Luttinger Liquid

Alexander Chudnovskiy
Hamburg Unversity

Luttinger Liquid

- Luttinger liquid concept
- Bosonization in operator approach
- Conformal field theory approach
- Examples of correlation functions

References: J. Voit, Rep. Prog. Phys. **57***,* **977-1116** (1994).

A.O. Gogolin, A.A. Nersesyan, A.M. Tsvelik, "Bosonization and strongly correlated systems", Cambridge University Press, 1998.

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Fermi liquid in 1D and in higher dimensions

- quasiparticles are electrons and holes (fermions)
- quasiparticles do not decay if close to the Fermi surface (decay rate $\sim T^2$)

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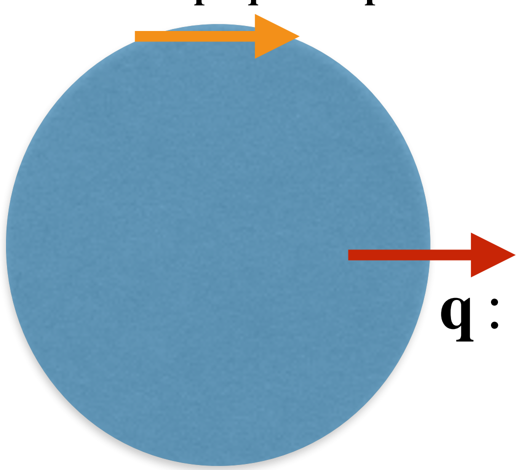
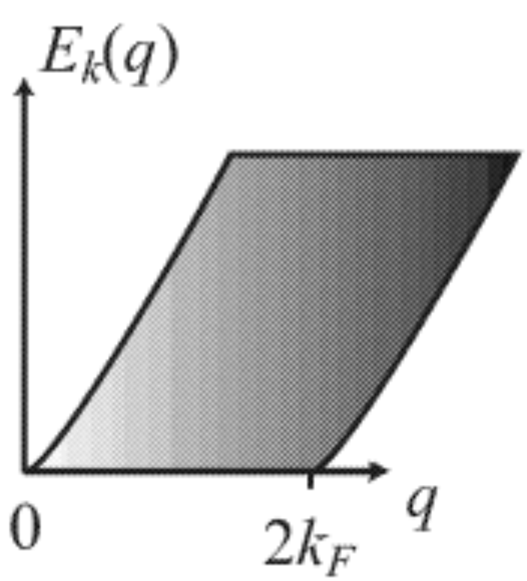


$\mathbf{q}: \quad E_{\mathbf{p}+\mathbf{q}} - E_{\mathbf{p}} \approx \upsilon_F q$

the momentum does not fix the energy
ph-excitations are not coherent



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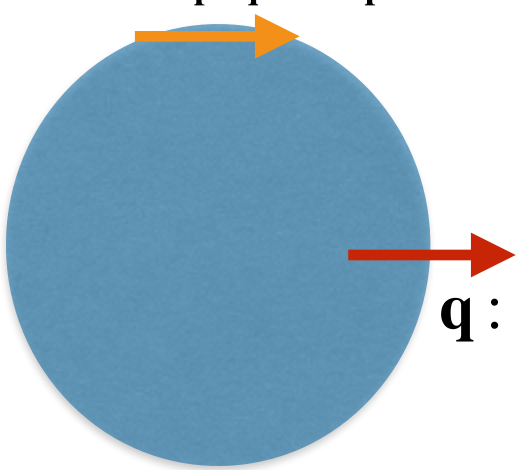

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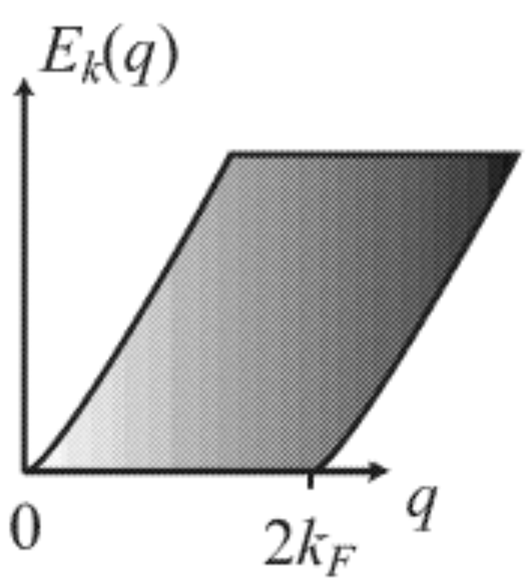

Particle-hole excitations in 1D

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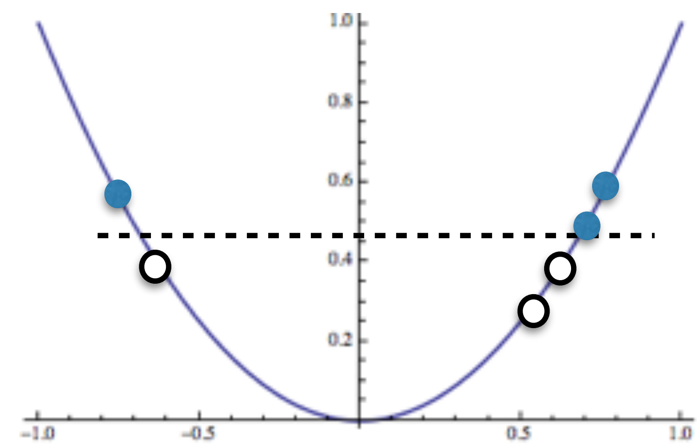

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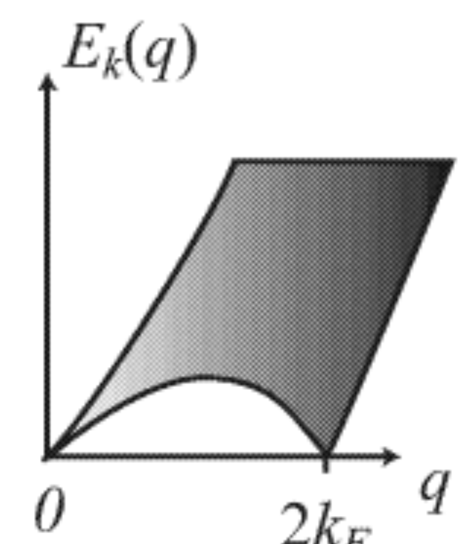

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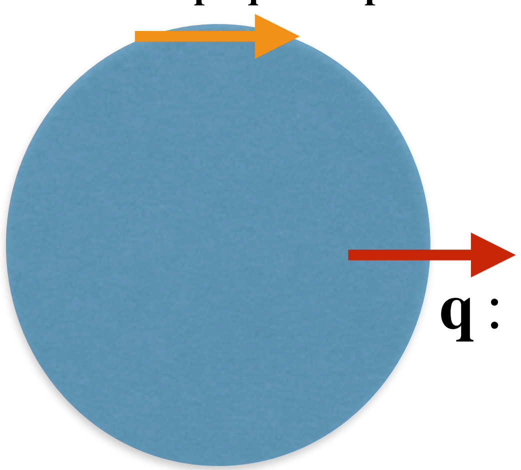

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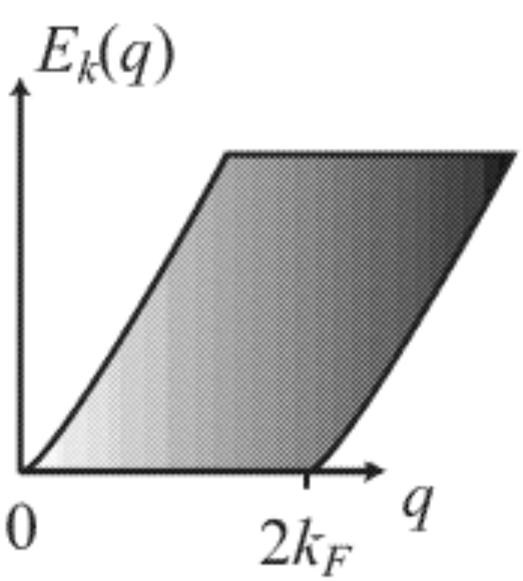

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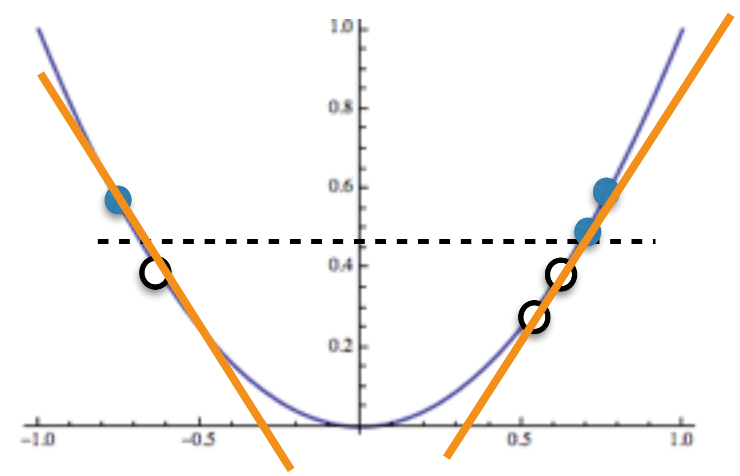

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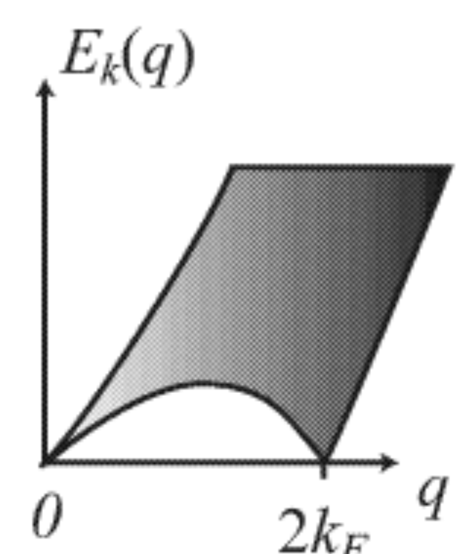
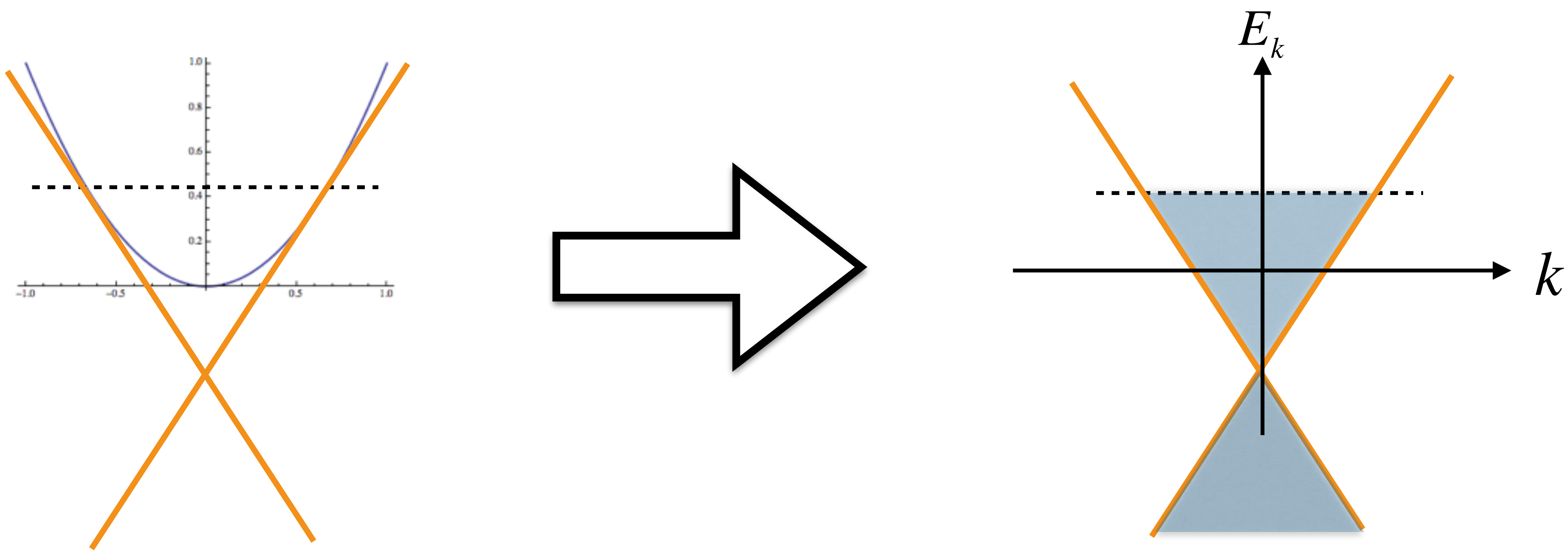
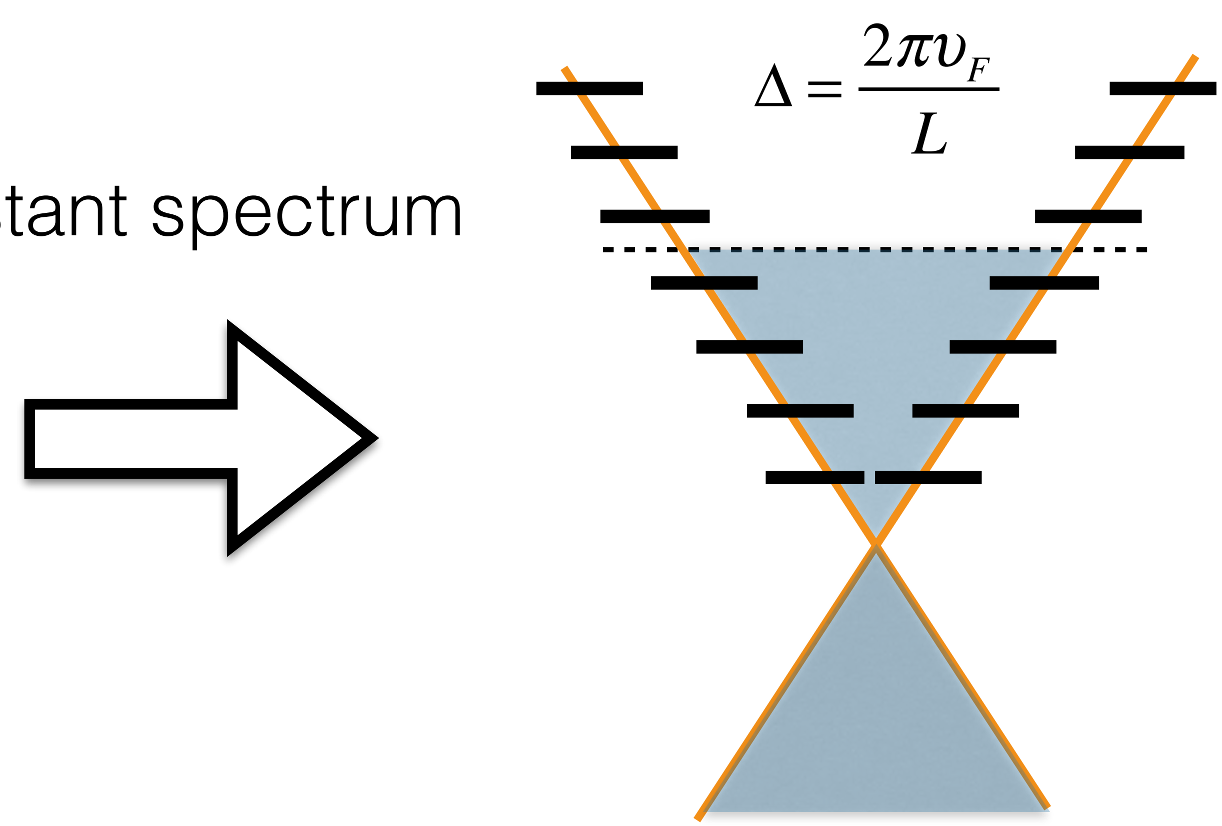


Fermi liquid in 1D: linearization of dispersion



Finite system: equidistant spectrum



Hamiltonian of 1D interacting electron system: Luttinger liquid model

$$\widehat{H}_0 = \sum_{\sigma} \sum_{k} \left\{ v_{\mathrm{F}} (k - k_{\mathrm{F}}) : c^{+}_{R,k\sigma} c_{R,k\sigma} : - v_{\mathrm{F}} (k + k_{\mathrm{F}}) : c^{+}_{L,k\sigma} c_{L,k\sigma} : \right\}$$

: … : - normal ordering with respect to the Fermi see: $:\widehat{O} := \widehat{O} - \left\langle \widehat{O} \right\rangle_{\mathrm{FS}}$.

Density fluctuations: $\rho_{r,\sigma}(p) = \sum_{k} : c^{+}_{r,k+p,\sigma} c_{r,k,\sigma} : = \sum_{k} c^{+}_{r,k+p,\sigma} c_{r,k,\sigma} - \delta_{p,0} \left\langle c^{+}_{r,k,\sigma} c_{r,k,\sigma} \right\rangle .$

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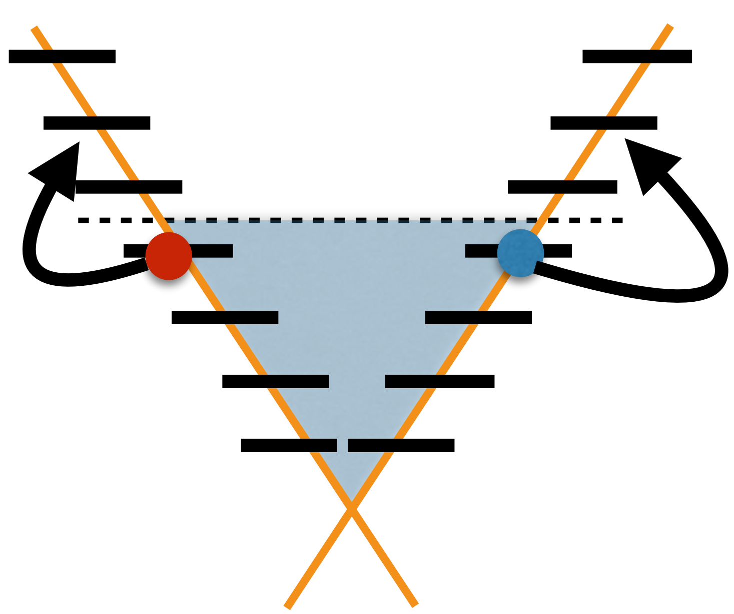
Interactions: take into account the small momentum transfer only

$$\widehat{H}_2 = \frac{1}{L} \sum_{p,\sigma,\sigma'} \left[g_{2\parallel}(p)\delta_{\sigma,\sigma'} + g_{2\perp}(p)\delta_{\sigma,-\sigma'} \right] \rho_{R\sigma}(p)\rho_{L\sigma}(-p)$$



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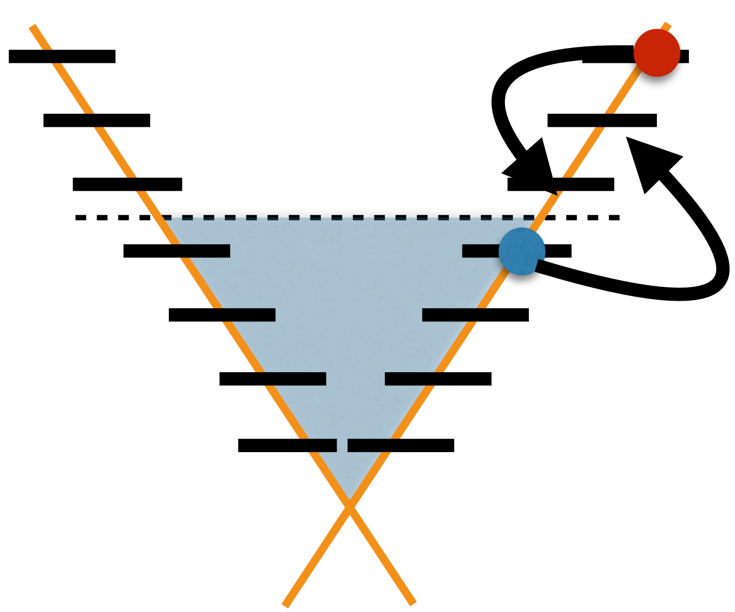
Interactions: take into account the small momentum transfer only

$$\widehat{H}_4 = \frac{1}{2L} \sum_{r=R,L} \sum_{p,\sigma,\sigma'} \left[g_{4\|}(p)\delta_{\sigma,\sigma'} + g_{4\perp}(p)\delta_{\sigma,-\sigma'} \right] : \rho_{r\sigma}(p)\rho_{r\sigma'}(-p) :$$



Symmetries of the Hamiltonian

The Hamiltonian commutes with the total number of particles in each branch and with a given spin-projection

$$N_{r\sigma} = \sum_k c^+_{r,k,\sigma} c_{r,k,\sigma} : \qquad \left[N_{r\sigma}, \widehat{H} \right] = 0.$$

$$\left[N_{r\sigma}, \rho_{r,p,\sigma} \right] = 0;$$

$$\sum_{k,k'} \left[c^+_{k'} c_{k'}, c^+_{k+p} c_k \right] = \sum_{k,k'} (c^+_{k'} (\delta_{k',k+p} + c^+_{k+p} c_{k'}) c_k - c^+_{k+p} (\delta_{kk'} + c^+_{k'} c_k) c_{k'}) =$$

$$\sum_k \left(c^+_{k+p} c_k + c^+_{k'} c^+_{k+p} c_{k'} c_k - c^+_{k+p} c_k - c^+_{k+p} c^+_{k'} c_k c_{k'} \right) = 0$$

For the election wave function: $\psi_{r\sigma}(x) \to e^{i\theta_{r\sigma}} \psi_{r\sigma}(x)$

Symmetries of the Hamiltonian

U(1) SYMMETRY $$\psi_{r\sigma}(x) \to e^{i\theta}\psi_{r\sigma}(x)$$

CHIRAL SYMMETRY $$\begin{cases} \psi_{R\sigma}(x) \to e^{i\theta}\psi_{R\sigma}(x) \\ \psi_{L\sigma}(x) \to e^{-i\theta}\psi_{L\sigma}(x) \end{cases}$$

SU(2) SPIN SYMMETRY, IF $$g_{2,\parallel} = g_{2,\perp}$$

$$\psi_{r\sigma}(x) \to \sum_{\sigma'} \left(e^{i\mathbf{\Omega}\cdot\vec{\sigma}} \right)_{\sigma\sigma'} \psi_{r\sigma'}(x)$$

Boson solution of the Luttinger model

Map the fermion problem with interactions to the problem of noninteracting bosons

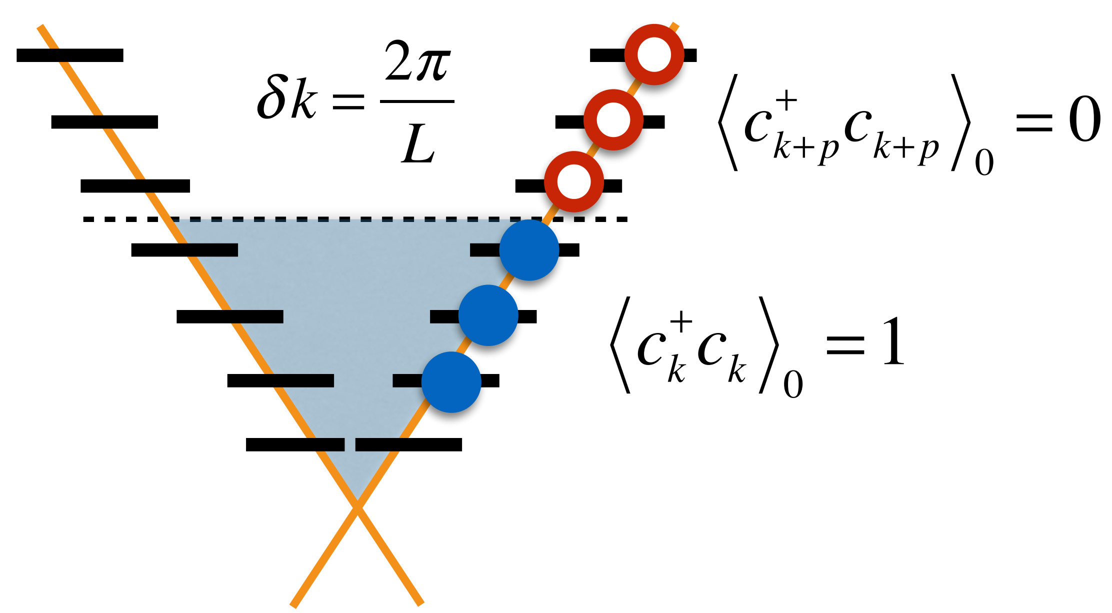
- Density operators obey bosonic commutation relations: relation to canonical bosonic creation/annihilation operators
- Hamiltonian can be represented as a bilinear of the density operators
- Explicit construction of a fermion through the bosonic operators

Commutator of the density operators

Right-movers: $$\left[\hat{\rho}_p, \hat{\rho}_{-p'}\right] = \sum_{k,k'}\left(c^+_{k+p}c_k c^+_{k'-p'}c_{k'} - c^+_{k'-p'}c_{k'}c^+_{k+p}c_k\right) = \sum_k \left(c^+_{k+p}c_{k+p'} - c^+_{k+p-p'}c_k\right)$$

$$= \sum_k \left(:c^+_{k+p}c_{k+p'}: - :c^+_{k+p-p'}c_k: + \left\langle c^+_{k+p}c_{k+p'}\right\rangle_0 - \left\langle c^+_{k+p-p'}c_k\right\rangle_0\right)$$

$$= \delta_{pp'}\sum_k \left(\left\langle c^+_{k+p}c_{k+p}\right\rangle_0 - \left\langle c^+_k c_k\right\rangle_0\right) = -\frac{pL}{2\pi}\delta_{pp'}$$



Commutator of the density operators

U(1) Kac-Moody algebra

Right-movers: $\left[\hat{\rho}_p, \hat{\rho}_{-p'}\right] = -\frac{pL}{2\pi}\delta_{pp'}$ Left-movers: $\left[\hat{\rho}_{L,p}, \hat{\rho}_{L,-p'}\right] = \frac{pL}{2\pi}\delta_{pp'}$

$$\left[\hat{\rho}_R(x), \hat{\rho}_R(x')\right] = \frac{i}{2\pi}\partial_x \delta(x-x') \qquad \left[\hat{\rho}_L(x), \hat{\rho}_L(x')\right] = -\frac{i}{2\pi}\partial_x \delta(x-x')$$

Relation between the density operators and canonical bosons

$$\left[\hat{\rho}_{L,p}, \hat{\rho}_{L,-p'}\right] = \frac{pL}{2\pi}\delta_{pp'} \Rightarrow \hat{\rho}_{L,p} = \sqrt{\frac{2\pi}{pL}}\hat{b}_L(p), \; \hat{\rho}_{L,-p} = \sqrt{\frac{2\pi}{pL}}\hat{b}_L^+(p), \; (p>0).$$

$$\left[\hat{\rho}_{R,p}, \hat{\rho}_{R,-p'}\right] = -\frac{pL}{2\pi}\delta_{pp'} \Rightarrow \hat{\rho}_{R,p} = \sqrt{\frac{2\pi}{pL}}\hat{b}_R^+(p), \; \hat{\rho}_{R,-p} = \sqrt{\frac{2\pi}{pL}}\hat{b}_R(p), \; (p>0).$$

Equivalence of the free fermion Hamiltonian to the Hamiltonian quadratic in density operators

$$\widehat{H}_0 = \sum_\sigma \sum_k \left\{ \upsilon_\mathrm{F}(k-k_\mathrm{F}) : c^+_{R,k\sigma} c_{R,k\sigma} : - \upsilon_\mathrm{F}(k+k_\mathrm{F}) : c^+_{L,k\sigma} c_{L,k\sigma} : \right\}$$

Easy to check: $\left[\widehat{H}_0, \rho_{R,\sigma}(p)\right] = \upsilon_F p \rho_{R,\sigma}(p)$ cf. $\left[\hat{\rho}_{R,\sigma,-p}, \hat{\rho}_{R,\sigma,p}\right] = \frac{pL}{2\pi}$

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$$\widehat{H}_0 = \frac{\pi \upsilon_F}{L} \sum_{p\neq 0, r=(R,L),\sigma} : \rho_{r,\sigma}(p) \rho_{r,\sigma}(-p) : + \mathrm{const.}$$

"const." is the ground state energy: $E^{\mathrm{GS}}_{N+1} - E^{\mathrm{GS}}_N = \upsilon_F \Delta p = \frac{2\pi \upsilon_F}{L}$

It changes with the total particle number:

$$\widehat{H}_0 = \frac{\pi \upsilon_F}{L} \sum_{p\neq 0, r=(R,L),\sigma} : \rho_{r,\sigma}(p) \rho_{r,\sigma}(-p) : + \frac{\pi \upsilon_F}{L} \sum_\sigma (N^2_{R\sigma} + N^2_{L\sigma})$$

Luttinger liquid Hamiltonian

$$\widehat{H} = \widehat{H}_0 + \widehat{H}_2 + \widehat{H}_4 = \frac{\pi v_F}{L} \sum_{\sigma} (N_{R\sigma}^2 + N_{L\sigma}^2) + \frac{\pi v_F}{L} \sum_{p \neq 0, r=(R,L), \sigma} :\rho_{r,\sigma}(p) \rho_{r,\sigma}(-p):$$

$$+ \frac{1}{L} \sum_{p,\sigma,\sigma'} \left[g_{2\parallel}(p) \delta_{\sigma,\sigma'} + g_{2\perp}(p) \delta_{\sigma,-\sigma'} \right] :\rho_{R\sigma}(p) \rho_{L\sigma'}(-p):$$

$$+ \frac{1}{2L} \sum_{r=R,L} \sum_{p,\sigma,\sigma'} \left[g_{4\parallel}(p) \delta_{\sigma,\sigma'} + g_{4\perp}(p) \delta_{\sigma,-\sigma'} \right] :\rho_{r\sigma}(p) \rho_{r\sigma'}(-p):$$

Separation of spin and charge

Charge and spin variables

$$\rho_r(p) = \frac{1}{\sqrt{2}}\left[\rho_{r\uparrow}(p) + \rho_{r\downarrow}(p)\right] \qquad N_{r\rho} = \frac{1}{\sqrt{2}}\left[N_{r\uparrow} + N_{r\downarrow}(p)\right]$$

$$\sigma_r(p) = \frac{1}{\sqrt{2}}\left[\rho_{r\uparrow}(p) - \rho_{r\downarrow}(p)\right] \qquad N_{r\sigma} = \frac{1}{\sqrt{2}}\left[N_{r\uparrow} - N_{r\downarrow}(p)\right], \quad r = R, L$$

Interaction constants: $g_{i\rho} = \frac{1}{2}\left(g_{i\parallel} + g_{i\perp}\right), \quad g_{i\sigma} = \frac{1}{2}\left(g_{i\parallel} - g_{i\perp}\right), \qquad i = 2,4.$

The Hamiltonian

$$\widehat{H} = \widehat{H}_\rho + \widehat{H}_\sigma .$$

$$\widehat{H}_\nu = \widehat{H}_0 + \widehat{H}_2 + \widehat{H}_4 , \quad \nu = \rho, \sigma .$$

same structure with new interaction constants

Diagonalization of LL-Hamiltonian by a Bogoliubov transformation

$$\tilde{\rho}_R(p) = \rho_R(p)\cosh[\xi(p)] + \rho_L(p)\sinh[\xi(p)]$$
$$\tilde{\rho}_L(p) = \rho_L(p)\cosh[\xi(p)] + \rho_R(p)\sinh[\xi(p)]$$

Choice of $\xi(p)$: $K(p) \equiv e^{2\xi(p)} = \sqrt{\dfrac{\pi v_F + g_4(p) - g_2(p)}{\pi v_F + g_4(p) + g_2(p)}}.$

For a typical interaction: $H_{\text{int}} = \int dx\,dx'\; V(x-x')\hat{n}(x)\hat{n}(x') = \int \frac{dq}{2\pi}\hat{n}(q)V(q)\hat{n}(-q)$

$$g_4 \propto \tilde{V}(q=0), \qquad g_2 \propto \tilde{V}(q=2k_F).$$

Repulsive interactions: $K < 1.$ Attractive interactions: $K > 1.$

The diagonalized Hamiltonian

$$\widetilde{H} = \frac{\pi}{L}\sum_{p\neq 0} \upsilon(p) : \widetilde{\rho}(p)\widetilde{\rho}(-p) : + \frac{\pi}{2L}\left[\upsilon_N(p)(N_R + N_L)^2 + \upsilon_J(p)(N_R - N_L)^2\right]$$

$$\upsilon(p) = \sqrt{\left[\upsilon_F + \frac{g_4(p)}{\pi}\right]^2 - \left[\frac{g_2(p)}{\pi}\right]^2}. \qquad K(p) \equiv e^{2\xi(p)} = \sqrt{\frac{\pi\upsilon_F + g_4(p) - g_2(p)}{\pi\upsilon_F + g_4(p) + g_2(p)}}.$$

$$\upsilon_N = \upsilon / K = \upsilon_F + (g_4 + g_2)/\pi,$$

$$\upsilon_J = \upsilon K = \upsilon_F + (g_4 - g_2)/\pi,$$

$$\upsilon_N \upsilon_J = \upsilon^2.$$

$$g_2, g_4 \quad \leftrightarrow \quad \upsilon, K \quad \leftrightarrow \upsilon_N, \upsilon_J$$

Velocities in the Luttinger liquid model

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υ - sound velocity of bosonic excitations

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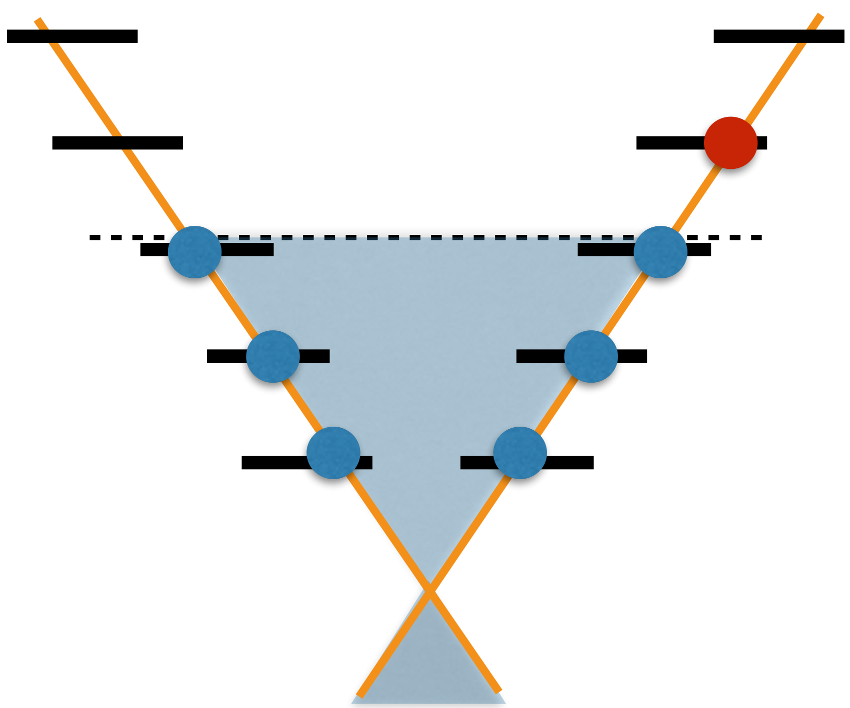
υ_N
- energy to add a particle (a fermion!) in the ground state
- shift of the chemical potential by creating charge excitation



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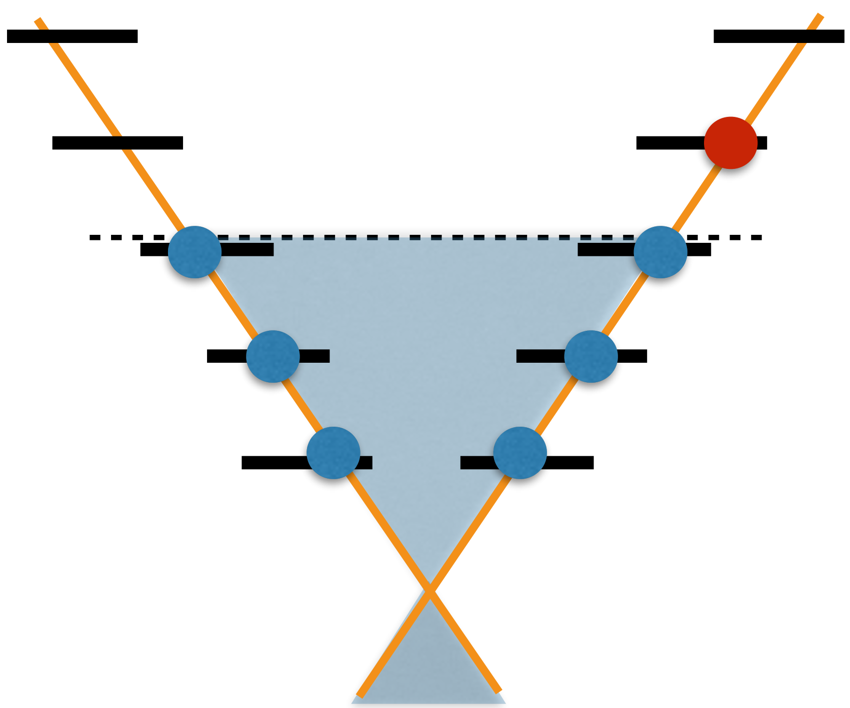

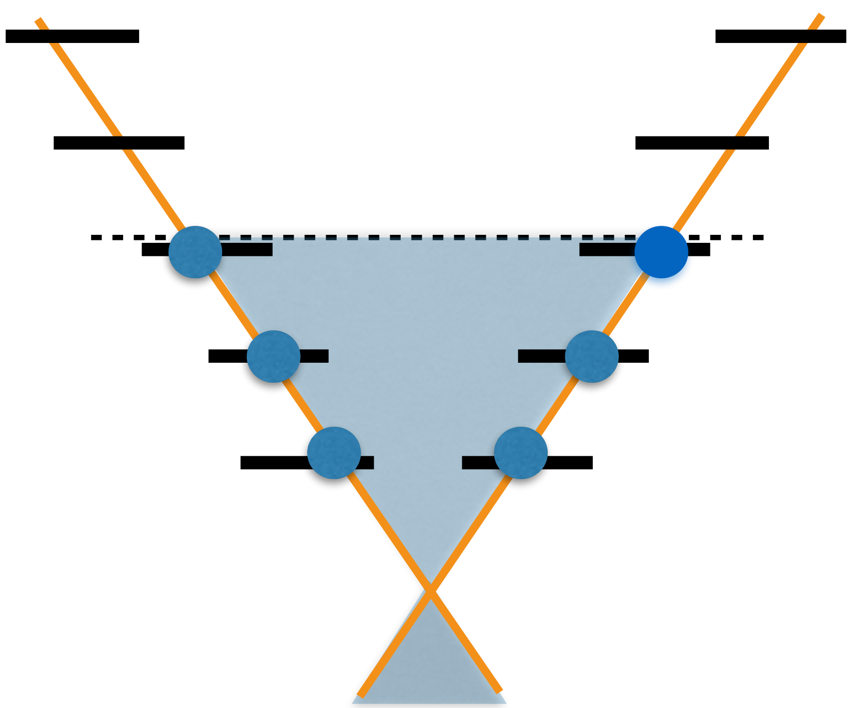

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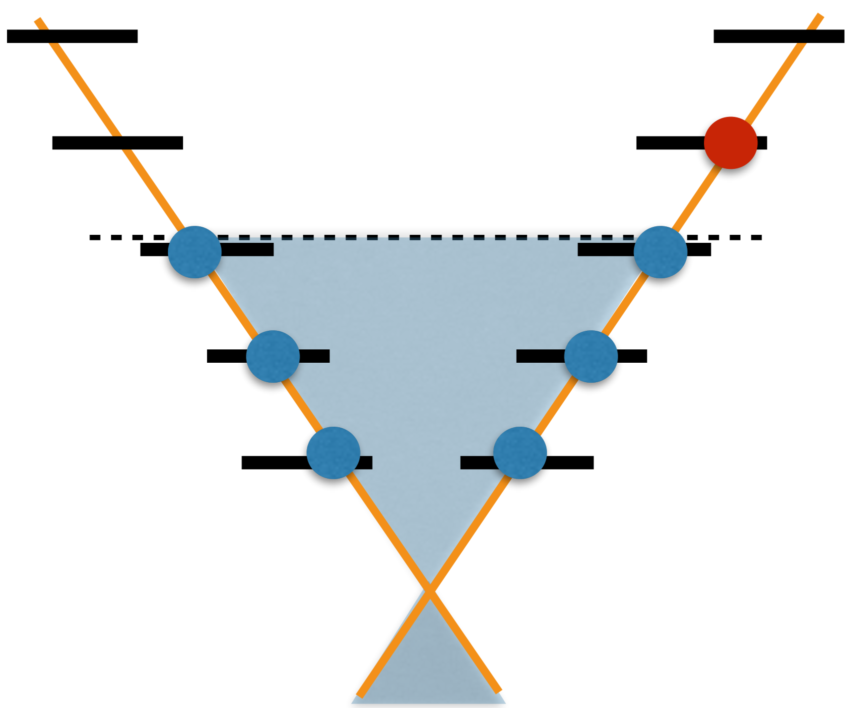

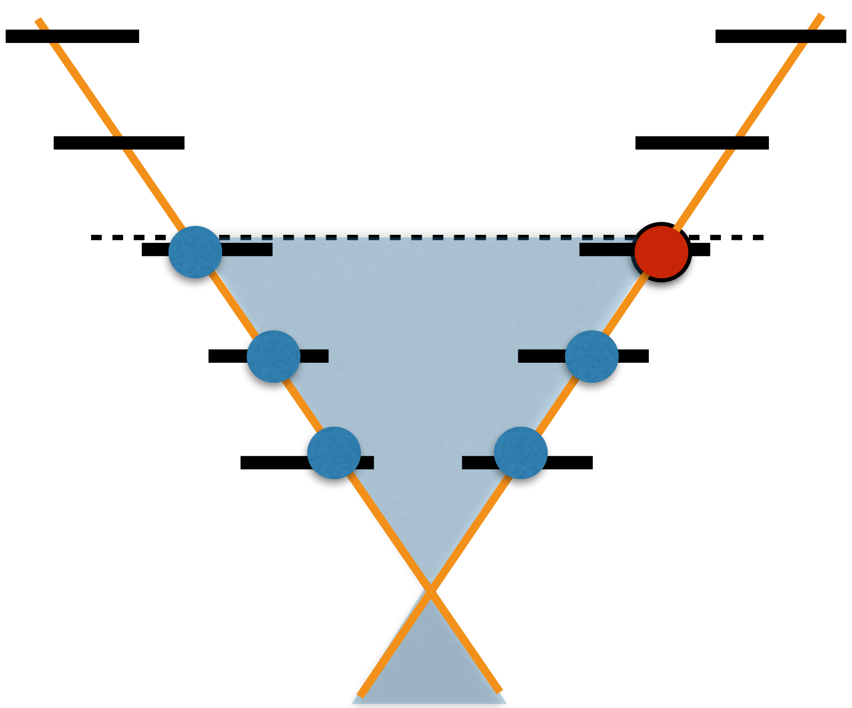

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υ_J - energy to create a persistent current in the periodic chain

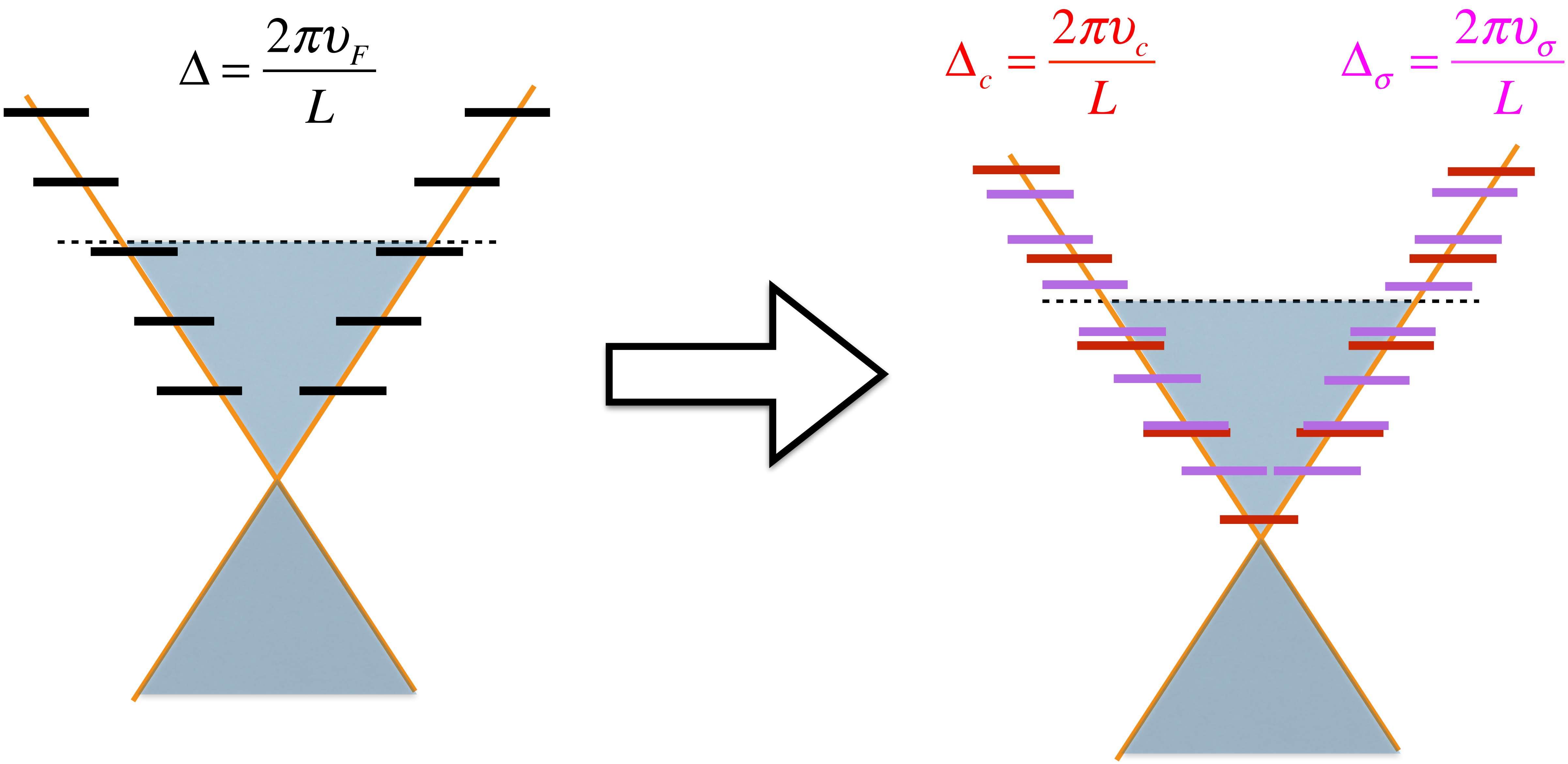
$$\upsilon_N, \upsilon_J \quad \longleftrightarrow \quad \upsilon, K$$

are completely determined by the spectrum of the model

determine all correlation functions

All correlation functions are completely determined by the spectrum

Separation of spin and charge: influence on the spectrum



Construction of the fermion operator in terms of bosonic fields: Bosonization

Fermion creation operator: $\widehat{\psi}^{+}(x)$

- increases the total number of fermions by 1
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For right-moving particles :

$$\widehat{\psi}^{+}(x) = \widehat{U}^{+} e^{i\hat{\phi}(x)} : \qquad \widehat{U}^{+}|N\rangle = |N+1\rangle, \quad \left[\widehat{U}^{+}, \widehat{\rho}_{p}\right] = 0.$$

$$\hat{\phi}(x) = -\frac{\pi x}{L}\widehat{N} + \lim_{\alpha\to 0}\left(\frac{2\pi i}{L}\sum_{p\neq 0}\frac{e^{-\alpha p/2 - ipx}}{p}\widehat{\rho}_{-p}\right)$$

Luttinger liquid Hamiltonian in terms of canonically conjugated operators

Phase fields:

$$\Phi(x) = \frac{1}{2}\left(\phi_R + \phi_L\right) = -\frac{i\pi}{2}\sum_{p\neq 0}\frac{e^{-\alpha|p|/2-ipx}}{p}\left[\rho_R(p)+\rho_L(p)\right] - \frac{\pi x}{L}\left(N_R + N_L\right),$$

$$\Theta(x) = \frac{1}{2}\left(\phi_R - \phi_L\right) = \frac{i\pi}{2}\sum_{p\neq 0}\frac{e^{-\alpha|p|/2-ipx}}{p}\left[\rho_R(p)-\rho_L(p)\right] - \frac{\pi x}{L}\left(N_R - N_L\right).$$

For the charge and spin sectors:

$$\Phi_\rho(x) = \frac{1}{\sqrt{2}}\left(\Phi_\uparrow(x)+\Phi_\downarrow(x)\right), \quad \Theta_\rho(x) = \frac{1}{\sqrt{2}}\left(\Theta_\uparrow(x)+\Theta_\downarrow(x)\right),$$

$$\Phi_s(x) = \frac{1}{\sqrt{2}}\left(\Phi_\uparrow(x)-\Phi_\downarrow(x)\right), \quad \Theta_s(x) = \frac{1}{\sqrt{2}}\left(\Theta_\uparrow(x)-\Theta_\downarrow(x)\right).$$

Charge density operator: $\rho(x) = \sqrt{2}\left[\rho_R(x)+\rho_L(x)\right] = -\frac{\sqrt{2}}{\pi}\frac{\partial \Phi_\rho(x)}{\partial x}.$

Canonically conjugated operators

$$\left.\begin{aligned}\left[\hat{\rho}_{L,p},\hat{\rho}_{L,-p'}\right]&=\frac{pL}{2\pi}\delta_{pp'},\\ \left[\hat{\rho}_{R,p},\hat{\rho}_{R,-p'}\right]&=-\frac{pL}{2\pi}\delta_{pp'}\end{aligned}\right\}\Rightarrow\left[\Phi(x),\Theta(x')\right]=i\pi\theta(x'-x).$$

$$\left[\Phi(x),\frac{1}{\pi}\partial_{x'}\Theta(x')\right]=i\delta(x'-x).$$

$\frac{1}{\pi}\partial_x\Theta(x)=\Pi\left(x\right)$ - the momentum, conjugated to the field $\Phi(x)$

Luttinger liquid Hamiltonian in terms of canonically conjugated fields

$$\hat{H} = \frac{1}{2\pi}\sum_{\nu=\rho,\sigma}\left\{\upsilon_{J\nu}\pi^2\Pi_\nu^2(x) + \upsilon_{N\nu}\left(\frac{\partial\Phi_\nu(x)}{\partial x}\right)^2\right\} = \frac{1}{2\pi}\sum_{\nu=\rho,\sigma}\left\{\upsilon_{J\nu}\left(\frac{\partial\Theta_\nu(x)}{\partial x}\right)^2 + \upsilon_{N\nu}\left(\frac{\partial\Phi_\nu(x)}{\partial x}\right)^2\right\}$$

$$\hat{H} = \frac{\upsilon}{2\pi}\sum_{\nu=\rho,\sigma}\left\{K_\nu\left(\frac{\partial\Theta_\nu(x)}{\partial x}\right)^2 + \frac{1}{K_\nu}\left(\frac{\partial\Phi_\nu(x)}{\partial x}\right)^2\right\}$$

Diagonalization by canonical transformation

$$\tilde{\Phi}_\nu = \Phi_\nu / \sqrt{K}, \quad \tilde{\Theta}_\nu = \Theta_\nu \sqrt{K}, \quad u = \upsilon / K.$$

$$\hat{H} = \frac{u}{2\pi}\sum_{\nu=\rho,\sigma}\left\{\left(\frac{\partial\tilde{\Theta}_\nu(x)}{\partial x}\right)^2 + \left(\frac{\partial\tilde{\Phi}_\nu(x)}{\partial x}\right)^2\right\} = \sum_p u|p|\left(\hat{b}_p^+\hat{b}_p + \frac{1}{2}\right) + \frac{\pi}{2L}\left(K\upsilon_J^2 + \frac{1}{K}\upsilon_N^2\right)$$

Important operators

Charge density: $\hat{\rho}(x) = \sqrt{2}\left(\hat{\rho}_R(x) + \hat{\rho}_L(x)\right) = -\frac{\sqrt{2}}{\pi}\frac{\partial \Phi_\rho(x)}{\partial x}.$

Fermion operator: $\hat{\psi}^+_{rs}(x) = \hat{U}^+_{rs} e^{i\hat{\phi}_{rs}(x)}, \quad \left(r = R, L; \quad s = \uparrow, \downarrow\right).$

$$\psi^+_{rs}(x) = \lim_{\alpha \to 0} \frac{e^{ir(k_F - \pi/L)x}}{\sqrt{2\pi\alpha}} \hat{U}^+_{rs} \exp\left[-\frac{i}{\sqrt{2}}\left(r\Phi_\rho(x) - \Theta_\rho(x) + s\left\{r\Phi_\sigma(x) - \Theta_\sigma(x)\right\}\right)\right]$$

Using $\Pi(x) = \frac{1}{\pi}\partial_x \Theta(x) \Rightarrow \Theta(x) = \int_{-\infty}^{x} dz \Pi(z)$

$$\psi^+_{rs}(x) \approx \lim_{\alpha \to 0} \frac{1}{\sqrt{2\pi\alpha}} \exp\left[irk_F x - ir\Phi_{rs}(x) + i\pi \int_{-\infty}^{x} dz \Pi_{rs}(z)\right].$$

Properties of Luttinger liquid

Specific heat: $C=\gamma T,\quad \gamma=\frac{1}{2}\gamma_0\left(\frac{\upsilon_F}{\upsilon_\rho}+\frac{\upsilon_F}{\upsilon_\sigma}\right).$

$\gamma_0=\frac{\pi^2 k_B}{3}N(E_F)=\frac{2\pi k_B}{3\upsilon_F}$ - the coefficient for the noninteracting LL

Spin susceptibility: $\chi^{-1}=\frac{1}{L}\frac{\partial^2 E}{\partial\sigma^2},\quad \chi=\frac{2K_\sigma}{\pi\upsilon_\sigma}=\frac{2}{\pi\upsilon_{N\sigma}}.$

Compressibility: $\kappa^{-1}=\frac{1}{L}\frac{\partial^2 E}{\partial\rho^2},\quad \kappa=\frac{2K_\rho}{\pi\upsilon_\rho}=\frac{2}{\pi\upsilon_{N\rho}}.$

Single particle (fermion) density of states: $N(\omega)\sim\omega^\alpha,\quad \alpha=\frac{1}{4}\sum_{\nu=\rho,\sigma}\left(K_\nu+1/K_\nu-2\right).$

Field theory approach: Gaussian model in 2D

Diagonalized Hamiltonian - free bosons

$$\widehat{H} = \frac{u}{2\pi} \sum_{\nu=\rho,\sigma} \left\{ \left(\frac{\partial \tilde{\Theta}_\nu(x)}{\partial x} \right)^2 + \left(\frac{\partial \tilde{\Phi}_\nu(x)}{\partial x} \right)^2 \right\} = \sum_p u|p| \left(\hat{b}_p^+ \hat{b}_p + \frac{1}{2} \right) + \frac{\pi}{2L} \left(K \upsilon_J^2 + \frac{1}{K} \upsilon_N^2 \right)$$

Action - the Gaussian model

$$S = \frac{1}{2} \int_A d\tau \, dx \left[\upsilon^{-1} \left(\partial_\tau \Phi \right)^2 + \upsilon \left(\partial_x \Phi \right)^2 \right], \qquad A = (0 < \tau < \beta, \;\; 0 < x < L)$$

Field theory approach: Gaussian model in 2D

$$S = \frac{1}{2}\int_A d\tau\, dx \left[\upsilon^{-1}(\partial_\tau \Phi)^2 + \upsilon(\partial_x \Phi)^2 \right], \qquad A = (0 < \tau < \beta,\;\; 0 < x < L)$$

Generating function: $Z[\eta] = \int D\Phi(\mathbf{x}) e^{-S[\Phi] - \int d\tau\, dx\, \eta(x)\Phi(x)}$

Integrate out $\boldsymbol{\Phi}(\mathbf{x})$: $\mathbf{p} = (\omega, q),\quad \Phi(\mathbf{p}) = \int d\tau\, dx\, e^{i\omega\tau} e^{iqx} \Phi(\tau, x)$

$$Z[\eta] = \int \prod_{\mathbf{p}} d\Phi(\mathbf{p}) \exp\left[-\frac{1}{2}\sum_{\mathbf{p}} \Phi(-\mathbf{p})(\upsilon p^2 + \upsilon^{-1}\omega^2)\Phi(\mathbf{p}) + \sum_{\mathbf{p}} \eta(-\mathbf{p})\Phi(\mathbf{p}) \right]$$

Shift of variables to eliminate the linear term in $\boldsymbol{\Phi}(\mathbf{p})$:

$$\tilde{\Phi}(\mathbf{p}) = \Phi(\mathbf{p}) + G(\mathbf{p})\eta(\mathbf{p}); \qquad G(\mathbf{p}) = (\upsilon q^2 + \upsilon^{-1}\omega^2)^{-1}$$

$$Z[\eta] = Z[0]\exp\left[\frac{1}{2\beta V} \sum_{\mathbf{p}\neq 0} \eta(-\mathbf{p}) G(\mathbf{p}) \eta(\mathbf{p}) \right]$$

Correlators of exponentials

$$Z[\eta] = Z[0]\exp\left[\frac{1}{2\beta V}\sum_{\mathbf{p}\neq 0}\eta(-\mathbf{p})G(\mathbf{p})\eta(\mathbf{p})\right]$$

Real space: $-(\upsilon\partial_x^2 + \upsilon^{-1}\partial_\tau^2)G(x,\tau;x',\tau') = \delta(x-x')\delta(\tau-\tau')$

Complex coordinates: $z = \tau + ix/\upsilon, \quad \bar{z} = \tau - ix/\upsilon.$

Laplace operator: $\nabla^2 = 4\upsilon^{-1}\partial_z\partial_{\bar{z}}, \quad \partial_z = \frac{1}{2}(\partial_\tau - i\upsilon\partial_x), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_\tau + i\upsilon\partial_x).$

Green's function far away from the boundaries

$$G(z,\bar{z}) = \frac{1}{4\pi}\ln\left(\frac{R^2}{z\bar{z} + a^2}\right)$$

R - the typical size of the system (the long-distance cutoff)

a - the short distance cutoff (lattice spacing)

Correlators of exponentials

Let $\eta(\xi) = i\sum_{n=1}^{n} \beta_n \delta(\xi - \xi_n).$

Then

$$Z[\eta(\xi)]/Z[0] = \langle \exp[i\beta_1 \Phi(\xi_1)] \exp[i\beta_2 \Phi(\xi_2)] ... \exp[i\beta_N \Phi(\xi_N)] \rangle$$

$$= \exp\left[-\sum_{i>j} \beta_i \beta_j G(\xi_i, \xi_j)\right] \exp\left[-\frac{1}{2}\sum_i \beta_i^2 G(\xi_i, \xi_i)\right] = \prod_{i>j} \left(\frac{z_{ij}\bar{z}_{ij}}{a^2}\right)^{\frac{\beta_i \beta_j}{4\pi}} \left(\frac{R}{a}\right)^{-\frac{\left(\sum_n \beta_n^2\right)}{4\pi}}.$$

For the infinite system $\frac{R}{a} \to \infty \Rightarrow \sum_n \beta_n = 0.$

Correlators of exponentials

Let $\eta(\xi)=i\sum_{n=1}^{n}\beta_n\delta(\xi-\xi_n).$

Then

$$Z[\eta(\xi)]/Z[0]=\langle\exp[i\beta_1\Phi(\xi_1)]\exp[i\beta_2\Phi(\xi_2)]...\exp[i\beta_N\Phi(\xi_N)]\rangle$$

$$=\exp\left[-\sum_{i>j}\beta_i\beta_j G(\xi_i,\xi_j)\right]\exp\left[-\frac{1}{2}\sum_i\beta_i^2 G(\xi_i,\xi_i)\right]=\prod_{i>j}\left(\frac{z_{ij}\bar{z}_{ij}}{a^2}\right)^{\frac{\beta_i\beta_j}{4\pi}}\left(\frac{R}{a}\right)^{-\frac{\left(\sum_n\beta_n^2\right)}{4\pi}}.$$

For the infinite system $\frac{R}{a}\to\infty\ \Rightarrow\ \sum_n\beta_n=0.$

$F(\Phi)=\int d\beta F(\beta)e^{i\beta\Phi}\quad\Rightarrow$ we can calculate correlators of any local functionals!

Analytic and anti-analytic parts of correlation functions

$$\left\langle \exp\left[i\beta_1\Phi\left(\xi_1\right)\right]\ldots\exp\left[i\beta_N\Phi\left(\xi_N\right)\right]\right\rangle = G\left(z_1,\ldots,z_N\right)G\left(\bar{z}_1,\ldots,\bar{z}_N\right)\delta_{\sum\beta_n,0},$$

$$G\left(z_1,\ldots,z_N\right) = \prod_{i>j}\left(\frac{z_{ij}}{a}\right)^{\frac{\beta_i\beta_j}{4\pi}}.$$

One can study analytic and anti-analytic parts independently

$$\Phi(z,\bar{z}) = \phi(z) + \bar{\phi}(\bar{z}), \quad \exp\left[i\beta\Phi\left(z,\bar{z}\right)\right] = \exp\left[i\beta\phi\left(z\right)\right]\exp\left[i\beta\bar{\phi}\left(\bar{z}\right)\right]$$

(right- and left movers!)

One can consider different coefficients by $\phi(z)$ and $\bar{\phi}(\bar{z})$

Correlation functions of physical operators

Introduce a dual field $\Theta(z,\bar{z}) = \phi(z) - \bar{\phi}(\bar{z})$.

$$\partial_z \Phi = \partial_z \Theta, \qquad \partial_{\bar{z}} \Phi = -\partial_{\bar{z}} \Theta.$$

Instead of $\left(\phi, \bar{\phi}\right)$ consider $\left(\Phi, \Theta\right)$.

General functional periodic in Φ and Θ

$$F\left(\Phi,\Theta\right) = \sum_{n,m} \tilde{F}_{n,m} \exp\left[\frac{2i\pi n}{T_1}\Phi + \frac{2i\pi m}{T_2}\Theta\right] = \sum_{n,m} \tilde{F}_{n,m} \exp\left[i\beta_{nm}\phi(z) + i\bar{\beta}_{nm}\bar{\phi}(\bar{z})\right],$$

$$\beta_{nm} = 2\pi\left(\frac{n}{T_1} + \frac{m}{T_2}\right), \qquad \bar{\beta}_{nm} = 2\pi\left(\frac{n}{T_1} - \frac{m}{T_2}\right).$$

Correlation functions of physical operators

Pair correlation function:

$$\left\langle \exp\left[i\beta_{nm}\phi(z_1)\right]\exp\left[i\bar{\beta}_{nm}\bar{\phi}(\bar{z}_1)\right]\exp\left[-i\beta_{nm}\phi(z_2)\right]\exp\left[-i\bar{\beta}_{nm}\bar{\phi}(\bar{z}_2)\right]\right\rangle = (z_{12})^{-\frac{\beta_{nm}^2}{4\pi}}(\bar{z}_{12})^{-\frac{\bar{\beta}_{nm}^2}{4\pi}} = \frac{1}{|z_{12}|^{2d}}\left(\frac{z_{12}}{\bar{z}_{12}}\right)^S$$

$$d = \Delta + \bar{\Delta} = \frac{1}{8\pi}\left(\beta^2 + \bar{\beta}^2\right) \text{ - scaling dimension}$$

$$S = \Delta - \bar{\Delta} = \frac{1}{8\pi}\left(\beta^2 - \bar{\beta}^2\right) \text{ - conformal spin}$$

Correlation function uniquely defined if:

$$\left(\frac{z_{12}e^{i2\pi}}{\bar{z}_{12}e^{-i2\pi}}\right)^S = \left(\frac{z_{12}}{\bar{z}_{12}}\right)^S e^{i4\pi S} = \left(\frac{z_{12}}{\bar{z}_{12}}\right)^S \quad \Rightarrow \quad 2S = \left(\frac{\beta_{nm}^2}{4\pi} - \frac{\bar{\beta}_{nm}^2}{4\pi}\right) = (\text{integer}).$$

$$T_2 = \frac{4\pi}{T_1} = \sqrt{4\pi K}.$$

Possible conformal dimensions of physical operators

$$\Delta_{nm} = \frac{\beta_{nm}^2}{4\pi} = \frac{1}{8}\left(n\sqrt{K} + \frac{m}{\sqrt{K}}\right)^2,$$

$$\bar{\Delta}_{nm} = \frac{\bar{\beta}_{nm}^2}{4\pi} = \frac{1}{8}\left(n\sqrt{K} - \frac{m}{\sqrt{K}}\right)^2.$$

K is the Luttinger liquid interaction constant

In Luttinger liquid models with different interaction constants operators of the same physical observables have representations with different K.

Example: electron in non-interacting LL

$$K = 1, \quad S = \frac{1}{2}: \quad n = m = 1$$

$$\psi_R \sim \exp\left[i2\sqrt{\pi}\phi(z)\right]$$

In the interacting LL

$$K \neq 1: \text{ n=m=1}. \quad \beta_{11} = \sqrt{\pi}\left(\sqrt{K} + 1/\sqrt{K}\right),$$

$$\psi_R \sim \exp\left[i\sqrt{\pi}\left\{\left(\sqrt{K} + \frac{1}{\sqrt{K}}\right)\phi(z) + \left(\sqrt{K} - \frac{1}{\sqrt{K}}\right)\bar{\phi}(\bar{z})\right\}\right].$$

Sine-Gordon model

Operators of physical quantities - periodic functions of Φ, Θ

Stability of LL model with respect to perturbations - Sine-Gordon model

$$S = \frac{1}{2}\int d^2\mathbf{x}(\nabla\Phi)^2 + g\int \frac{d^2x}{a^2}\cos(\beta\Phi).$$

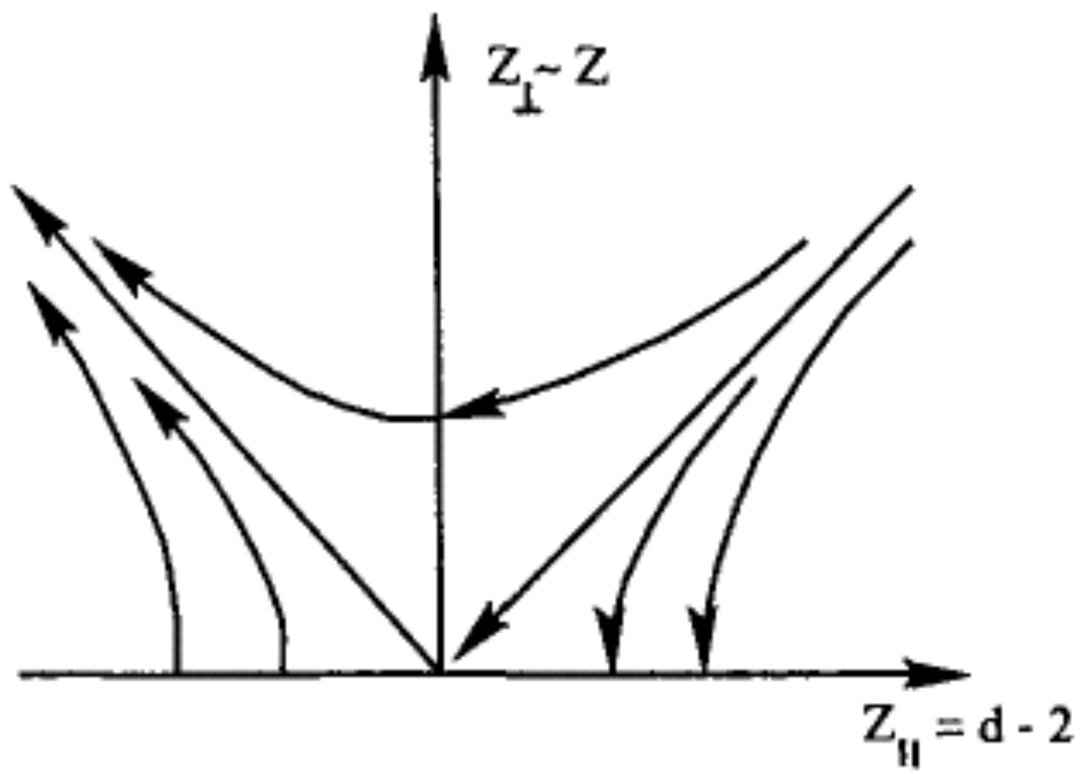
RG-analysis: RG flow for the dimension $d = \frac{\beta^2}{4\pi}$ and for the coupling constant g

$$d(l) = 2 + z_{\parallel}(l), \quad z_{\perp}(l) = \sqrt{8A}g(l). \quad A\text{- a nonuniversal constant}$$

$$\frac{dz_{\parallel}}{dl} = -z_{\perp}^2, \qquad \frac{dz_{\perp}}{dl} = -z_{\parallel}z_{\perp}.$$

$d < 2$ - relevant

$d > 2$ - irrelavant



Summary

- Luttinger liquid model: main approximation
- Bosonization:

    - operator approach
    - field theory approach
    - translation between the two languages