varphi_i - \varphi_j - 2\pi \frac{\Phi_{ij}}{\Phi_0}) \quad q_i = 2e\, i \frac{d}{d\varphi_i}$$

C_{ij} - capacitance matrix E_J - Josephson energy

JOSEPHSON ARRAYS

Elementary building block



More realistic Hamiltonian:

$$H = \frac{1}{2}\sum_{i,j} C_{ij}^{-1}(q_i + Q_i)(q_j + Q_j) + (E_J + \delta E_J)\cos(\varphi_i - \varphi_j - 2\pi\frac{\Phi_{ij} + \delta\Phi}{\Phi_0}) \quad q_i = 2e\, i\frac{d}{d\varphi_i}$$

C_{ij} - capacitance matrix E_J - Josephson energy

$Q_i = Q_i^0 + Q_i(t)$ - induced charge (static and fluctuating)

$\delta\Phi = \delta\Phi^0 + \delta\Phi(t)$ - static flux due to area scatter and flux noise

$\delta E_J = \delta E_J^{\,0} + \delta E_J(t)$ - static scatter of Josephson energies and their time dependent fluctuations.

Conclusion for granular superconductive arrays:

1) most basic features of SIT are understood

2) really quantitative SIT theory is not constructed yet

3) the origin of the “Bose metal” is fully unclear

II. Homogeneously disordered SC films

1) Suppression of Tc by disorder-enhanced Coulomb
2) Mesoscopic fluctuations of Tc near the QPT

Amorphous v/s Granular films

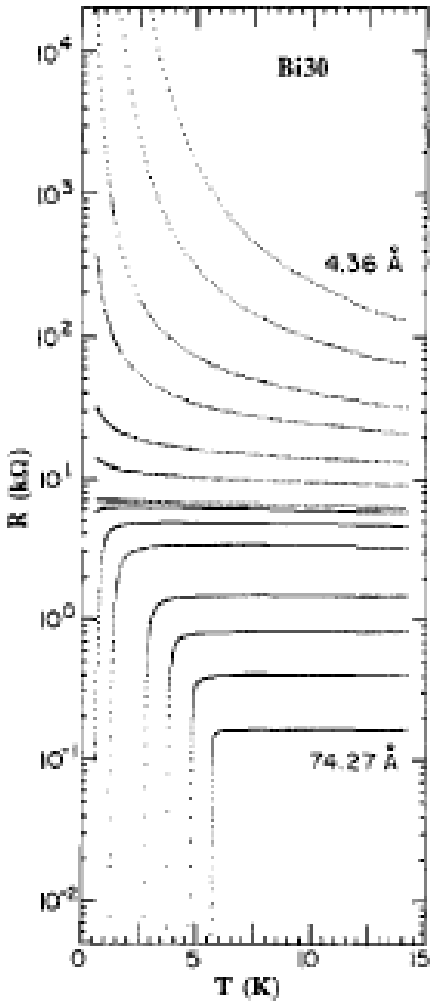


Fig. 5. Temperature dependence of the sheet resistance $R(T)$ for Bi films deposited onto Ge [15]. The films are considered to be homogeneous.

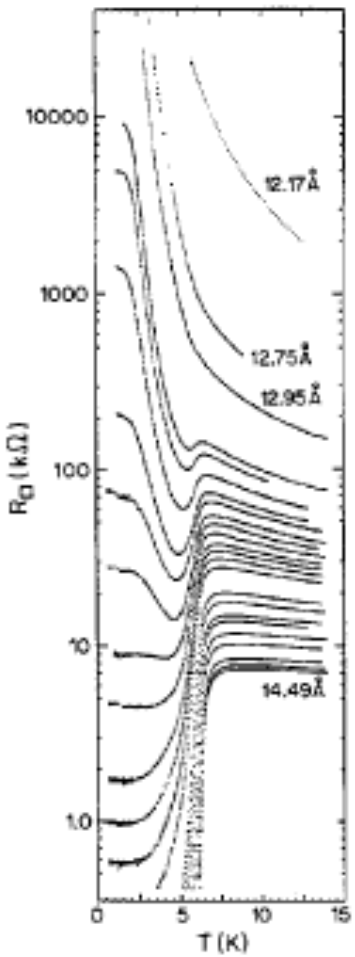


Fig. 6. Evolution of the resistive transition $R_{\square}(T)$ for granular Ga films [16].

1) Suppression of T_c in amorphous thin films by disorder-enhanced Coulomb interaction

Theory

S. Maekawa and H. Fukuyama, J. Phys. Soc. Jpn. **51**, 1380 (1982).
H. Takagi and Y. Kuroda, Solid Stat. Comm. **41**, 643 (1982).
A. M. Finkel'stein, Pis'ma ZhETF **45**, 37 (1987), [JETP Lett **45**, 46 (1987)]; A. M. Finkel'stein in *Proc. Int. Symp. on Anderson Localization*, edited by T. Ando and H. Fukuyama (Springer-Verlag,Berlin, 1988), p. 230, Springer Proc. in Physics Vol. 28.

Experiment

J. M. Graybeal and M. R. Beasly, Phys. Rev. B **29**, 4167 (1984), J. M. Graybeal, M. R. Beasly, and R. L. Green, Physica B+C **126** 731 (1984).

P. Xiong, A. V. Herzog, and R. C. Dynes, Phys. Rev. Lett. **78**, 927 (1997).

Review: A. M. Finkel'stein, Physica B **197**, 636 (1994).

Generalization to quasi-1D stripes: Yu. Oreg and A. M. Finkel'stein
Phys. Rev. Lett. **83**, 191 (1999)

Similar approach for 3D poor conductor near Anderson transition:
P. W. Anderson, K. A. Muttalib, and T. V. Ramakrishnan,
Phys. Rev. B **28**, 117 (1983)

Why disorder leads to “Coulomb revival” ?

Matrix element of Coulomb repulsion:

$$U_{ij} = \int \Psi_i^{\,2}(r)\, \Psi_j^{\,2}(r')\, U(r-r')\, dr\, dr'$$

U(r) is short-range due to RPA screening

$$\langle \Psi_i^{\,2}(r)\, \Psi_j^{\,2}(r) \rangle = V^{-1}\, [1 + (1/g)\, \ln (1/\tau\omega_{ij})\,]$$

Uniform term subject to Tolmachev Log

Enhancement due to disorder

Suppression of T_c in amorphous thin films: qualitative picture

- Disorder increases Coulomb interaction and thus decreases the pairing interaction (sum of Coulomb and phonon attraction). In perturbation theory:

$g = 2\pi \hbar\sigma / e^2$

$$\lambda(\varepsilon) = \lambda_0 - \frac{1}{24\pi g} Log\left(\frac{1}{\varepsilon\tau}\right)$$

Return probability in 2D

Roughly, $$\frac{\delta T_c}{T_c} = -\frac{\delta\lambda}{\lambda^2}$$

But in fact BCS-type problem with <u>energy-dependent</u> coupling must be solved

The origin of the correction term ?

It is "revival" of strong Coulomb repulsion, due to slow diffusion at $g \sim 1$

Coulomb suppression of T_c

Critical temperature

$$\frac{1}{|\lambda_0|} = \frac{1}{\lambda_g}\tanh(\lambda_g \zeta_c)$$

$$T_c\tau = \left(\frac{1 - \frac{1}{\sqrt{2\pi g}}\ln\frac{1}{T_{c0}\tau}}{1 + \frac{1}{\sqrt{2\pi g}}\ln\frac{1}{T_{c0}\tau}}\right)^{\sqrt{\pi g/2}}$$

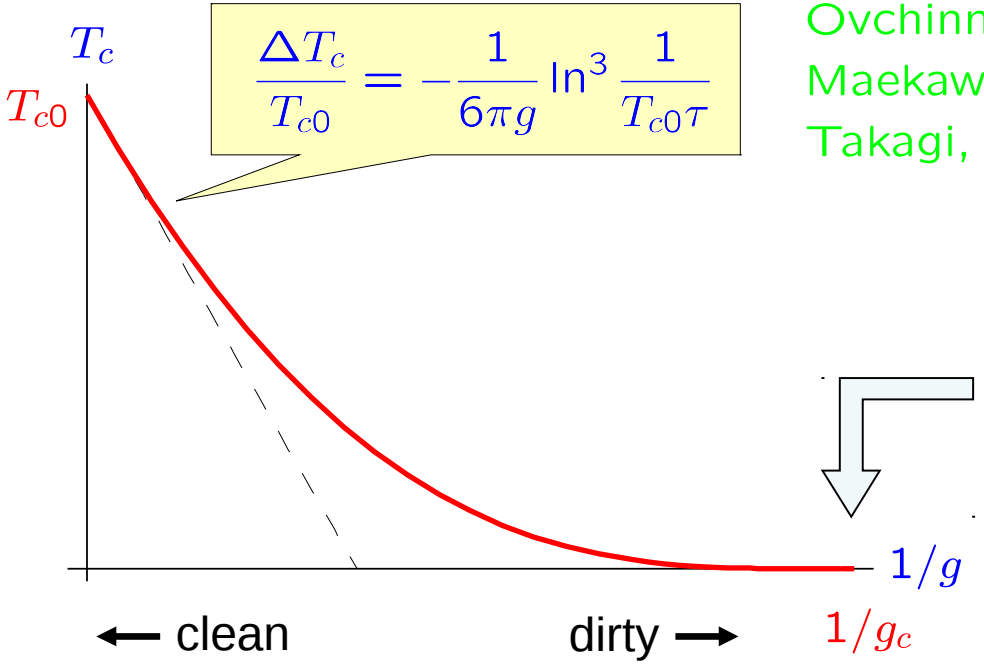
Finkelstein (1987)

$$\frac{\Delta T_c}{T_{c0}} = -\frac{1}{6\pi g}\ln^3\frac{1}{T_{c0}\tau}$$

Ovchinnikov (1973) **(wrong sign)**

Maekawa, Fukuyama (1982)

Takagi, Kuroda (1982)



$$g_c = \frac{1}{2\pi}\ln^2\frac{1}{T_{c0}\tau}$$

At $\ln(1/T_{c0}\tau) > 5$

$g_c > 4$ i.e. $R_c < R_Q$

Conclusion:

Superconductor – Metal

Quantum phase transition

2) Mesoscopic fluctuations of T_c near the Finkelstein's Quantum Phase Transition

M. Feigelman and M. Skvortsov, *Phys. Rev. Lett*. **95** 057002 (2005)

Fluctuations of effective attraction "constant"

$$\lambda(\varepsilon) = \lambda_0 - \boxed{\frac{1}{24\pi g} Log\left(\frac{1}{\varepsilon\tau}\right)} \longleftarrow \lambda_c$$

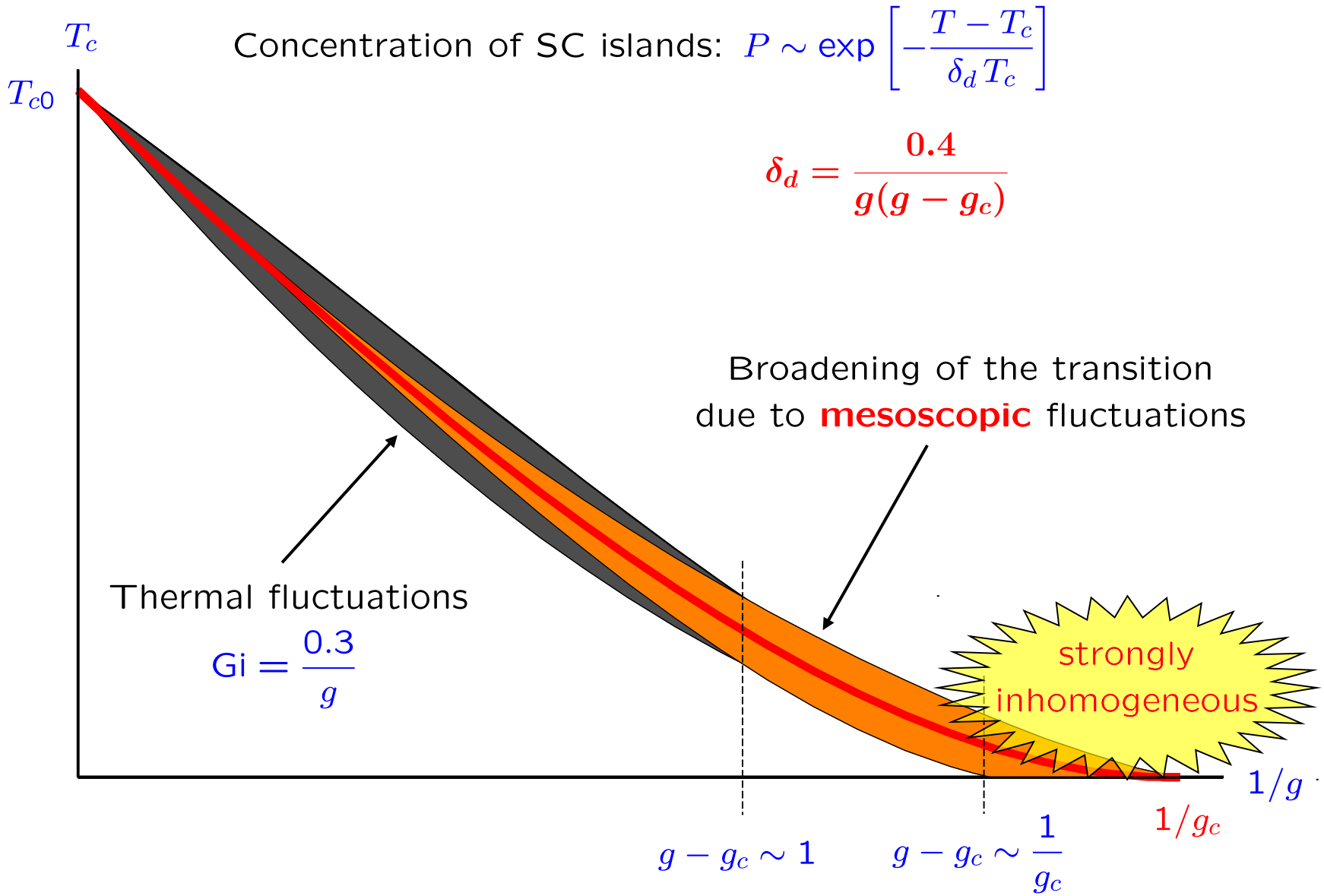
$$\frac{\delta T_c}{T_c} = -\frac{\delta\lambda}{\lambda^2}$$

λ_c Is prone to Universal Mesoscopic Fluctuations

$$\Delta\lambda_c \sim (1/g^2)\ Log\ (1/\varepsilon\tau)$$

Close to critical point $<\lambda>$ decreases and becomes comparable to $\Delta\lambda_c$

Mesoscopic vs. thermal fluctuations



Experiment on TiN films

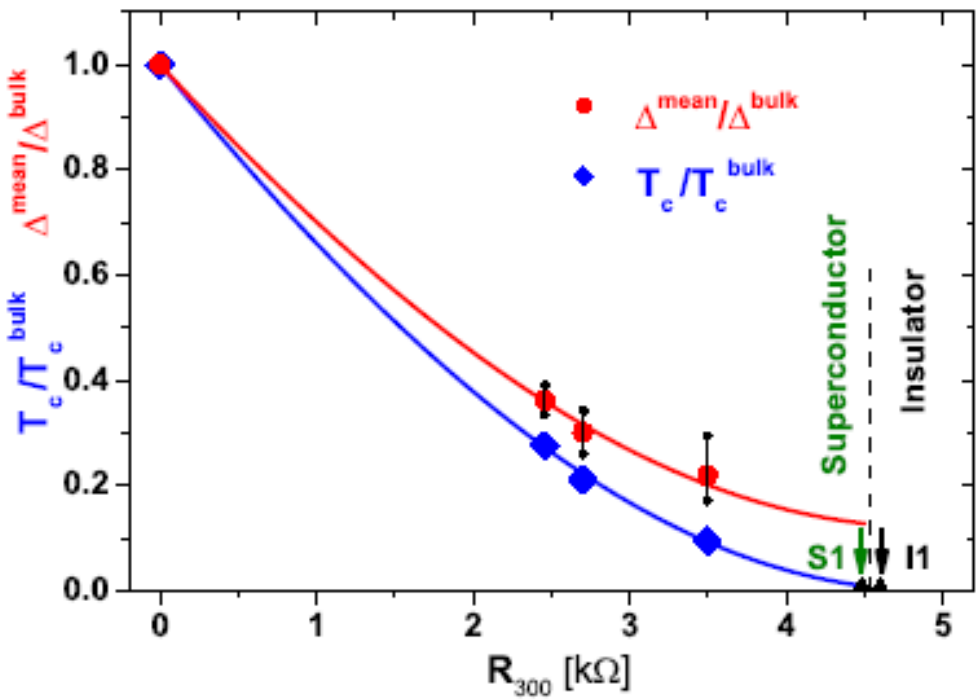


FIG. 3: Critical temperature T_c and mean spectral gap $\overline{\Delta}$ at $T = 50\,\mathrm{mK}$ divided by their bulk value versus the room temperature sheet resistance. Black bars represent the measured gap ranges for each sample ($\sim 4\sigma$ for TiN1 and TiN2). Blue solid line is a fit according to Finkelstein's model [24]. Red solid line is a guide to the eye. Black triangles are the room temperature sheet resistances for the "last"superconducting sample (S1 - $R_{300} = 4.48\,\mathrm{k}\Omega$) and "first"insulating sample (I1 - $R_{300} = 4.60\,\mathrm{k}\Omega$) from the Ref. [8].

B.Sacepe, C.Chapelier, T.Baturina et al Phys. Rev.Lett. **101**,157006 (2008)

Conclusion for uniformly disordered metal films: SC-M transition is not uniform

Close to the quantum critical point, at $g - g_c \leq 1$,

the superconducting phase transition

is due to overlapping superconducting droplets

created by mesoscopic fluctuations

Quantitative theory of this transition is not developed yet

Quantum S-M transition: SC islands on the top of a poor metal film

M. V. Feigel'man and A. I. Larkin
Chem. Phys. Lett. 235, 107 (1998)
B. Spivak, A. Zyuzin, and M. Hruska
Phys. Rev. B **64**, 132502 (2001)
M. V. Feigel'man, A. I. Larkin and M.A.Skvortsov
Phys. Rev. Lett. **86**, 1869 (2001)

III. Superconductivity v/s Localization

1) Motivation from the experiments
2) BCS-like theory for critical eigenstates
3) Superconductivity with pseudo-gap
4) Quantum phase transition: Bethe lattice model

M.Feigelman, L.Ioffe, V.Kravtsov, E.Yuzbashyan,
Phys Rev Lett.**98**, 027001, 2007

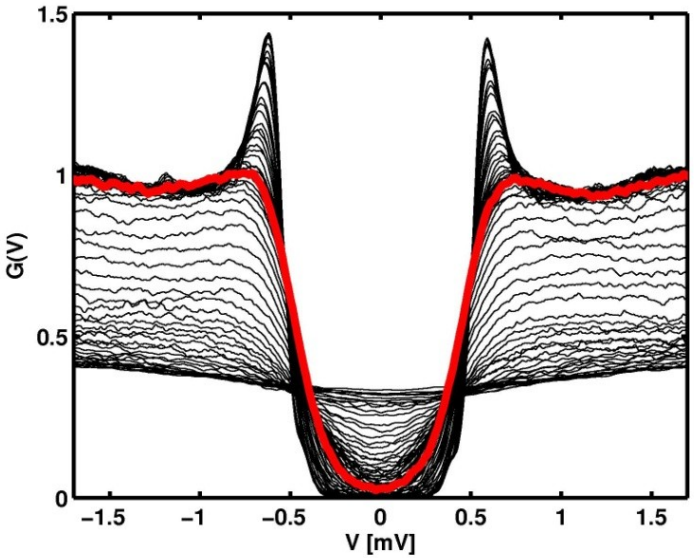
M.Feigelman, L.Ioffe, V.Kravtsov, E.Cuevas, Ann.Phys. **325**, 1390 (2010)

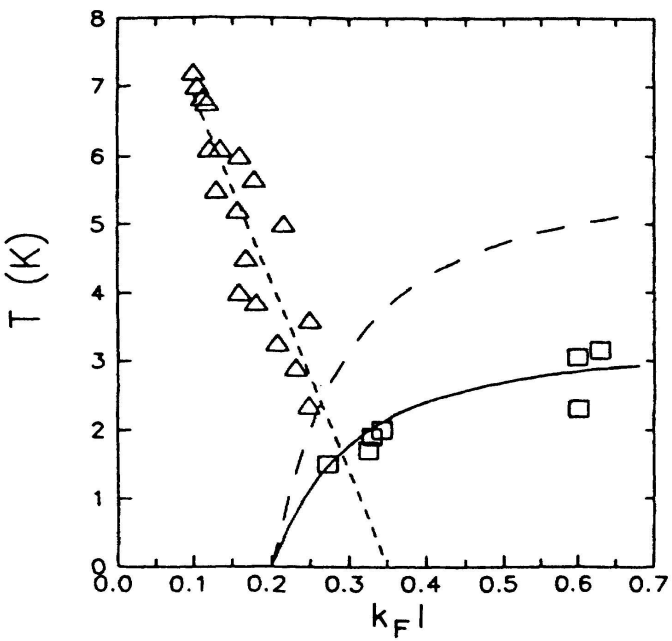
L.Ioffe and M. Mezard Phys.Rev.Lett. **105**, 037001 (2010)

M.Feigelman, L.Ioffe and M.Mezard, Phys. Rev.B **82**, 184534 (2010)

B. Sacépé, T. Dubouchet, C. Chapelier, M. Sanquer, M. Ovadia, D. Shahar, M. Feigel'man, and L. Ioffe, Nature Physics **7**, 239 (2011)

SUPERCONDUCTOR-INSULATOR: EXPERIMENTAL EVIDENCE





Direct evidence for the gap above the transition (Chapelier, Sacepe 2007). Activation behavior does not show gap suppression at the critical point as a function of the disorder (Shahar & Ovaduyahu, 1992)!

Class of relevant materials

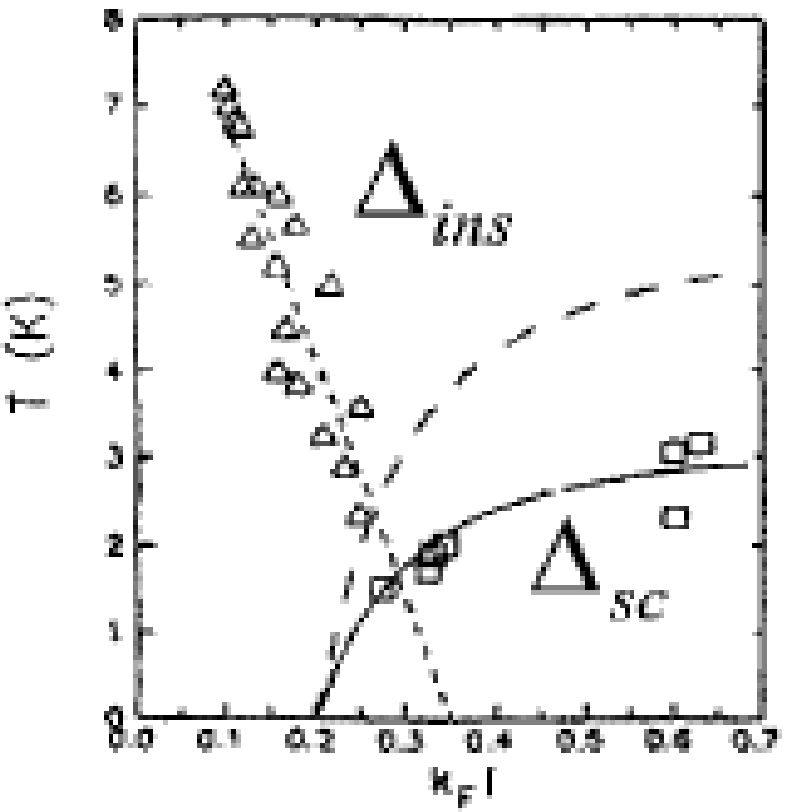
- Amorphously disordered
  (no structural grains)
- Low carrier density
  (around 10^{21} cm^{-3} at low temp.)

Examples:

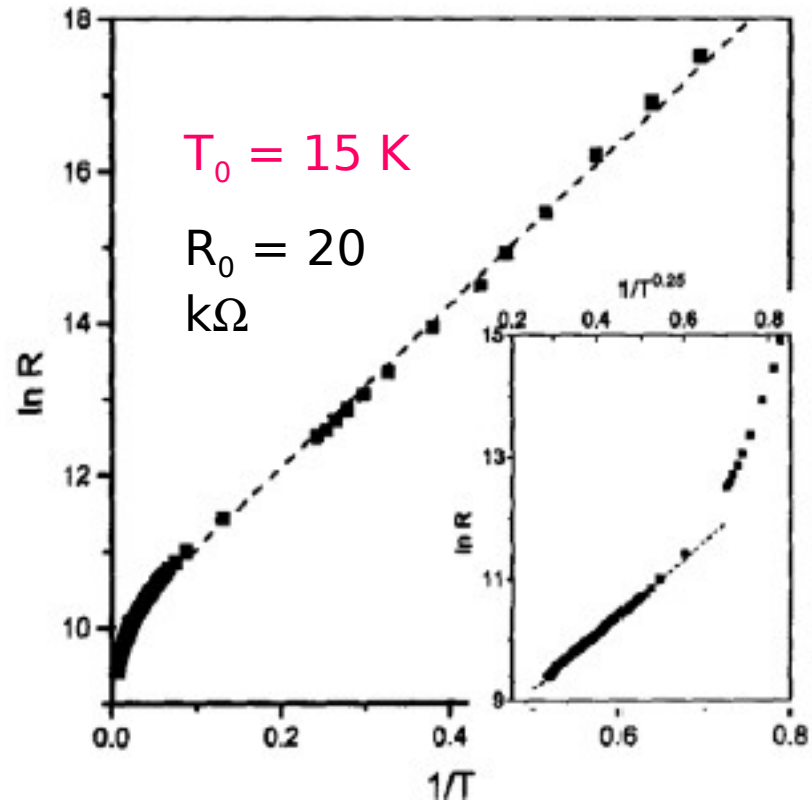
InO_x NbN_x thick films or bulk (+ B-doped Diamond?)

TiN thin films Be (ultra thin films)

D.Shahar &
Z.Ovadyahu
amorphous InO 1992



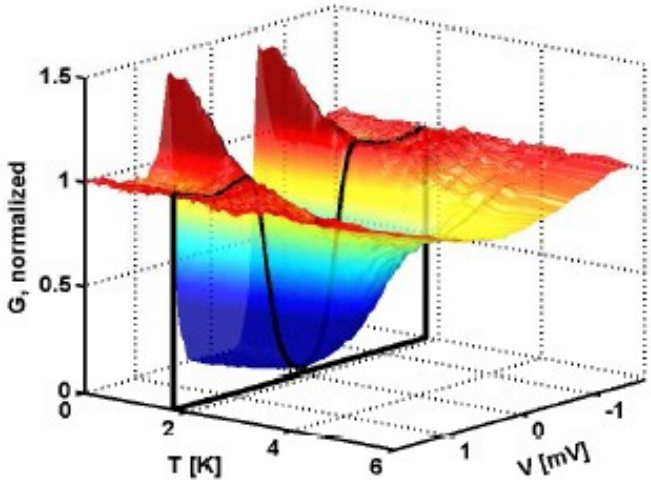
On insulating side (far enough):
Kowal-Ovadyahu 1994

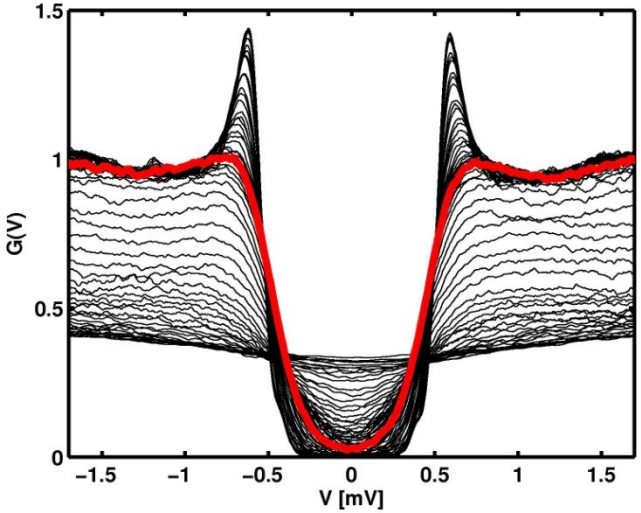


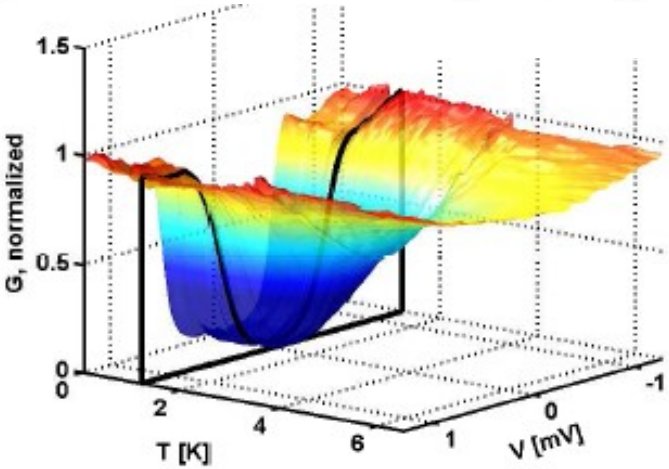
SC side: local tunneling conductance

Spectral signature of localized Cooper pairs in disordered superconductors.

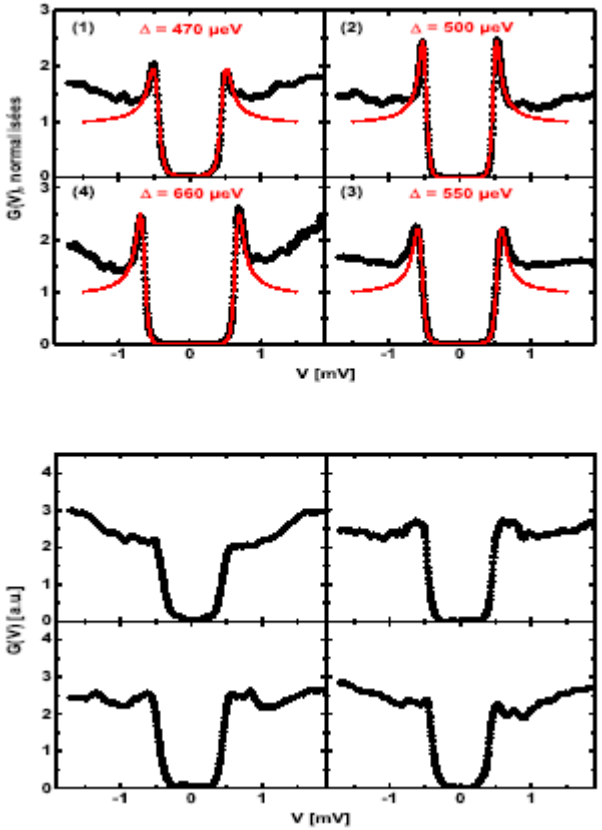
Benjamin Sacépé,[1,*] Thomas Dubouchet,[1] Claude Chapelier,[1] Marc Sanquer,[1] Maoz Ovadia,[2] Dan Shahar,[2] Mikhail Feigel'man,[3] and Lev Ioffe[4,3]



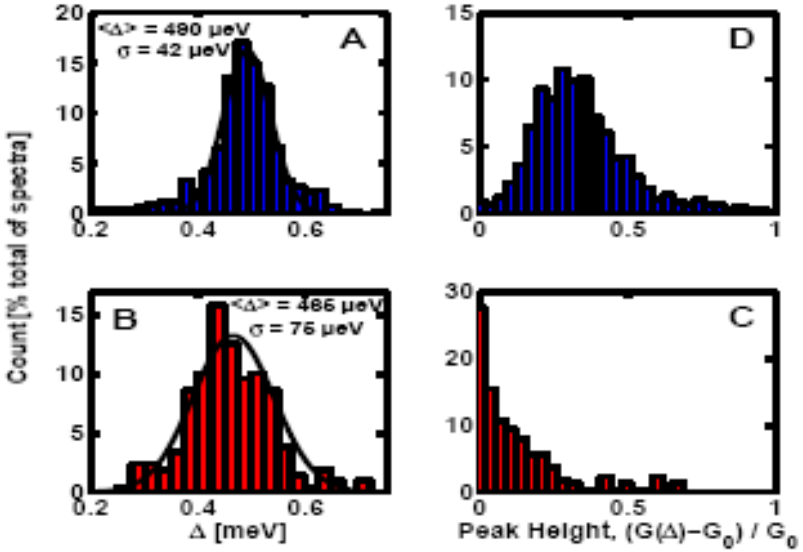




Local tunneling conductance-2



Gap widths Peak heights



Less disorder

More disorder

Superconductive state near SIT is very unusual:

- the spectral gap appears much before (with T decrease) than superconductive coherence does

- Coherence peaks in the DoS appear together with resistance vanishing

- Amplitude of coherence peaks strongly fluctuates in space

Bosonic v/s Fermionic scenario ?

None of them is able to describe data on InO_x and TiN

3-d scenario: competition between Cooper pairing and localization

(without any role of Coulomb interaction)

Theoretical model

Simplest BCS attraction model,
but for critical (or weakly localized) electron eigenstates

$$H = H_0 - g \int d^3r \, \Psi_\uparrow^\dagger \Psi_\downarrow^\dagger \Psi_\downarrow \Psi_\uparrow$$

$\Psi = \Sigma \, c_j \, \Psi_j \, (r)$ Basis of localized eigenfunctions

M. Ma and P. Lee (1985) : S-I transition at $\delta_L \approx T_c$

We will see that in fact SC state survives far into the region $\delta_L >> Tc$

Superconductivity at the Localization Threshold: $\delta_L \to 0$

Consider Fermi energy very close to the mobility edge:

single-electron states are extended

but **fractal** and populate small fraction of the whole volume

How BCS theory should be modified to account for eigenstate's fractality ?

Method: combination of analitic theory and numerical data for Anderson mobility edge model

Mean-Field Eq. for T_c

$$\Delta(r) = \int K_T(r, r')\Delta(r')d^d r' \tag{9}$$

where kernel $\hat{K}_T$ is equal to

$$K_T(r, r') = \frac{\lambda}{2\nu_0} \sum_{ij} \frac{\tanh\frac{\xi_i}{2T} + \tanh\frac{\xi_j}{2T}}{\xi_i + \xi_j} \psi_i(r)\psi_j(r)\psi_i(r')\psi_j(r') \tag{10}$$

Standard averaging over space $\Delta(r) \to \overline{\Delta}$ leads to "Anderson theorem" result: totally incorrect in the present situation.

The reason: critical eigenstates $\psi_j(r)$ are strongly correlated in real 3D space, they fill some small **submanifold** of the whole space only.

In fact one should define T_c as the divergence temperature of the Cooper ladder

$$\mathcal{C} = \left(1 - \hat{K}\right)^{-1}$$

Thus averaging procedure should be applied to $\mathcal{C}$ instead of K

We expand $\mathcal{C}$ in powers of K and average over disorder realizations. Keeping main sequence of resulting diagramms only, we come to the following equation for determination of T_c:

$$\Phi(\xi) = \frac{\lambda}{2} \int \frac{d\xi' \tanh(\xi'/2T)}{\xi'} M(\xi - \xi')\Phi(\xi') \tag{11}$$

$$M(\omega) = \nu \overline{M_{ij}} = \int \overline{\psi_i^2(r)\psi_j^2(r)} d^d r \quad \text{for} \quad |\xi_i - \xi_j| = \omega$$

Fractality of wavefunctions

For critical eigenstates

$$L_{\text{loc}} \to \infty$$

one finds

$$M(\omega) = \left(\frac{E_0}{\omega}\right)^{\gamma}$$

where

$$\gamma = 1 - \frac{D_2}{d}$$

is a measure of fractality

Usual "dirty superconductor":

$$M(\omega) = 1 \qquad \gamma = 0$$

IPR: $M_i = \int |\psi_i(\mathbf{r})|^4 d\mathbf{r}$

$$\langle M_i \rangle \approx 3\ell^{-(d-d_2)} L^{-d_2}.$$

$$E_0 = 1/\nu_0 \ell^3$$

$D_2 \approx 1.3$ in 3D

l is the short-scale cut-off length

3D Anderson model: $\gamma = 0.57$

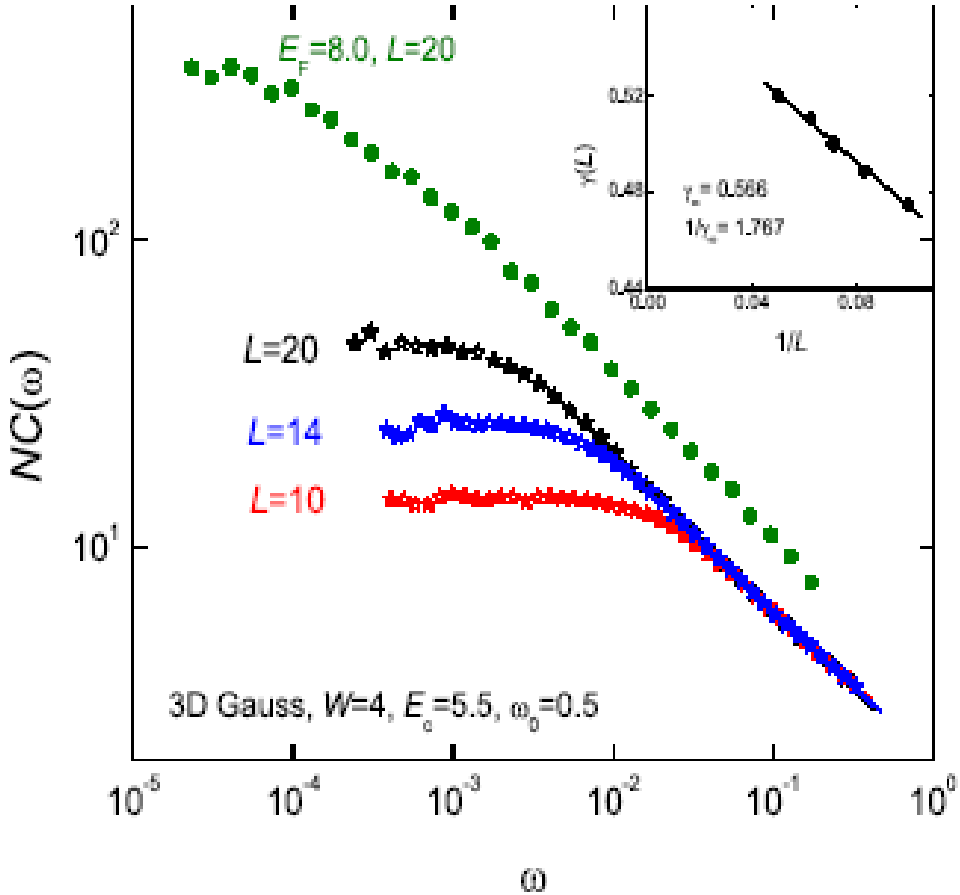


FIG. 2: (Color online) Correlation function $M(\omega)$ for 3DAM with Guassian disorder and lattice sizes $L = 10, 14, 20$ at the mobility edge $E = 5.5$ (red, blue and black points) and at the energy $E = 8$ inside localized band (green points). Inset shows γ values for $L = 10.12.14.16.20$.

Modified mean-field approximation for critical temperature T_c

$$\Delta(\xi) = \frac{\lambda}{2}\int d\zeta \eta(\zeta) M(\xi-\zeta)\Delta(\zeta)$$

$$\eta_i \equiv \eta_{ii} = \xi_i^{-1}\tanh(\xi_i/2T).$$

$$T_c^0(\lambda,\gamma) = E_0\lambda^{1/\gamma}C(\gamma)$$

For small λ this T_c is higher than BCS value !

More details in the lecture by Vladimir Kravtsov, 20.06.15

Similar result via RG in 2D: I. Burmistrov, I.Gornyi & A.Mirlin, 2011,2015

Critical temperature T_c is well-defined through the whole system in spite of strong $\Delta(r)$ fluctuations

Superconductivity with Pseudogap

Now we move Fermi- level into the range of localized eigenstates

Local pairing develops
in addition to collective pairing

Local pairing energy

Parity gap in ultrasmall grains

K. Matveev and A. Larkin 1997

------- E_F
--↑↓--
-- ↓--

$\Delta \ll \delta$: **No many-body correlations**

$$\Delta_P = \frac{1}{2}\lambda\delta$$

$$\lambda_R = \lambda/(1 - \lambda \log(\epsilon_0/\delta)).$$

$$\Delta_P = \frac{\delta}{2\ln\frac{\delta}{\Delta}}$$

Correlations between pairs of electrons localized in the same “orbital”

Parity gap for Anderson-localized eigenstates

The increase of thermodynamic potential Ω due to addition of *odd* electron to the ground-state is

$$\delta\Omega_{\rm oe} = \xi_{m+1} = \xi_{m+1} - \tilde{\xi}_{m+1} + \tilde{\xi}_{m+1} = \frac{g}{2} M_{m+1} + O(\mathcal{V}^{-1})$$

$$\tilde{\xi}_j = \xi_j - \frac{g}{2} M_j$$

Energy of two single-particle excitations after depairing:

$$2\Delta_P = \xi_{m+1} - \xi_m + gM_m = \frac{g}{2}(M_m + M_{m+1}) + O(\mathcal{V}^{-1})$$

$$\langle M_i \rangle = 3\ell^{-(d-d_2)} L_{\rm loc}^{-d_2}, \qquad \Delta_P = \frac{3}{2} g \ell^{-3} (L_{loc}/\ell)^{-d_2} = \frac{3\lambda}{2} E_0 \left(\frac{E_c - E_F}{E_0} \right)^{\nu d_2}$$

Δ_P plays the role of the activation gap

Critical temperature in the pseudogap regime

MFA:

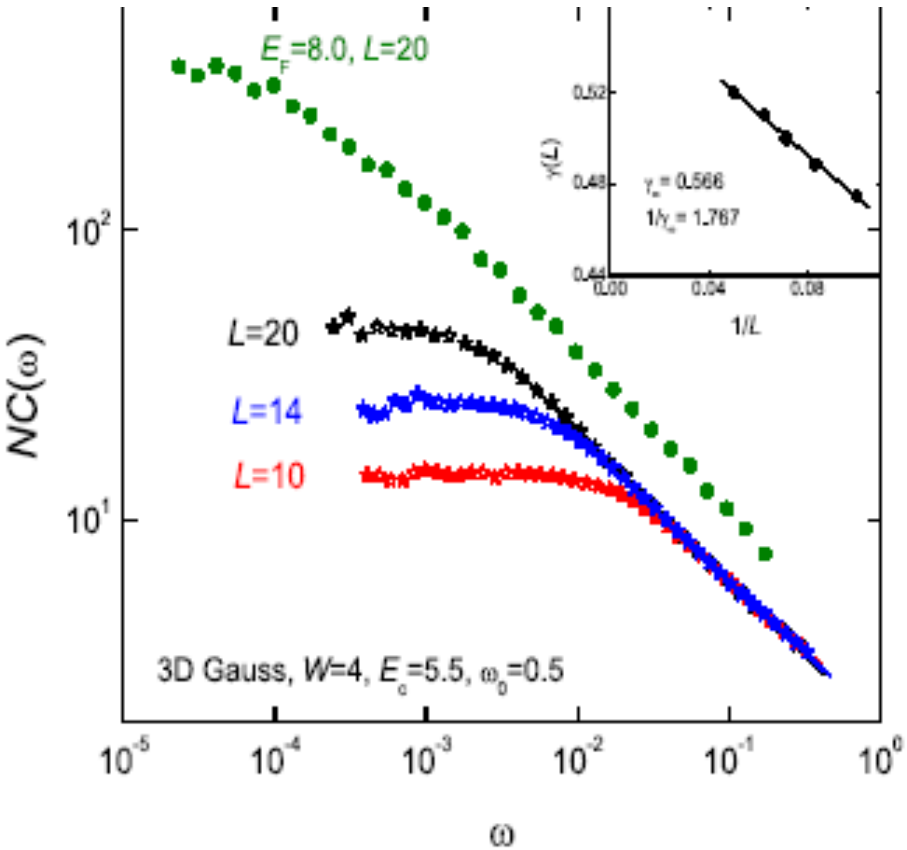
$$\Delta(\xi) = \frac{\lambda}{2} \int d\zeta \eta(\zeta) M(\xi - \zeta) \Delta(\zeta)$$

$$\eta_i \equiv \eta_{ii} = \xi_i^{-1} \tanh(\xi_i / 2T).$$

Here we use M(ω) specific for localized states

MFA is OK as long as $Z \sim \nu_0 T_c L_{loc}^d$ **is large**

Correlation function M(ω)



No saturation at $\omega < \delta_L$:

$M(\omega) \sim \ln^2(\delta_L / \omega)$

(Cuevas & Kravtsov PRB,2007)

Superconductivity with $T_c < \delta_L$ is possible

This region was not found previously

Here "local gap" exceeds **SC gap :**

$$\Delta_P = \frac{1}{2D^\gamma(\gamma)}\delta_L\left(\frac{\Delta(0)}{\delta_L}\right)^\gamma$$

FIG. 2: (Color online) Correlation function $M(\omega)$ for 3DAM with Guassian disorder and lattice sizes $L = 10, 14, 20$ at the mobility edge $E = 5.5$ (red, blue and black points) and at the energy $E = 8$ inside localized band (green points). Inset shows γ values for $L = 10.12.14.16.20$.

Critical temperature in the pseudogap regime

MFA:

$$\Delta(\xi) = \frac{\lambda}{2}\int d\zeta \eta(\zeta) M(\xi-\zeta)\Delta(\zeta)$$

$$\eta_i \equiv \eta_{ii} = \xi_i^{-1}\tanh(\xi_i/2T).$$

We need to estimate

$$Z_{\text{eff}} \sim \nu_0 T_c (R_\omega)^3$$

$$R_\omega \approx 2L_{\text{loc}} \ln\frac{\delta_L}{\omega} \gg L_{\text{loc}}$$

$$R_0^2 = \frac{\sum_{ij} r_{ij}^2 M_{ij}}{\sum_{ij} M_{ij}}$$

$$Z_{\text{eff}} \equiv \nu_0 T_c R_0^3 = 8\frac{T_c}{\delta_L}\ln^3\frac{\delta_L}{T_c}$$

It is nearly constant in a very broad range of $\frac{\delta_L}{T_c}$

T_c versus Pseudogap

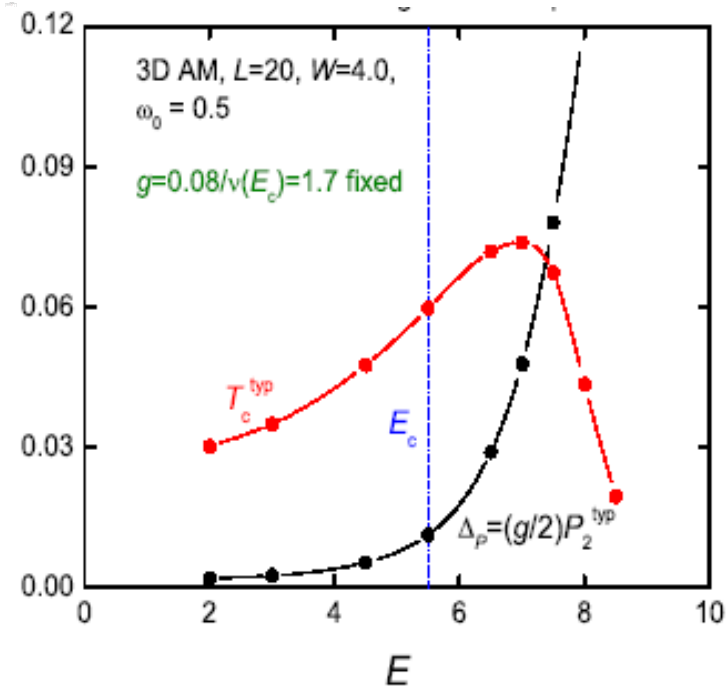


FIG. 25: (Color online) Virial expansion results for T_c (red points) and typical pseudogap Δ_P (black) as functions of E_F. The model with fixed value of the attraction coupling constant $g = 1.7$ was used; pairing susceptibilities were calculated using equations derived in Appendix B.

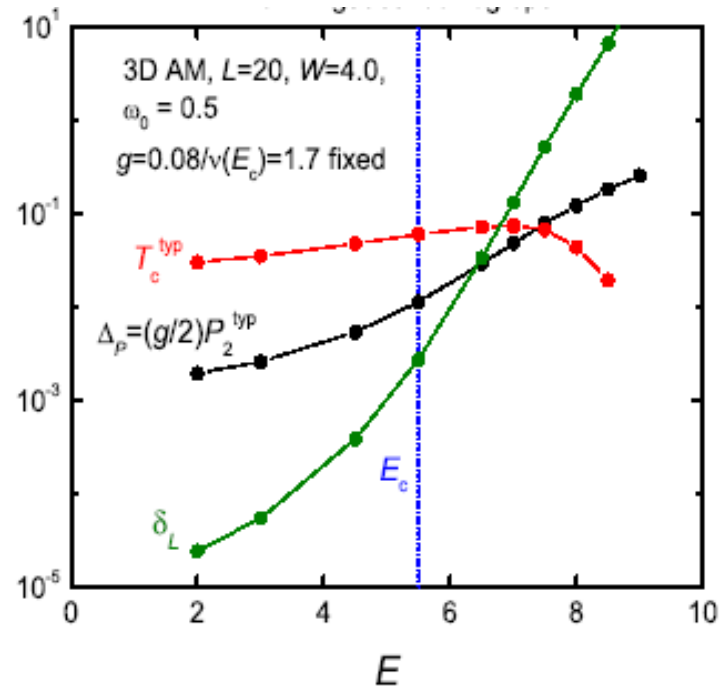


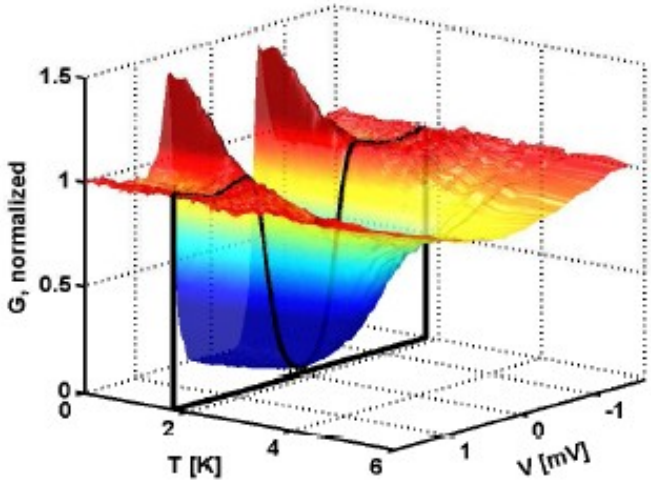
FIG. 26: (Color online) Virial results for T_c (red points), typical pseudogap Δ_P (black) and the corresponding level spacing δ_L (green), as functions of E_F on semi-logarithmic scale.

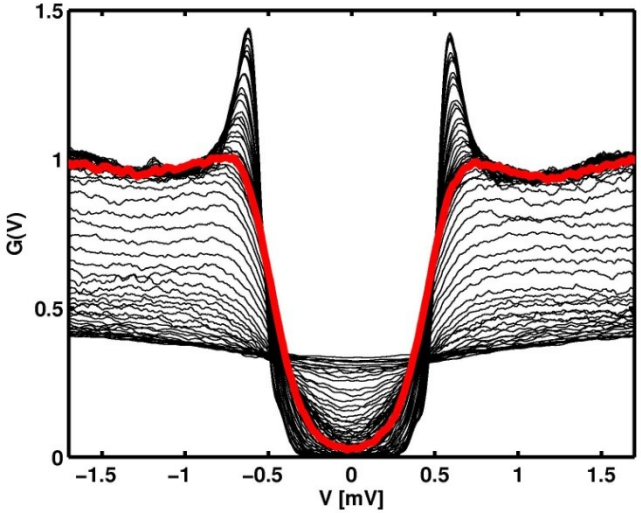
Transition exists even at $\delta_L >> T_{c0}$

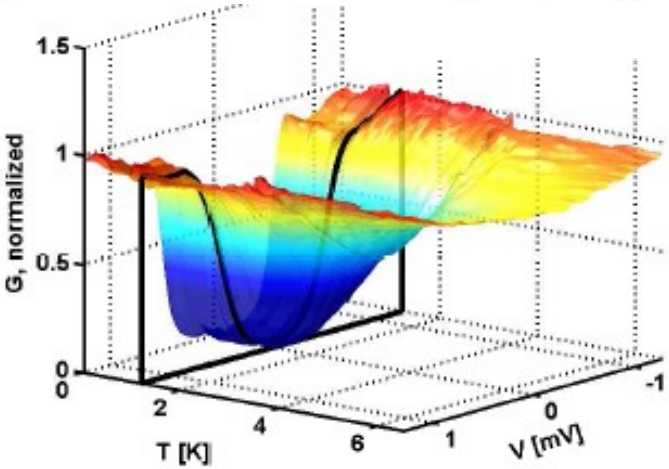
SC side: local tunneling conductance

Spectral signature of localized Cooper pairs in disordered superconductors.

Benjamin Sacépé,[1,*] Thomas Dubouchet,[1] Claude Chapelier,[1] Marc Sanquer,[1] Maoz Ovadia,[2] Dan Shahar,[2] Mikhail Feigel'man,[3] and Lev Ioffe[4,3]

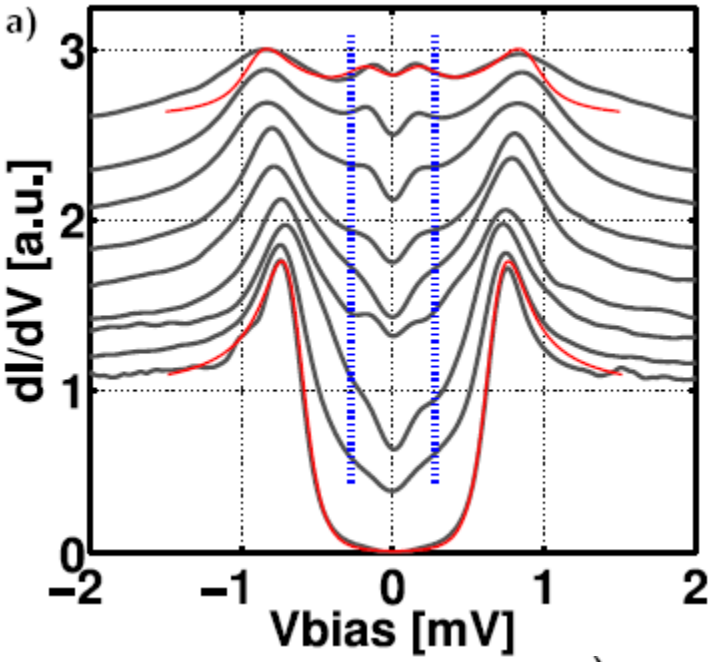






Andreev point-contact spectroscopy

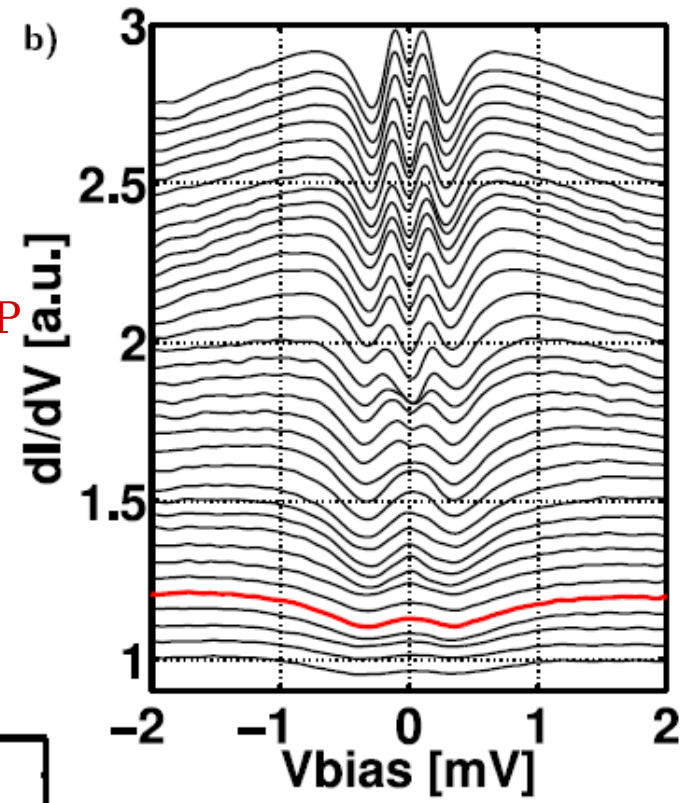
T. Dubouchet,[1,*] C. Chapelier,[1] M. Sanquer,[1] B. Sacépé,[2,3] Maoz Ovadia,[3] and Dan Shahar[3]

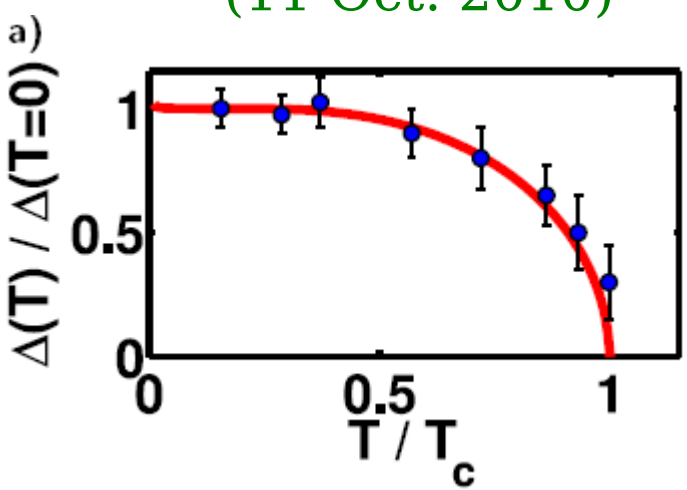


$$2eV_1 = 2\Delta$$

$$eV_2 = \Delta + \Delta_P$$

T.Dubouchet, thesis, Grenoble (11 Oct. 2010)





Low-energy effective Hamiltonian for pseudogaped SC state

$\Delta_P >> T_c$

Single-electron states suppressed by pseudogap

„Pseudo spin" representation:

$$S^+_\mu = a^+_{\mu\uparrow} a^+_{\mu\downarrow} \qquad S^-_\mu = a_{\mu\uparrow} a_{\mu\downarrow}$$

$$2S^z_\mu = a^+_{\mu\uparrow} a_{\mu\uparrow} + a^+_{\mu\downarrow} a_{\mu\downarrow}$$

H_{BCS} acts on Even sector: all states which are 2-filled or empty

$$\hat{H} = \sum_\mu 2\xi_\mu S^z_\mu - g \sum_{\mu,\nu} M_{\mu\nu} S^+_\mu S^-_\nu + \sum_{B_\mu} \left(\xi_\mu + \frac{G_\mu}{2}\right)$$

B: „blocked" states

$$\bar{M}_{\mu\nu} = \frac{1}{\nu V} M(\xi_\mu - \xi_\nu)$$

D.S ↑ ← total volume

"Pseudospin" approximation

$$Z \sim \nu_0 T_c L^d_{loc}$$

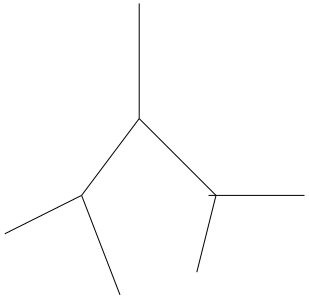
Effective number of interacting neighbours

S-I-T: Third Scenario

- Bosonic mechanism: preformed Cooper pairs + competition Josephson v/s Coulomb – S I T in arrays
- Fermionic mechanism: suppressed Cooper attraction, no paring – S M T

- Pseudospin mechanism: individually localized pairs

- S I T in amorphous media



SIT occurs at small Z and lead to paired insulator

$$H = 2\sum_i \xi_i s_i^z - \sum_{ij} M_{ij}(s_i^x s_j^x + s_i^y s_j^y)$$

How to describe this quantum phase transition ?

Bethe lattice model is solved (*M.F.,L.Ioffe & M.Mezard*)

MODEL SOLUTION 1: CAVITY EQUATIONS.

Main idea: cavity equations.
Introduce effective field that simulates the effect of spins at higher levels:

$$H = -\xi_0 \sigma_0^z - h_0 \sigma_0^x \qquad \left\langle \sigma_0^x \right\rangle_0 = \frac{h_0}{\sqrt{h_0^{\,2} + \xi_0^{\,2}}} \operatorname{Tanh}\left[\frac{\sqrt{h_0^{\,2} + \xi_0^{\,2}}}{T} \right]$$

$$H = -\xi_0 \sigma_0^z - \sum_j (\xi_j \sigma_j^z + \sigma_0^x \sigma_j^x + h_j \sigma_j^x) \qquad \text{Choose } h_0 \text{ so that } \left\langle \sigma_0^x \right\rangle_H = \left\langle \sigma_0^x \right\rangle_0$$

Roughly - this approximation is sufficient to get the transition temperature to O(1/K):

$$h_{k+1} = \frac{g}{K} \sum_j \frac{h_{k,j}}{\sqrt{\xi_{k,j}^{\,2} + h_{k,j}^{\,2}}} \operatorname{Tanh} \frac{\sqrt{\xi_{k,j}^{\,2} + h_{k,j}^{\,2}}}{T}$$

If averaged over uniform distribution of ξ we get usual BCS-like equation:

$$h = g \int_0^\infty \frac{d\xi h}{\sqrt{\xi^2 + h^2}} \operatorname{Tanh}\left[\frac{\sqrt{\xi^2 + h^2}}{T} \right]$$

that tells us that $T_c > 0$ for any $g > 0$.

MFA: $T_c = \exp(-1/g)$ Is it correct ?

MODEL SOLUTION 2: EQUATION FOR T_c.

To find T_c we need to find when infinitely small field applied at the boundary leads to large field in the center:

$$h_0 = Zh_N \qquad Z = \sum_{\{i[k]\}} \prod_k \frac{g}{K} \frac{\mathrm{Tanh}[\xi_{k,i[k]} / T]}{\xi_{k,i[k]}}$$

That is whether $Z=exp(fN)$ with $f>0$ ("magnet" or "superconductor") or $f<0$ (paramagnet)?

Non-trivial physics is due to the fact that Z is not necessarily self-averaging quantity! Consider higher moments:

$$K\left\langle \left[\frac{g}{K} \frac{\mathrm{Tanh}[\xi_{k,i} / T]}{\xi_{k,i}} \right]^n \right\rangle = \sqrt{\frac{3\pi}{4K}} K^{1-n} g^n / T^{n-1}$$

The moments diverge at $T=g/K$ which becomes higher than 'average' $T_c=exp(-1/g)$.

Generalized solution for the order parameter

Linear recursion ($T=T_c$)

$$B_i = (g/K)\sum_k (B_k/\xi_k)\tanh(\beta\xi_k) \quad ,(1)$$

One should look for the stationary distribution function for random order parameter, ***P(B)***

If **<B>** is finite, one could average recursion (1) and obtain MFA result

Actually 1st moment diverges: $P(B) = \dfrac{B_0^m}{B^{1+m}}$ **m < 1**

Laplace transform satisfies the equation:

$$\mathcal{P}(s) = \left[\int_0^1 d\xi\, \mathcal{P}\left(s\frac{g}{K}\frac{\tanh\beta\xi}{\xi}\right)\right]^K$$

We search the solution in the form: $\mathcal{P}(s) = 1 - As^x$ with $x < 1$

Distribution function for the order parameter

Laplace transform satisfies the equation:

$$\mathcal{P}(s) = \left[\int_0^1 d\xi\, \mathcal{P}\left(s\frac{g}{K}\frac{\tanh\beta\xi}{\xi}\right)\right]^K$$

Solution: $\mathcal{P}(s) = 1 - As^x$ with $x < 1$

Both *A* and *x* are variational parameters

$$P(B) = \frac{B_0^m}{B^{1+m}}$$

$$1 = K\int \frac{d\xi}{\xi}\left(\frac{g}{K}\frac{\tanh(\beta\xi)}{\xi}\right)^x$$

$$\int_{-1}^1 \frac{d\xi}{2}\frac{\tanh^x\beta\xi}{\xi^x}\ln\left(\frac{g}{\xi K}\tanh\beta\xi\right) = 0$$

T=0

$$g_c e^{1/(eg_c)} = K$$

$$m = 1 - eg_c$$

Vicinity of the Quantum Critical Point

$$T_c(K) = \vartheta(y_c)\left(\frac{K}{K_c} - 1\right)^{1/y_c}$$

$$y_c = eg. \quad << 1$$

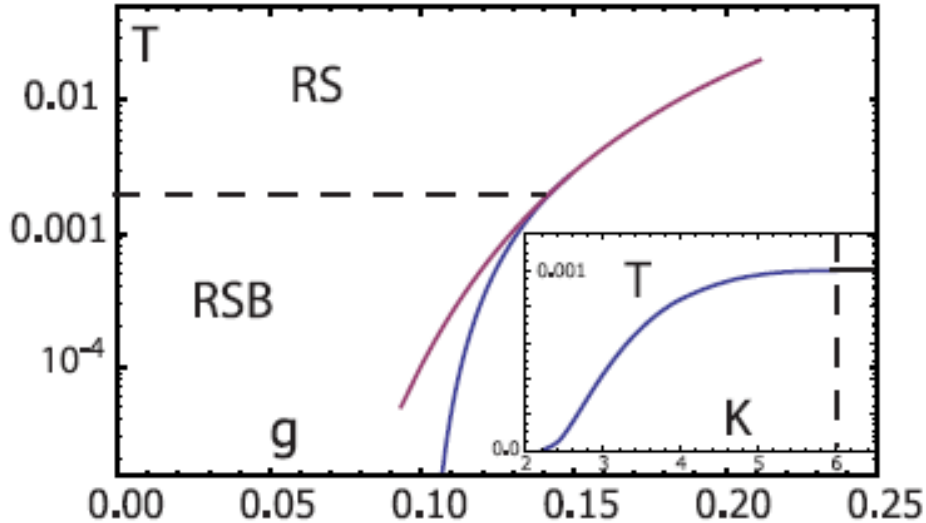


FIG. 2: Main panel: phase diagram in plane (g, T) for $K = 4$. Full lines show the critical temperature as function of g. The low temperature phase is superconducting, the high temperature phase is insulating. The top curve is the naive mean-field prediction which gives the correct result above $T_{RSB} = 0.0207$. The bottom curve is the result of the correct analysis on the Bethe lattice, including the RSB effects in the DP problem, which occur at temperatures $T < T_{RSB}$. The insert shows the phase diagram as function of K for $g = 0.129$. For this value of g the replica symmetric solution gives K-independent transition temperature $T_c = 0.001$; this value roughly correspond to the experimental situation in disordered InO films (see section VI). The prediction of replica symmetric theory is correct for $K > K^{RSB} \simeq 6$. For smaller K the transition temperature starts to drop, the quantum critical point corresponds to $K_c \simeq 2.2$. Notice that in a numerically wide regime the replica symmetry is broken but the effect on transition temperature is small.

Order parameter: scaling near transition

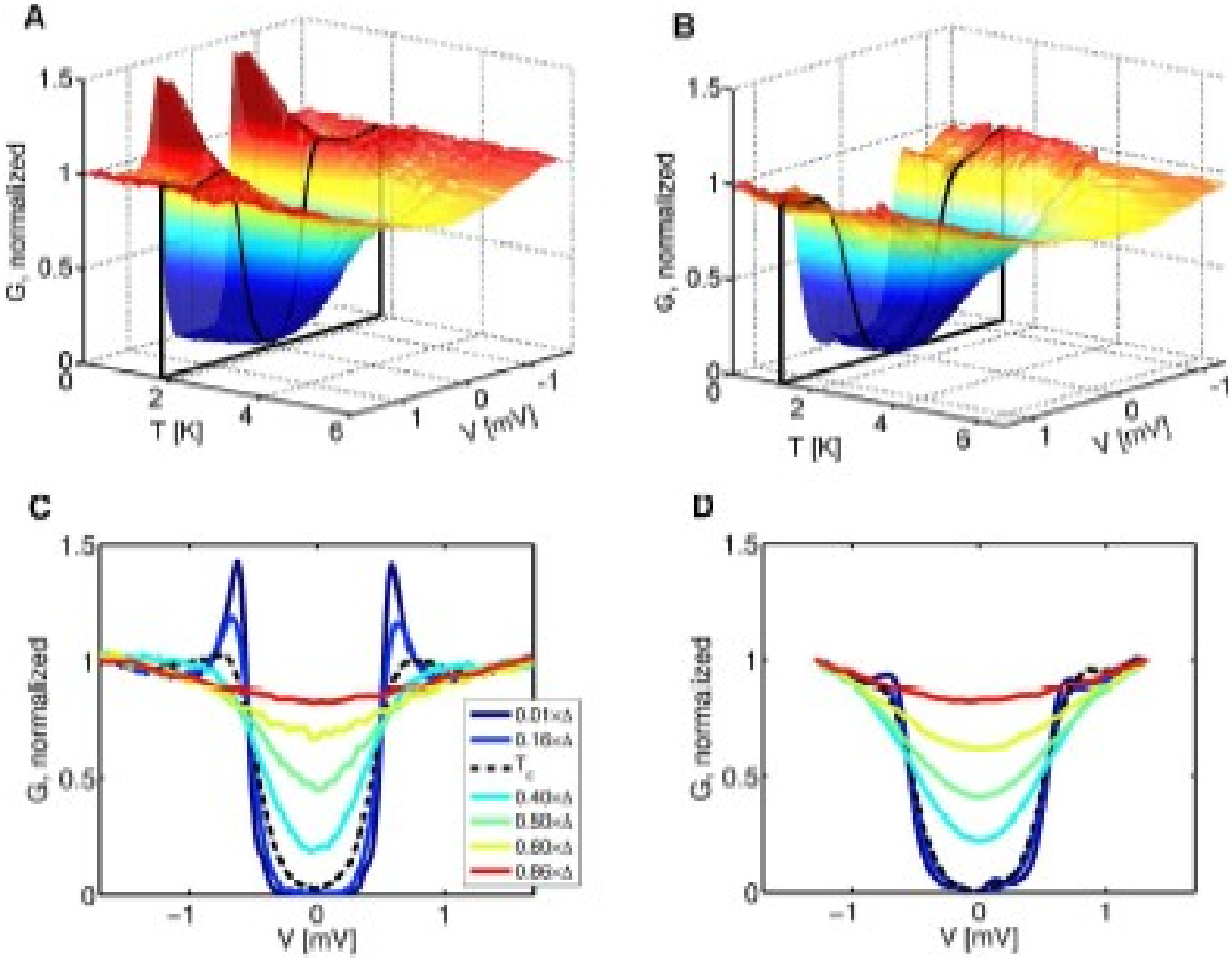
$$B_j = \frac{g}{K}\sum_{k=1}^{K}\frac{B_k}{\sqrt{B_k^2+\xi_k^2}}\tanh\beta\sqrt{B_k^2+\xi_k^2}\,.$$

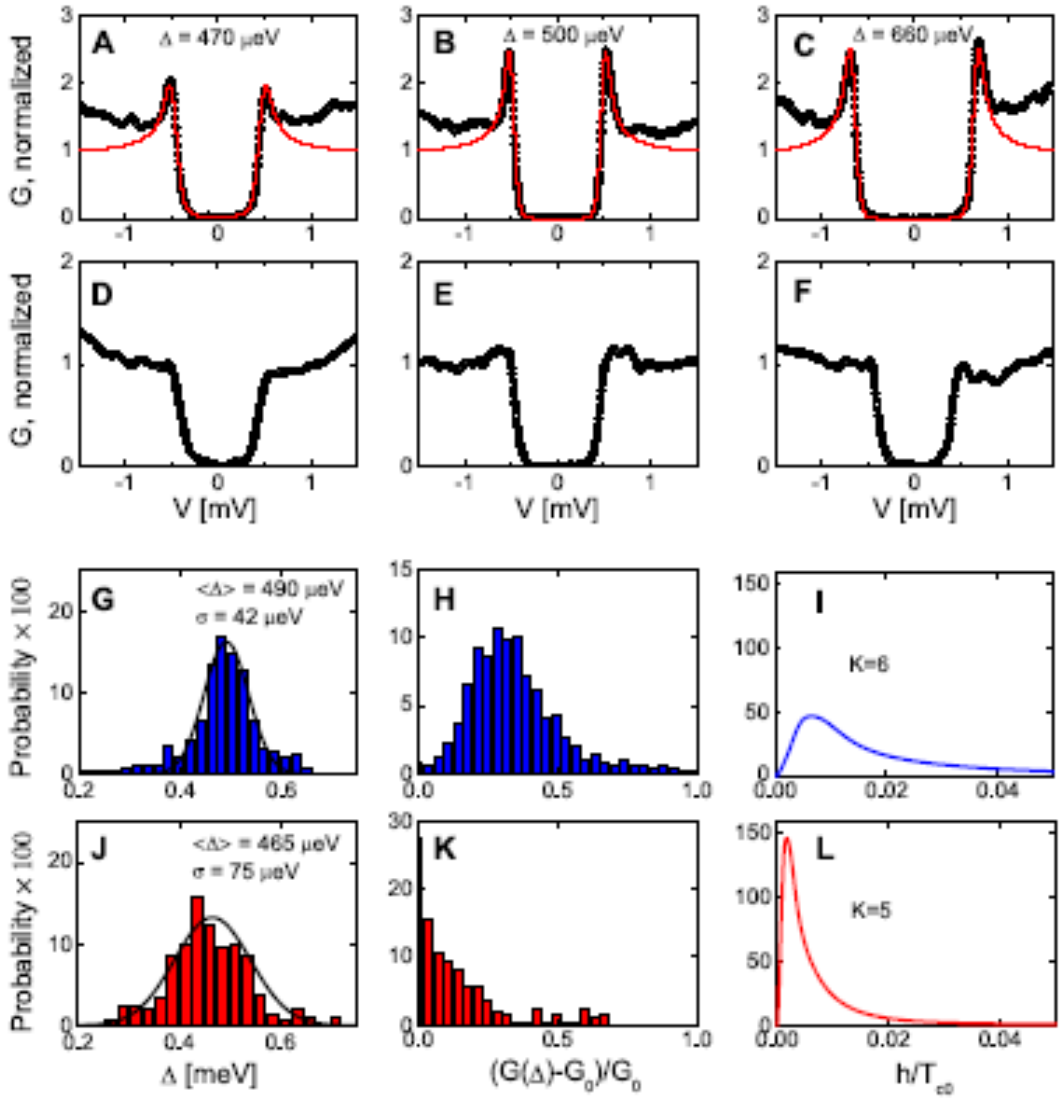
$$P(B) = \frac{B_0^m}{B^{1+m}}$$

$$B_i^m \simeq \sum_{k=1}^{K}\left(\frac{g}{K}\frac{B_k}{\sqrt{B_k^2+\xi_k^2}}\tanh\beta\sqrt{B_k^2+\xi_k^2}\right)^m$$

Typical value near the critical point:

$$B_0 \simeq e^{-1/(eg_c)}\exp\left[-\frac{C}{(g/g_c)^m-1}\right]$$


A
B
C
D
G, normalized
T [K]
V [mV]
0.01×Δ
0.16×Δ
T_c
0.40×Δ
0.53×Δ
0.80×Δ
0.86×Δ


A
Δ = 470 μeV
B
Δ = 500 μeV
C
Δ = 660 μeV
G, normalized
D
E
F
V [mV]
G
<Δ> = 490 μeV
σ = 42 μeV
H
I
K=6
Probability × 100
J
<Δ> = 465 μeV
σ = 75 μeV
K
L
K=5
Δ [meV]
(G(Δ)-G0)/G0
h/Tc0

Conclusion for part III

Pairing on nearly-critical states produces fractal superconductivity with relatively high T_c but very small superconductive density

Pairing of electrons on localized states leads to hard gap and Arrhenius resistivity for 1e transport

Pseudogap behaviour is generic near S-I transition, with "insulating gap" above T_c

Superconductivity is extremely inhomogeneous near SIT, for 2 different reasons:

i) fractality, ii) lack of self-averaging

The End

Alternative method to find Tc:

Virial expansion

(A.Larkin & D.Khmelnitsky 1970)

$$F = \sum_{n=1}^{\infty} \mathcal{F}^{(n)} = \sum_i F_i + \sum_{i>j} (F_{ij} - F_i - F_j)$$

$$+ \sum_{i>j>k} (F_{ijk} - F_{ij} - F_{jk} - F_{ik} + F_i + F_j + F_k) + \ldots$$

$$V_\Delta = -\sum_j (\Delta S_j^+ + \Delta^* S_j^-)$$

$$\chi(T) = -\frac{\partial^2 F}{\partial\Delta\partial\Delta^*} = \sum_{M=1} \chi_M(T)$$

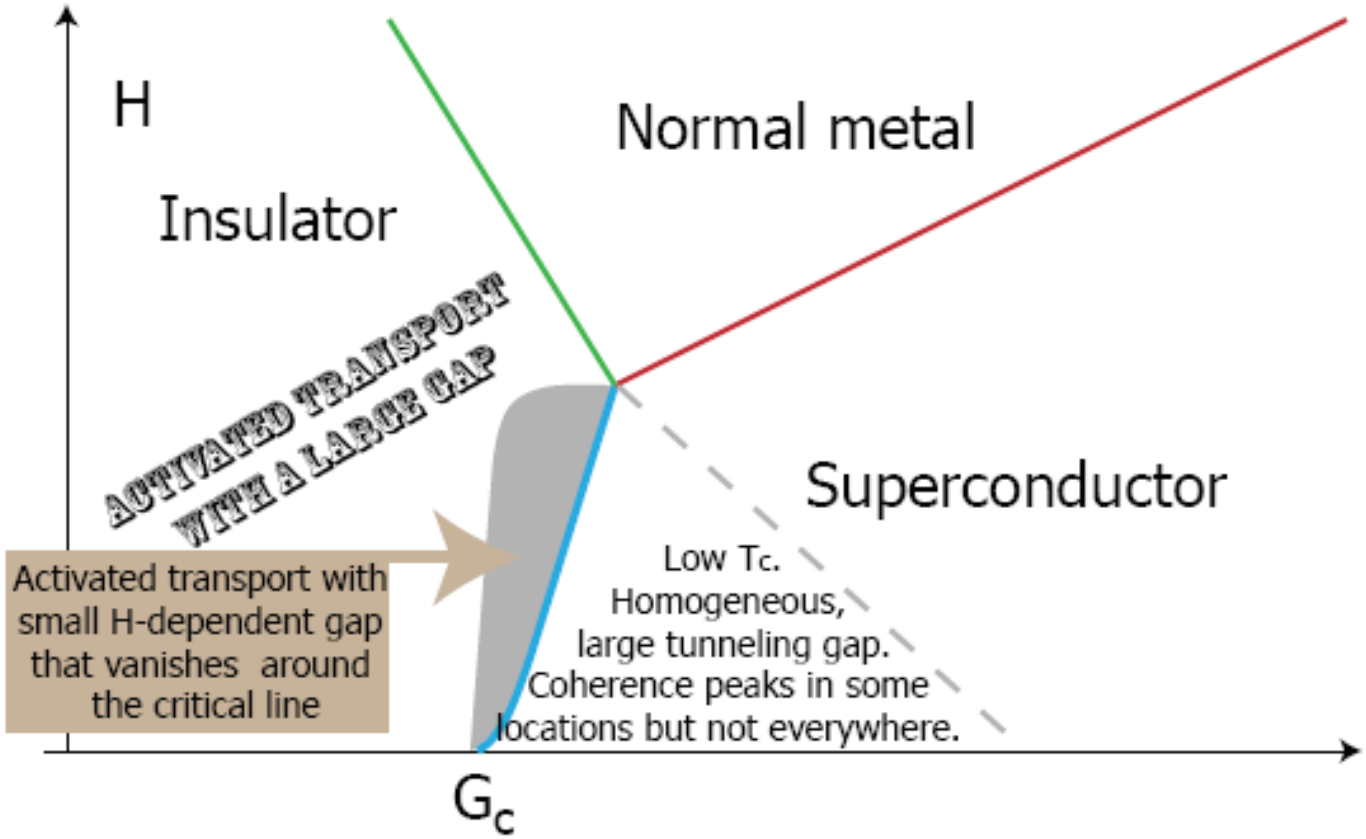
$$\chi_1 = \sum_i \chi_i^{(1)} \qquad (148)$$

$$\chi_2 = \sum_{n>m} (\chi_{nm}^{(2)} - \chi_n^{(1)} - \chi_m^{(1)})$$

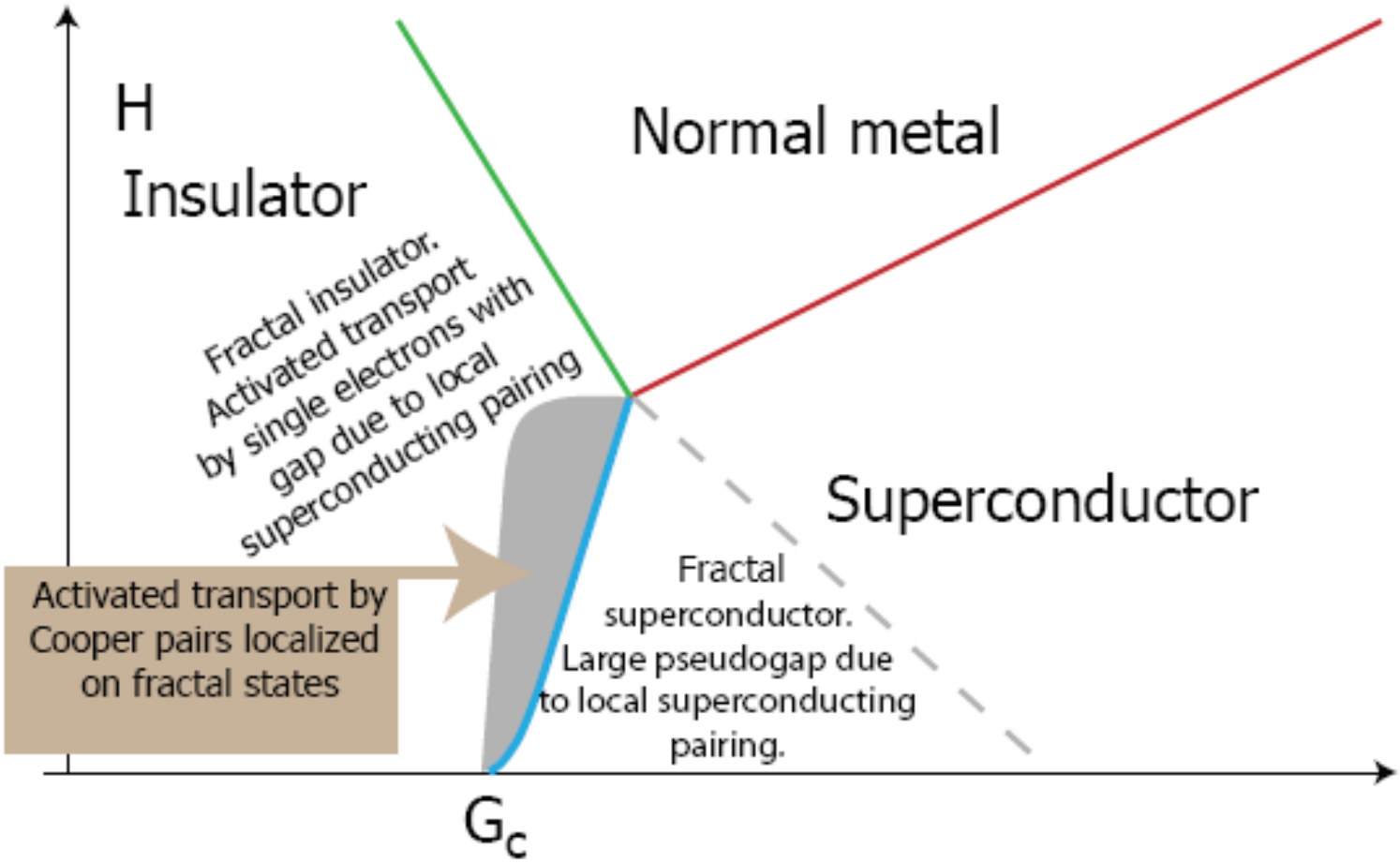
$$\chi_3 = \sum_{n>m>l} (\chi_{nml}^{(3)} - \chi_{nl}^{(2)} - \chi_{ml}^{(2)} - \chi_{nm}^{(2)} + \chi_n^{(1)} + \chi_m^{(1)} + \chi_l^{(1)})$$

$$\lim_{n\to\infty} \frac{\chi^{(n+1)}(T_c)}{\chi^{(n)}(T_c)} = 1 \quad \Longrightarrow \quad \frac{\chi^{(3)}(T_c)}{\chi^{(2)}(T_c)} = 1$$

Experimental phase diagram

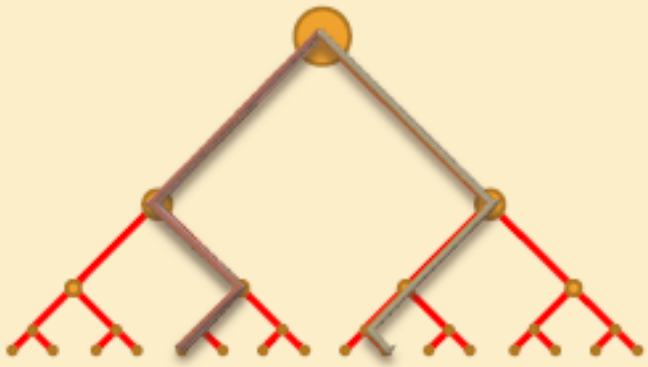


Our interpretation



EQUATION FOR T_c IN ONE STEP RSB.

Bundled average (n/m bundles):



$$Z^n = \left(K^{n/m} \left(\frac{g}{K} \right)^n \int_0^\infty \left[\frac{\mathrm{Tanh}(\xi / T)}{\xi} \right]^m d\xi \right)^N$$

$$f = \ln \frac{g}{K} + \frac{1}{m} \left[\ln K + \ln \int_0^\infty \left[\frac{\mathrm{Tanh}(\xi / T)}{\xi} \right]^m d\xi \right]$$

Before continuation to $n \to 0$ $n > m > 1$, after $n < m < 1$

RSB occurs when m^* minimizing $f(m)$ becomes m<1.

$$\ln \frac{K}{g} = \mathrm{g} \int_0^\infty \left[\frac{\mathrm{Tanh}(\xi / T_{RSB})}{\xi} \right] \ln \left[\frac{\mathrm{Tanh}(\xi / T_{RSB})}{\xi} \right] d\xi$$

Quantum critical point: $g(T=0) = K \exp(-\frac{1}{m^*} \ln \frac{K}{1-m^*})$

where m^*: $\ln \frac{K}{1-m^*} = \frac{m^*}{1-m^*}$

MODEL SOLUTION 3: EQUATION FOR T_c.

$$h_0 = Zh_N \qquad Z = \sum_{\{i[k]\}} \prod_k \frac{g}{K} \frac{\mathrm{Tanh}[\xi_{k,i[k]}/T]}{\xi_{k,i[k]}}$$

Z=exp(fN) with *f>0* ("magnet" or "superconductor") or *f<0* (paramagnet, non-supercond)

Reminder: Non-trivial physics is due to the fact that *Z* is not necessarily self-averaging quantity!

For T<g/K Z is not self-averaging and typical Z_{typ}=expN<f> might be different from <Z>.

Typical lattice shows the transition when <f> > 0.

To find average <f> use replica trick:

$$Z^n = \sum_{\{i_a[k]\}} \prod_k \frac{g}{K} \frac{\mathrm{Tanh}[\xi_{k,i_a[k]}/T]}{\xi_{k,i_a[k]}} \qquad a = 1..n$$

Solve the problem for n replicas and continue to n=0. Similar problems were solved in the context of directed polymer physics (*Derrida and Spohn*).

Replica symmetric solution (i.e. all replicas are independent) gives the BCS-like result.

However at low T(g) replica symmetry breaks down.

PROPERTIES OF THE INSULATING PHASE.

- **Perturbation theory argument.**
  Relaxation rate of the central spin at low frequency with low ξ is due to its coupling to the boundary spins:

$$\Gamma_0 = \sum_{i[k]} \prod_k \left(\frac{(g/K)^2}{\xi_{i[k]}^{\ 2} + (g/K)^2} \right) \Gamma_{i[N]}$$

1. Level width is zero if typical susceptibility diverges

$$Z_2^{\ n} = \sum_{\{i_a[k]\}} \prod_k \left(\frac{g}{K}\right)^2 \frac{1}{\xi^2_{k,i_a[k]} + \Gamma^2} \quad a = 1..n \quad n \to 0,\ \Gamma \to 0$$

2. In RSB phase $Z_2 = Z^2$, so it diverges at the same critical point g_c

- **Conclusion**: low energy levels have zero width in the insulating phase at T=0.

Threshold for activated transport

Nonzero line-width appears above

$$\omega(g,K) = 2(eg)^{1/eg}\left(1-\frac{K}{K_c}\right)^{1/eg}$$

threshold frequency only:

$$\Gamma^{typ}(\omega) \simeq \Gamma_0(\omega) \simeq e^{-1/eg} exp\left(-\frac{\omega_1}{\omega-\omega_c}\right)$$

$$\omega_1 = \frac{C}{e(eg)^3}\frac{\omega_c}{1-K/K_c} \gg \omega_c$$

This is T = 0 result !

Nonzero activation energy for transport of pairs

is due to the absence of thermal bath at low

Nonzero but low temperatures:

ω

$$\Gamma^{typ}(\omega,T) \approx \max\left(\eta(T),\Gamma_0(\omega)\right)$$

$$\eta(T) = \left(\frac{2g}{K}\right)^2 \frac{4\sqrt{\pi}\omega_1 e^{-1/eg}}{(\omega_c+\sqrt{\omega_1 T})^2}\left(\frac{T}{\omega_1}\right)^{3/4} \exp\left[-\frac{\omega_c}{T}-2\sqrt{\frac{\omega_1}{T}}\right]$$

PROPERTIES OF THE INSULATING PHASE.

- **Perturbation theory argument.**
  Levels with finite energies have non-zero width if typical susceptibility at finite frequencies diverge.

$$Z_2^{\,n}(\omega) = \sum_{\{i_a[k]\}} \prod_k \left(\frac{g}{K}\right)^2 \frac{1}{(\xi - \omega/2)^2_{k,i_a[k]} + \Gamma^2} \qquad a = 1..n \quad n \to 0,\ \Gamma \to 0$$

1 step RSB computation for finite frequencies:

$$g(\omega) = g_c \frac{1}{(1-\omega/2)^{1-m(\omega)} + (\omega/2)^{1-m(\omega)}}$$

$$m(\omega):\ \min_\omega \frac{1}{m} \log\left\{\frac{K}{1-m}\left[(1-\omega/2)^{1-m} + (\omega/2)^{1-m}\right]\right\}$$

- **Conclusion:** at $g<g_c$ "mobility edge" appears at T=0. States with E<E(g) are sharp, states at higher energies have non-zero width.
- For g<g* all states are sharp

Coulomb enchancement near mobility edge ??

Normally, Coulomb interaction is overscreened,
with universal effective coupling constant ~ 1

Condition of universal screening: $2\sigma/(T_c\kappa) \sim (\xi_0/a_{\rm scr})^2/\kappa \gg 1$

$a_{\rm scr}$ is the Thomas-Fermi screening length, $\sigma = (e^2 k_F/6\pi^2)\,(k_F l)$

Example of a-InO_x : bad dirty metal with $k_F l \sim 0.3$

$e^2 k_F \sim 5000K$ the ratio $\sigma/T_c \sim 10$

dielectric constant $\kappa \geq 30$

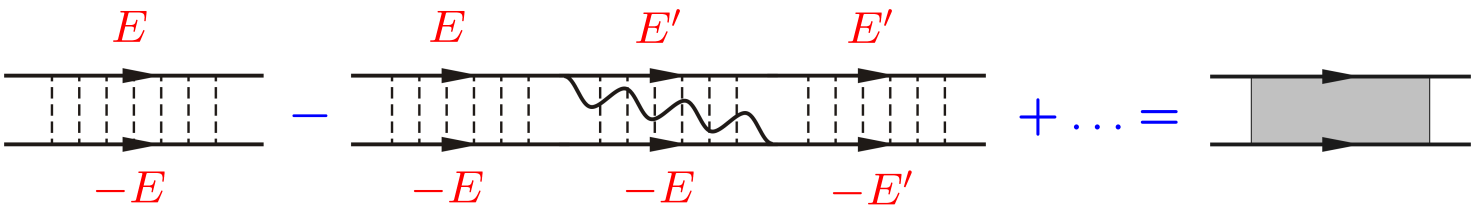
Effective Coulomb potential is weak:

$$\mu \sim 2\sigma/(T_c\kappa) < 1$$

Ginzburg-Landau expansion: step 1

$$F = \frac{\nu}{|\lambda_0|}\int d\mathbf{r}\,\Delta^*(\mathbf{r})\Delta(\mathbf{r}) - \int d\mathbf{r}\,d\mathbf{r}'\,\Delta^*(\mathbf{r})\Delta(\mathbf{r}')K(\mathbf{r},\mathbf{r}')$$
$$+\frac{1}{2}\int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 d\mathbf{r}_4\,\Delta^*(\mathbf{r}_1)\Delta(\mathbf{r}_2)\Delta^*(\mathbf{r}_3)\Delta(\mathbf{r}_4)K^{(4)}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) + \ldots$$

$\langle K(\mathbf{q})\rangle =$



$$T\sum_E \frac{1}{|E|} \qquad T^2\sum_{E,E'}\frac{1}{|E|}\frac{1}{|E'|}\frac{1}{g}\ln\frac{1}{|E+E'|\tau}$$

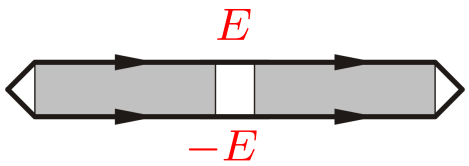
$$\int_0^{\zeta_T} d\zeta = \zeta_T \qquad \frac{1}{g}\int_0^{\zeta_T} d\zeta\,d\zeta'\min(\zeta,\zeta') = \frac{\zeta_T^3}{3g}$$

Uniform free energy:

$$F = \nu\int d\mathbf{r}\left(\frac{1}{|\lambda_0|} - \frac{1}{\lambda_g}\tanh(\lambda_g\zeta_T)\right)|\Delta(\mathbf{r})|^2 \;\Longrightarrow\; \nu(T/T_c - 1)\int d\mathbf{r}\,\frac{|\Delta(\mathbf{r})|^2}{\cosh^2(\lambda_g\zeta_c)}$$

Ginzburg-Landau expansion: step 2

- Gradient term:

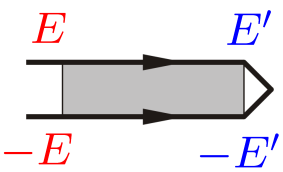


$$\frac{\partial}{\partial Dq^2}\frac{1}{Dq^2+2|E|}\bigg|_{q=0} \sim \frac{1}{E^2}$$

$$\gamma \propto T\sum_E \frac{w^2(\zeta)}{E^2} \approx w^2(\zeta_T)\, T\sum_E \frac{1}{E^2}$$

Screening factor

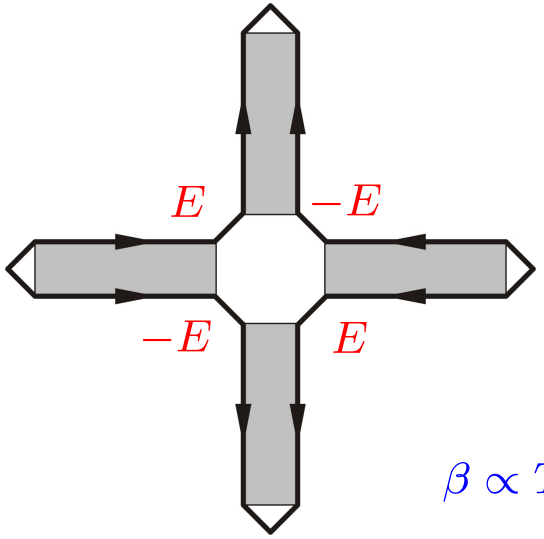
$$\frac{w(E)}{|E|} = T\sum_{E'}$$



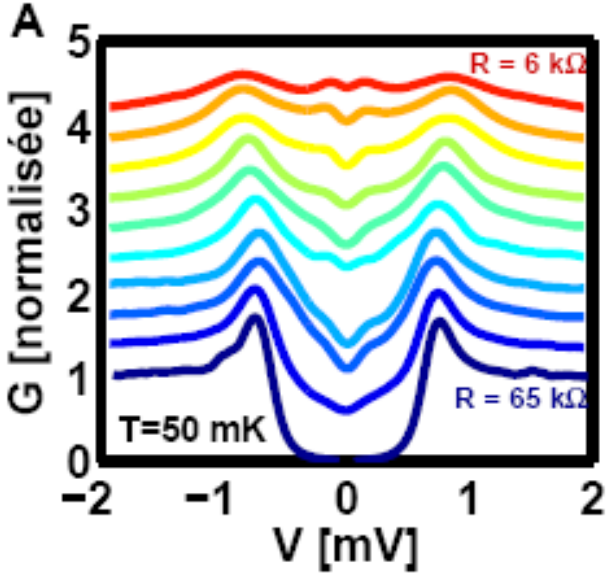
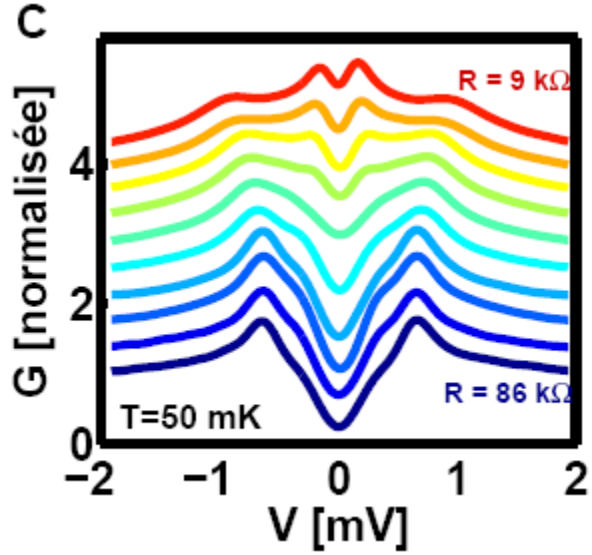
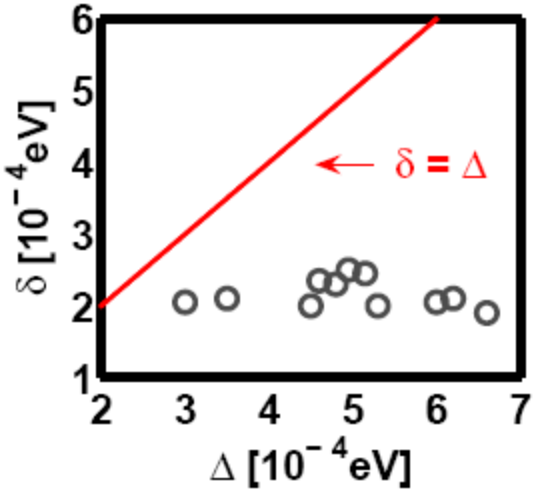
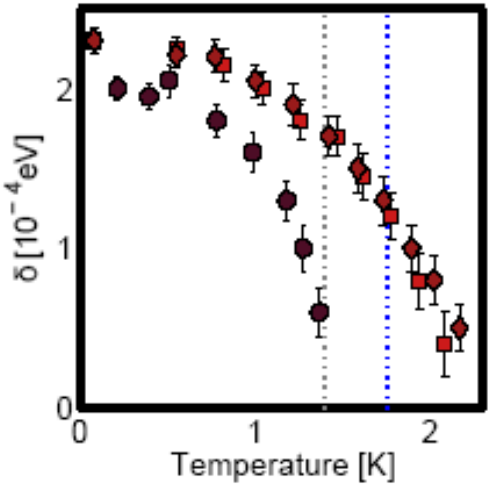
$$w(\zeta) = \cosh\lambda_g\zeta - \tanh\lambda_g\zeta_T\,\sinh\lambda_g\zeta$$

$$w(\zeta_T) = \frac{1}{\cosh\lambda_g\zeta_T}$$

- Quartic term:



$$\beta \propto T\sum_E |E|\frac{w^4(\zeta)}{E^4} \approx w^4(\zeta_T)\, T\sum_E \frac{1}{|E|^3}$$

Andreev contact spectroscopy

T.Dubouchet, thesis, Grenoble (11 Oct. 2010)