Quench Dynamics in Integrable Quantum Many-Body Systems

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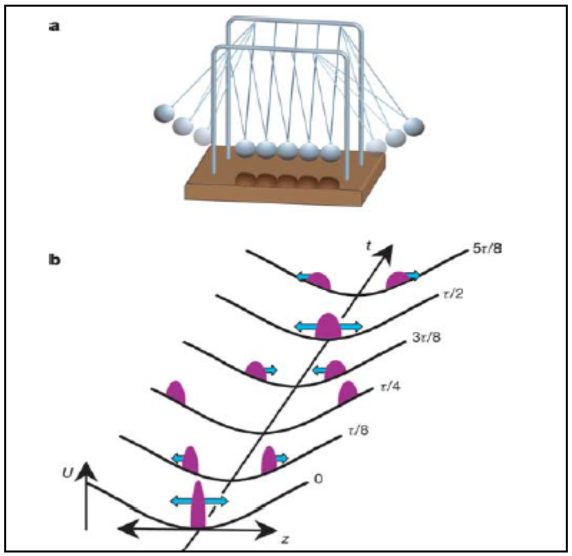
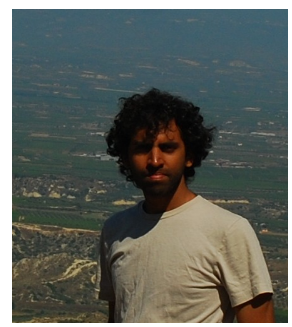
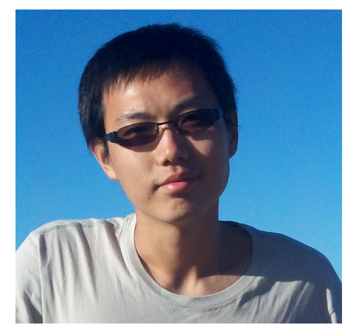


Kinoshita, Wenger, Weiss (Nature '06)





Garry Goldstein **Deepak Iyer** **Wenshuo Liu**

Novel Theories and Materials - Trieste, Aug 2015

Quenching and Time Evolution

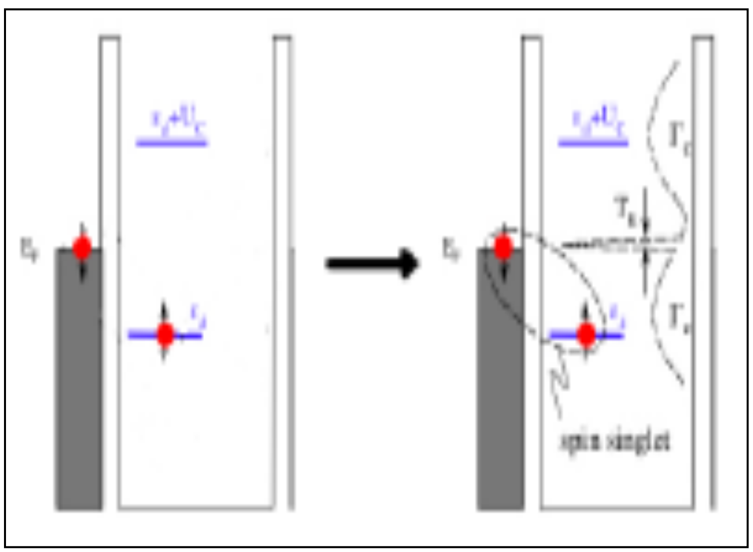
- **Prepare an isolated quantum many-body system in state $|\Phi_0\rangle$,** typically eigenstate of H_0
- **At $t=0$ turn on interaction H_1, and evolve system with $H = H_0 + H_1$:**

$$|\Phi_0, t\rangle = e^{-iHt}|\Phi_0\rangle$$

- **Many experiments: cold atom systems, nano-devices, molecular electronics, photonics :** *new systems, old questions*

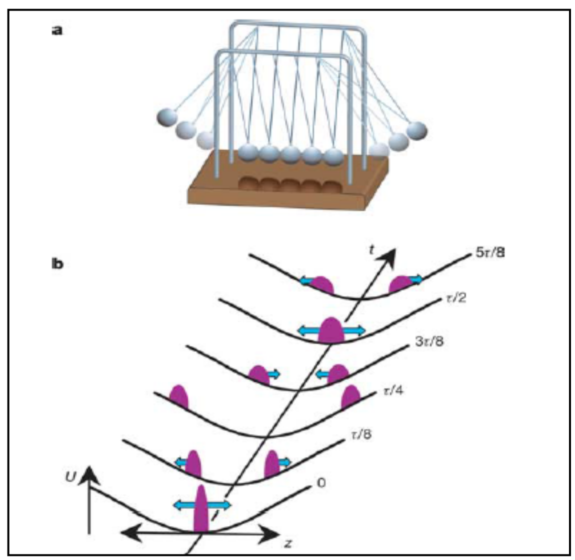
Time evolution of observables:	$\langle O(t)\rangle = \langle \Phi_0, t\vert O\vert \Phi_0, t\rangle$

- **Manifestation of interactions**



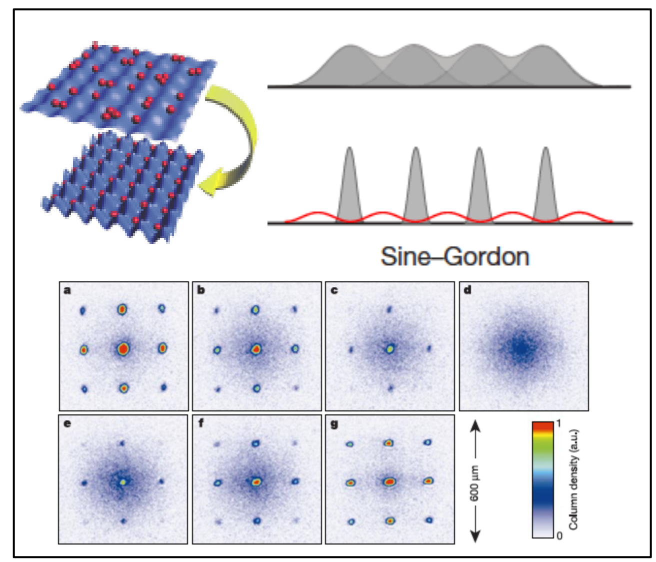
Time evolution of the Kondo peak.
- Time resolved photo emission spectroscopy

Kinoshita et al '06



Newton's Cradle

Greiner et al '02 *Bloch et al '08*



Mott insulator ←→ superfluid :2d, 1d

Quenching – long time limit, equlibration, thermalization

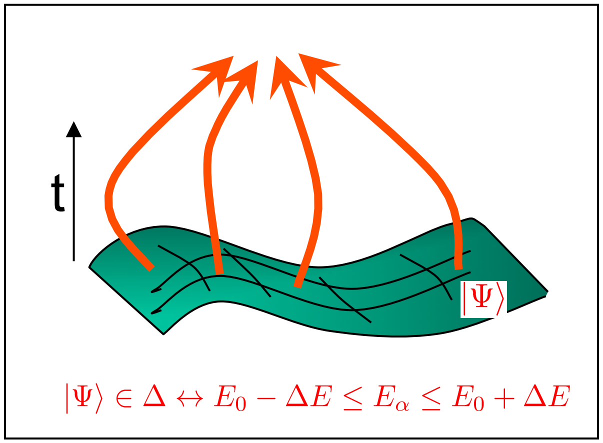
Time evolution and statistical mechanics:

use: $1 = \sum_\alpha |\alpha\rangle\langle\alpha|$

$$\langle A(t)\rangle = \langle\Phi_0|e^{iHt}Ae^{-iHt}|\Phi_0\rangle = \sum_{\alpha,\beta}\langle\Phi_0|\alpha\rangle A_{\alpha\beta}\langle\beta|\Phi_0\rangle e^{i(E_\alpha - E_\beta)t} \underset{t\to\infty}{\longrightarrow}$$

Does it equilibrate? $\rightarrow$

$$\sum_\alpha |\langle\Phi_0|\alpha\rangle|^2 A_{\alpha\alpha} \overset{?}{=} \langle A\rangle_{\text{microcan}}(E_0) \equiv \frac{1}{N_{E_0,\Delta E}}\sum_{\alpha\in\Delta E} A_{\alpha\alpha}$$



- **Long time limit and thermalization:** **(Gibbs ensemble - GE)**
- is there a limit for *local* op. $\bar{A} = \lim_{t\to\infty}\langle A(t)\rangle$? (equilibration)
- is there a density operator ρ such that $\bar{A} = Tr(\rho A)$?
- does it depend only on $E_0 = \langle\Phi_0|H|\Phi_0\rangle$, not on $|\Phi_0\rangle$?

- **Thermalization (ETH** and others)
  - ETH (non integrable models): $A_{\alpha\alpha} = Tr|\alpha\rangle\langle\alpha|A = f_A(E_\alpha)$, with f smooth funct. of E_α

  Deutsch '92, Srednicki '94, Rigol et al '08, Lebowitz et al '10, …

  - In **translation** inv. systems, $t\to\infty$ leads to $|\alpha\rangle = |\beta\rangle$, **diagonal ensemble, MC ensemble**

  → Gibbs ensemble: $\langle A(t\to\infty)\rangle = \frac{1}{Z}Tr(e^{-\beta H}A)$, with β det. by $\langle H\rangle_{t=0} = \frac{1}{Z}Tr(e^{-\beta H}H)$

  - Strong ETH: $Tr\{\prod_{i=1}^{n}|\alpha_i\rangle\langle\alpha_i|O_i\} = f_{O_1,\ldots,O_n}(E_{\alpha_1},\ldots,E_{\alpha_n})_{distinct}$, smooth function

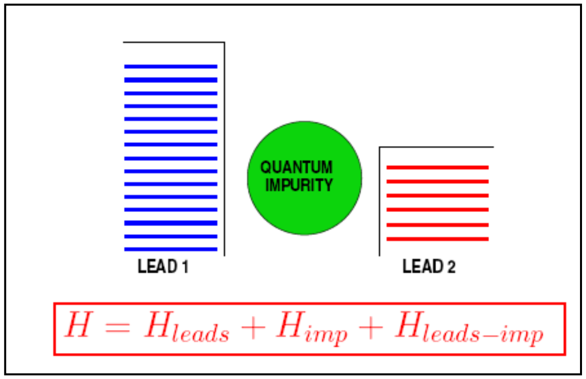
  generalization of ETH to projections and non-local measurements (G. Goldstein, NA '14)

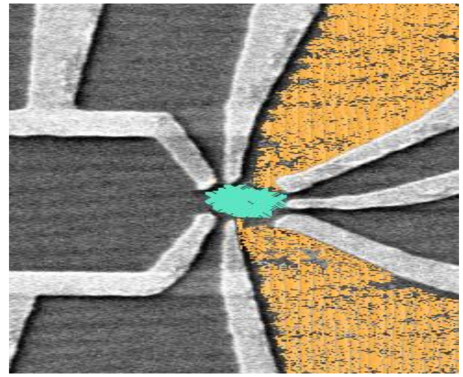
Quenching and non-thermalization

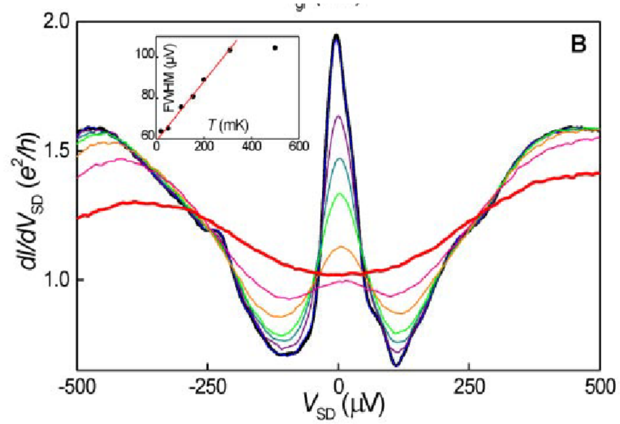
Nonequilibrium currents

Goldhaber-Gordon *et al*, Cronenwett *et al*, Schmid *et al*

- **Two baths or more:**
  - time evolution in a nonequilibrium set up:

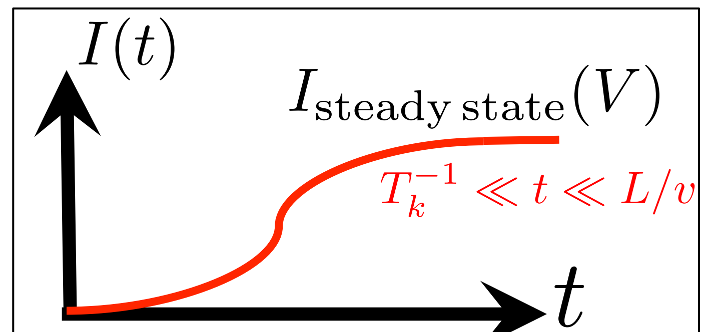






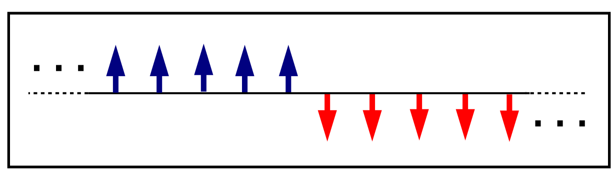
- $t \leq 0$, leads decoupled, system described by: ρ_o
- $t = 0$, couple leads to impurity
- $t \geq 0$, evolve with $H(t) = H_0 + H_1$

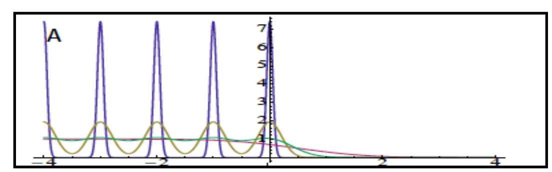
Interplay - strong correlations and nonequilibrium



- **What is the time evolution of the current** $\langle I(t) \rangle$ **?**
- **Long time limit: Under what conditions is there a nonequilibrium steady state (NESS)? Dissipation mechanism?**
- **Steady state – is there a non-thermal** ρ_{NESS} **? Voltage dependence?**
- **New effects out of equilibrium? New scales? Phase transitions, universality?**

- **Domain wall:** spin currents, NESS
  $t \to \infty, L \to \infty$





- ***Newton's Cradle*** (no NESS)

Outline

Wish to study these questions exactly and confront them with experiment

0. Introduction to integrable many body systems and their experimental realization

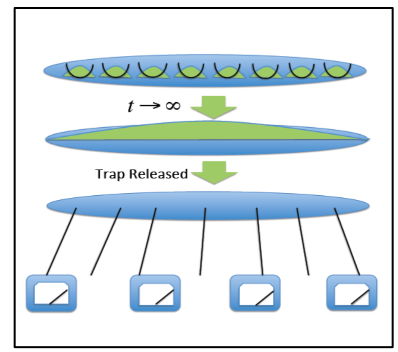
1. Quench evolution in quantum integrable many-body systems

- *Yudson's contour approach (infinite volume system):* *(Yudson '85)* $L = \infty, N$ fixed
- *Generalization to thermodynamic systems:* *(Goldstein, NA '13)*

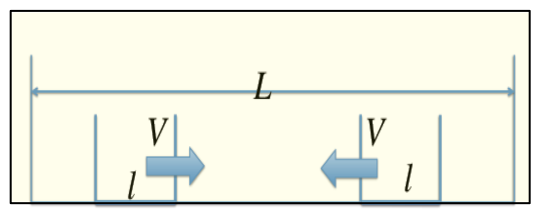
$$N, L \to \infty, \; n = N/L \text{ fixed}, \; t \ll L/v_{typ}$$

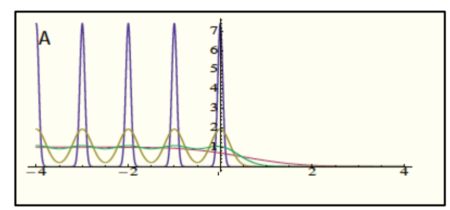
2. Bosons on the continuous line with short range interactions *(Lieb-Liniger model)*

- Time evolution of the thermodynamic system *(monster formula)*



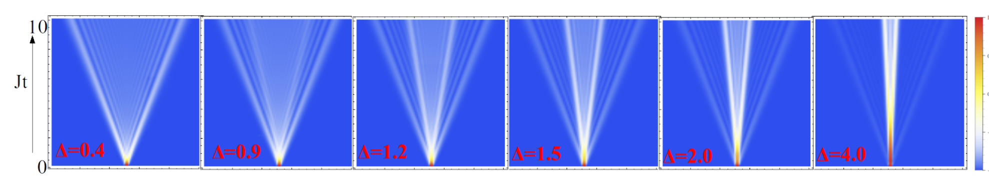
- Equilibration and GGE for repulsive interaction:
  - Quench from Mott to Lieb-Liniger fluid
  - Newton's Cradle on the average (poor man's version)





- Non equilibration (NESS): the Domain Wall
- Failure of GGE for attractive Lieb-Liniger, Hubbard, Kondo..

3. Quench dynamics of the XXZ magnet



Quenching in 1-d systems

Physical Motivation:

- **Natural dimensionality of many systems:**

  - wires, waveguides, optical traps, edges

- **Impurities: Dynamics dominated by s-waves, reduces to 1D system**
- **Many experimental realizations: ultracold atom traps, nano-systems..**

Special features of 1- d : theoretical

- **Strong quantum fluctuations for any coupling strength**
- **Powerful mathematical methods:**

  - RG methods, Bosonization, CFT methods, **Bethe Ansatz approach**

- Bethe Ansatz approach: allows complete diagonalization of *H*

- Experimentally realizable: *Hubbard model, Kondo model, Anderson model, Lieb-Liniger model, Sine-Gordon model, Heisenberg model, Richardson model..*

- BA ⟶ Quench dynamics of many body systems? Exact!

Others approaches: Keldysh, t-DMRG, t-NRG, t-RG

Much work in context of Luttinger Liquid: Cazalilla et al, Mitra et al

The Bethe Ansatz - Review

- **General N - particle state**

$$|F^\lambda\rangle_N = \int d^N x \, F^\lambda(\vec{x}) \prod_{j=1}^{N} \psi^\dagger(x_j)|0\rangle$$

- **Eigenfunctions very complicated in general**
- **The BA - wave function much simpler:**

Product of single particles wave functions $f_\lambda(x)$ and S-matrices S_{ij} ,

Example:

$$H = -\sum_{j=1}^{N} \partial_{x_j}^2 + c \sum_{i<j} \delta(x_i - x_j)$$

$$f_\lambda(x) \sim e^{i\lambda x}$$

$$S_{ij}(\lambda_i - \lambda_j) = \frac{\lambda_i - \lambda_j + ic}{\lambda_i - \lambda_j - ic}$$

i. divide configuration space into N! regions $Q, \{x_{Q1} \leq \cdots, \leq x_{QN}\}$

ii particles interact only when crossing: inside a region product of single particle wave funct.

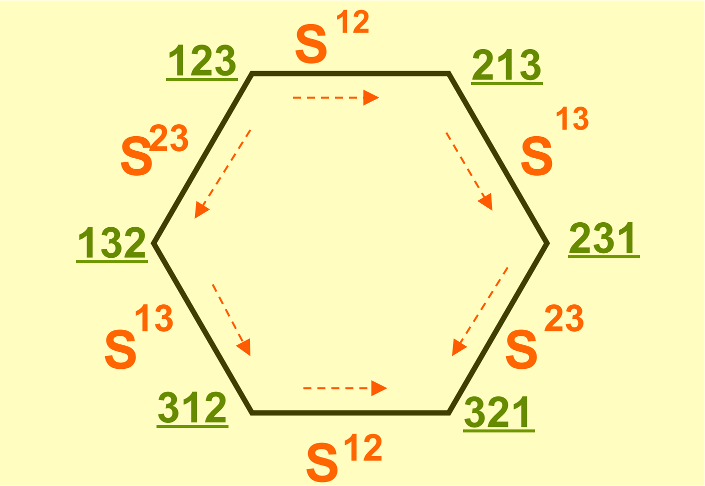
iii. assign amplitude A^Q to region Q

iv. amplitudes related by S-matrices S_{ij} (*e.g.* $A^{132} = S^{23} \, A^{123}$)

$$\rightarrow \quad F^\lambda(\vec{x}) = \sum_{Q \in S_N} A^Q \prod_j f_{\lambda_{Qj}}(x_j)$$

v. do it consistently: ***Yang-Baxter relation***

$$S^{12} \, S^{13} \, S^{23} = S^{23} \, S^{13} \, S^{12}$$



Integrability

- **The model is *integrable:*** infinite number of conserved charges $I_n, \; n = 1, 2, ..$

  - They commute among themselves $[I_n, I_m] = 0$
  - They commute with the Hamiltonian $[I_n, H] = 0$ typically $I_2 = H, \;\; I_1 = P$
  - The action of the charges on the eigenstates: $I_n \, |\lambda_1, .., \lambda_N\rangle = \sum_{j=1}^{N} (\lambda_j)^n \, |\lambda_1, .., \lambda_N\rangle$

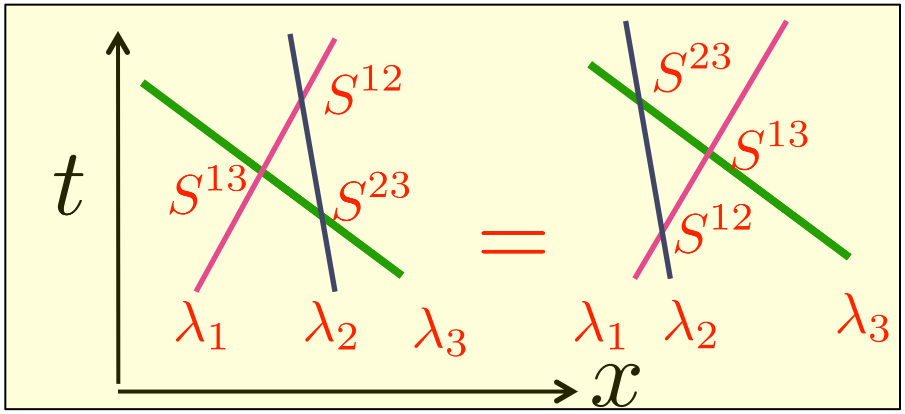
I_n acting on a particle moves it by an amount that depends on its momentum

$$e^{-icI_n} \int d\lambda \; e^{i\lambda(x-a)} e^{-\frac{(\lambda-\lambda_0)^2}{2\sigma}} = \int d\lambda \; e^{-ic\lambda^n} \; e^{i\lambda(x-a)} e^{-\frac{(\lambda-\lambda_0)^2}{2\sigma}}$$

by saddle point approximation*:*

$$x = a + c(n-1)\lambda_0^{n-1}$$

$$\boxed{S^{23} \; S^{13} \; S^{12} = S^{12} \; S^{13} \; S^{23}}$$



Trans. Invariant systems expected to evolve

- **GGE, generalized Gibbs ensemble**:

$$\langle A(t \to \infty) \rangle = Tr(\hat{\rho} A) \qquad \text{with} \qquad \hat{\rho} = Z^{-1} \exp(-\sum_n \beta_n I_n)$$

- Inverse temperatures β_n determined by initial conditions: $Tr(I_n \hat{\rho}) = \langle I_n \rangle_{t=0}$

Time Evolution and the Bethe Ansatz

- A given state $|\Phi_0\rangle$ can be formally time evolved in terms of a complete set of energy eigenstates $|F^\lambda\rangle$

$$|\Phi_0\rangle = \sum_\lambda |F^\lambda\rangle\langle F^\lambda|\Phi_0\rangle \longrightarrow |\Phi_0, t\rangle = e^{-iHt}|\Phi_0\rangle = \sum_\lambda e^{-i\epsilon_\lambda t}|F^\lambda\rangle\langle F^\lambda|\Phi_0\rangle$$

If H integrable $\longrightarrow$ eigenstates $|F^\lambda\rangle$ are known via the Bethe-Ansatz

BA wave function: $F^\lambda(\vec{x}) = \sum_{Q\in S_N} A^Q \prod_j f_{\lambda_{Qj}}(x_j)$ *S-matrix* $S_{ij}(\lambda_i - \lambda_j) = \dfrac{\lambda_i - \lambda_j + ic}{\lambda_i - \lambda_j - ic}$

- Use *Bethe Ansatz* to study quench evolution and nonequilibrium
- New technology is necessary:

- **Standard approach**: **PBC** $\longrightarrow$ **Bethe Ansatz eqns** $\longrightarrow$ **spectrum** $\longrightarrow$ **thermodynamics**

- **Non equilibrium entails *additional* difficulties:**

i. Compute overlaps (form factors) ii. Compute matrix elements iii. Sum over complete basis

Progress: J. S. Caux et al, P. Calabrese at al

Yudson's contour representation

Instead of $|\Phi_0\rangle = \sum_\lambda |F^\lambda\rangle\langle F^\lambda|\Phi_0\rangle$ introduce (directly in infinite volume):

Contour representation of $|\Phi_0\rangle$

$$|\Phi_0\rangle = \int_\gamma d^N\lambda \, |F^\lambda\rangle(F^\lambda|\Phi_0\rangle$$

V. Yudson, sov. phys. *JETP* (1985)

Computed S-matrix of Dicke model

with: $|F^\lambda\rangle$ Bethe eigenstate

$|F^\lambda)$ obtained from Bethe eigenstate by setting $S = I$ - One quadrant suffices

γ contour in momentum space $\{\lambda\}$ determined by **pole structure** of $S(\lambda_i - \lambda_j)$

Note: in the infinite volume limit momenta $\{\lambda\}$ are not quantized
- no Bethe Ansatz equations, $\{\lambda\}$ free parameters

then:

$$|\Phi_0, t\rangle = \int_\gamma d^N\lambda \, e^{-iE(\lambda)t}|F^\lambda\rangle(F^\lambda|\Phi_0\rangle$$

- Describes systems in the zero density limit
- Generalize to thermodynamic systems with finite density

Interacting bosonic system

Bosons in a 1-d with short range interactions

$$H = -\int dx b^\dagger(x)\partial^2 b(x) + c\int dx\, b^\dagger(x)b(x)b^\dagger(x)b(x)$$

c **- coupling constant**

$c > 0$ *repulsive*

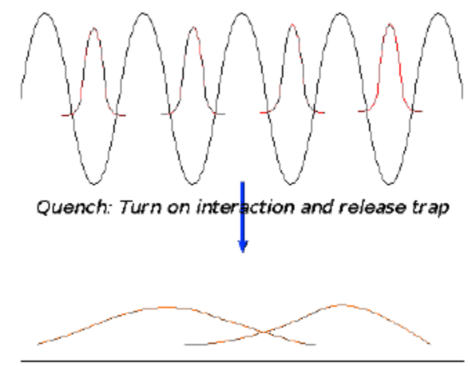
$c < 0$ *attractive*

Can be tune by Feshbach resonance

Equivalently:

$$H = -\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + 2c\sum_{j<l}\delta(x_j - x_l)$$

- **Initial condition I : bosons in a periodic optical lattice (Mott state)**



Initial condition and quench

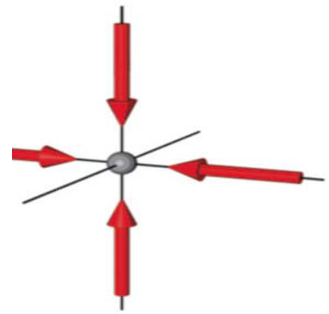
- **Initial condition II : bosons in a harmonic trap - condensate**

Ultracold Atoms – the Lieb Liniger model

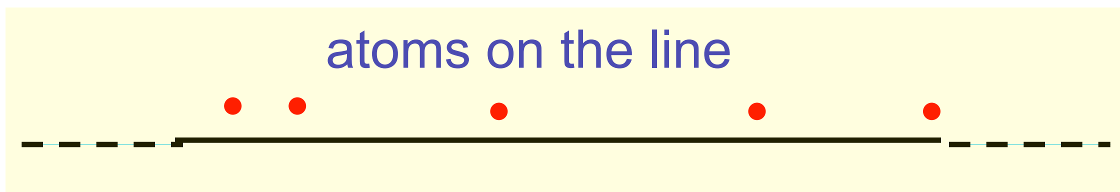
Gas of neutral atoms moving on the line and interacting with short-range interaction

Short range interaction among atoms: $V(x_1 - x_2) = c\delta(x_1 - x_2)$

$$H_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{j<l} \delta(x_j - x_l)$$



Bloch et al '08



Comment*:*

- Very short range interaction. Is it valid? For low densities, when

$$l = L/N \ll l_{\text{Van der Vaals}}$$

- The description of physics depends on the scale of observation

Bosonic system – BA solution

The N-boson eigenstatestate (Lieb-Liniger '67)

$$|\lambda_1, \cdots, \lambda_N\rangle = \int_y \prod_{i<j} Z^y_{ij}(\lambda_i - \lambda_j) \prod_j e^{i\lambda_j y_j}\; b^\dagger(y_j)|0\rangle$$

- **Eigenstates labeled by Momenta** $\lambda_1, \cdots, \lambda_N$

- ***Thermodynamics***: *impose PBC* $\longrightarrow$ *BA eqns* $\longrightarrow$ *momenta* $\begin{cases} \textit{real} & c > 0 \\ \textit{n-strings} & c < 0 \end{cases}$

- ***Dynamics*** *(infinite volume): momenta unconstrained*

- **Dynamic factor:** $Z^y_{ij}(\lambda_i - \lambda_j) = \dfrac{\lambda_i - \lambda_j - ic\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic} \quad = \begin{cases} 1 & y_i > y_j \\ S^{ij} & y_i < y_j \end{cases}$

- *The 2-particle S-matrix:* $S_{ij}(\lambda_i - \lambda_j) = \dfrac{\lambda_i - \lambda_j + ic}{\lambda_i - \lambda_j - ic}$ *enters when the particles cross*

- poles of the S-matrix at: $\lambda_i = \lambda_j + ic$

- **The energy eigenvalues**

$$H|\lambda_1, \cdots, \lambda_N\rangle = \sum_j \lambda_j^2\, |\lambda_1, \cdots, \lambda_N\rangle$$

bosonic system: contour representation

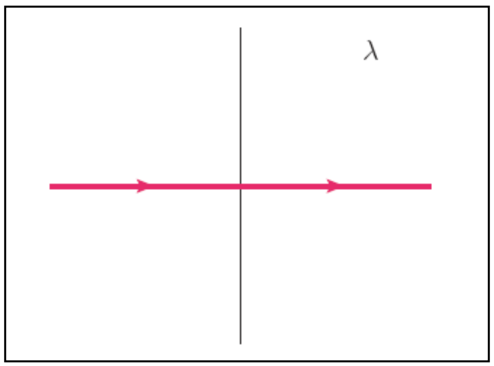
"Central theorem"

denote:

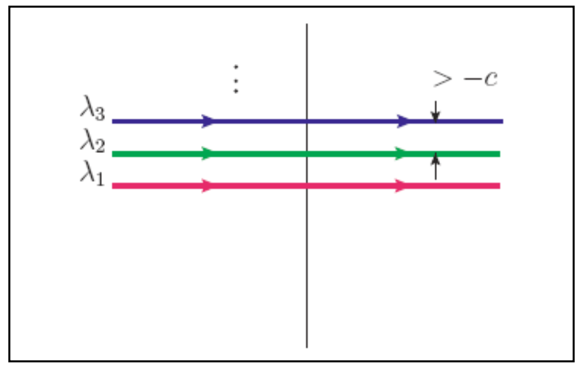
$$\theta(\vec{x}) = \theta(x_1 > x_2 > \cdots .x_N)$$

$$|\Phi_0\rangle = \int_x \Phi_0(\vec{x}) b^\dagger(x_N) \cdots b^\dagger(x_1)|0\rangle =$$

$$= \int_{x,y} \int_\lambda \theta(\vec{x})\Phi_0(\vec{x}) \prod_{i<j} \frac{\lambda_i - \lambda_j - ic\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic} \prod_j e^{i\lambda_j(y_j - x_j)} b^\dagger(y_j)|0\rangle$$



Repulsive c > 0



Attractive c < 0,

contour accounts for strings, bound states

It time evolves to:

$$|\Phi_0, t\rangle = \int_x \int_y \int_\lambda \theta(\vec{x})\Phi_0(\vec{x}) \prod_{i<j} \frac{\lambda_i - \lambda_j - ic\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic} \prod_j e^{-i\lambda_j^2 t} e^{i\lambda_j(y_j - x_j)} b^\dagger(y_j)|0\rangle$$

- **Expression contains full information about the dynamics of the system**

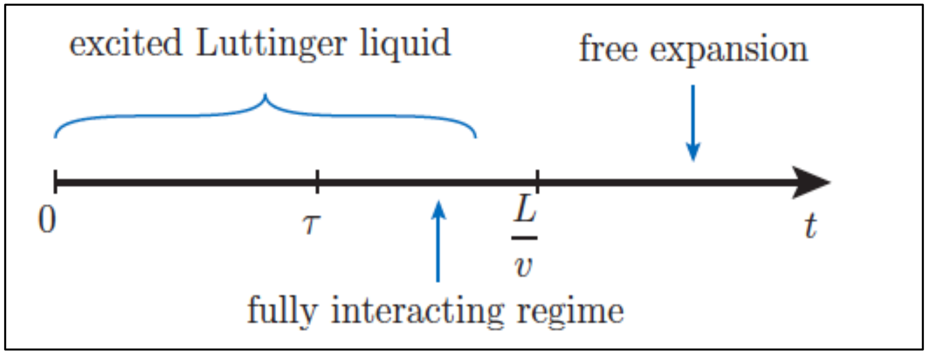
Yudson – finite density systems

Thermodynamic regime -

$N, L \to \infty, \;\; n = N/L \; fixed, \;\; t \ll L/v_{typ}$

Thermodynamic regime - constant density $t \ll L/v_{typ}$

Expansion of interacting gas - density decreases $t \gg L/v_{typ}$



Finite size Yudson representation:

- Claim: express any finite-size initial state (defined in a quadrant, e.g. bosons)

$$|\Phi_0\rangle = \int_{-L/2}^{L/2} dx_N \int_{-L/2}^{x_2} dx_1 \Phi\left(x_1..x_N\right) b^\dagger\left(x_N\right)..b^\dagger\left(x_1\right)|0\rangle\,,$$

in finite-size Yudson form (valid for repulsive interactions $c > o$) *(Goldstein, NA '13)*

$$|\Phi_0\rangle = \sum_{n_1=-\infty}^{\infty} \dots \sum_{n_N=-\infty}^{\infty} \mathcal{N}\left(\{k_n\}\right)^{-1} |k_{n_1}....k_{n_N}\rangle\left(k_{n_1}...k_{n_N}\right||\Phi_0\rangle\,.$$

- Less powerful than before: need solutions for the Bethe momenta $\{k_n\}$*, but not overlaps*

Allows the computation of time evolution of observable - Θ

$$G(\Theta, t; x_1...x_N; y_1...y_N) = \langle 0|b(y_1)..b(y_N)(\,e^{iHt}\,\Theta\,e^{-iHt}\,)b^\dagger(x_1)..b^\dagger(x_N)|0\rangle$$

Time evolution of an observable

- Evolution of the observable

$$G(\Theta, t; x_1 ... x_N; y_1 ... y_N) = \langle 0| b(y_1)..b(y_N)(\, e^{iHt}\,\Theta\, e^{-iHt}\,) b^\dagger(x_1)..b^\dagger(x_N)|0\rangle$$

- Inserted Yudson representation twice - overlaps simple plane waves

$$G\left(\Theta, t; x_1, x_2, ...x_N; y_1, ...y_N\right) =$$

$$= \sum_{n_1,...n_N} \frac{1}{\mathcal{N}\left(k_{n_1}....k_{n_N}\right)} \langle 0|\, b\left(y_1\right)...b\left(y_N\right)|k_{n_1}..k_{n_N}\rangle \times$$

$$\times \langle k_{n_1}...k_{n_N}|\,\Theta\,|q_{n_1},..q_{n_N}\rangle \times \sum_{n_1,..n_N} \frac{1}{\mathcal{N}\left(q_{n_1}...q_{n_N}\right)} \times$$

$$\times \left(q_{n_1}, ...q_{n_N}|\, b^\dagger\left(x_1\right)...b^\dagger\left(x_N\right)|0\rangle \prod_i e^{i\left(k_{n_i}^2 - q_{n_i}^2\right)t}$$

- Need matrix elements: $\langle k_{n_1}....k_{n_N}|\,\Theta\,|q_{n_1}...q_{n_N}\rangle$

Korepin, Bogoliubov, Izergin, Slavnov, Kitanine..

- Consider observable: $\Theta = e^{\alpha Q_{xy}}$ *with:* $Q_{xy} \equiv \int_x^y b^\dagger(z)\, b(z)\, dz$

Charge between x and y.

$\langle\Theta(t)\rangle$ *- generating function of density correlation function*

Time evolution of an observable

Consider: $Q_{xy} \equiv \int_x^y b^\dagger(z)\, b(z)\, dz$ *Charge between x and y.*

- Thermodynamic limit $L, N \to \infty, \quad t \to \infty \ll L/v_{typ}$

$$Exp\left(\alpha Q_{xy}(T)\right) = 1 + \frac{i\left(e^{\alpha}-1\right)}{2\pi}\left(\int dx_1\, dy_1 \frac{Exp\left(-\frac{i}{4T}\left(y_1^2 - x_1^2\right)\right)}{y_1 - x_1} \cdot \left(Exp\left(\frac{i(y_1-x_1)x}{2T}\right) - Exp\left(\frac{i(y_1-x_1)y}{2T}\right)\right)\left\langle b^+(y_1)\, Exp\left(i\pi\int_{x_1}^{y_1}\rho(z)dz\right) b(x_1)\right\rangle\right) -$$

$$-\frac{\left(e^{\alpha}-1\right)^2}{(2\pi)^2}\left(\int dx_1\, dy_1 \frac{Exp\left(-\frac{i}{4T}\left(y_1^2 - x_1^2\right)\right)}{y_1 - x_1} \cdot \left(Exp\left(\frac{i(y_1-x_1)x}{2T}\right) - Exp\left(\frac{i(y_1-x_1)y}{2T}\right)\right)\right) \times$$

$$\times\left(\int dx_2\, dy_2 \frac{Exp\left(-\frac{i}{4T}\left(y_2^2 - x_2^2\right)\right)}{y_2 - x_2} \cdot \left(Exp\left(\frac{i(y_2-x_2)x}{2T}\right) - Exp\left(\frac{i(y_2-x_2)y}{2T}\right)\right)\right) \times$$

$$\times\left\langle \operatorname{sgn}(y_2-y_1)\operatorname{sgn}(x_2-x_1)\, b^+(y_1)\, b^+(y_2)\, Exp\left(i\pi\int_{x_1}^{y_1}\rho(z)dz\right) Exp\left(i\pi\int_{x_2}^{y_2}\rho(z)dz\right) b(x_1)\, b(x_2)\right\rangle -$$

$$-\frac{i2\left(e^{2\alpha}-1\right)}{(2\pi)^2 c}\left(\int dx_1\, dy_1 \frac{Exp\left(-\frac{i}{4T}\left(y_1^2 - x_1^2\right)\right)}{y_1 - x_1} \cdot Exp\left(\frac{i(y_1-x_1)x}{2T}\right)\int dx_2\, dy_2 \frac{Exp\left(-\frac{i}{4T}\left(y_2^2 - x_2^2\right)\right)}{T} \cdot Exp\left(\frac{i(y_2-x_2)x}{2T}\right) -\right.$$

$$\left.\int dx_1\, dy_1 \frac{Exp\left(-\frac{i}{4T}\left(y_1^2 - x_1^2\right)\right)}{y_1 - x_1} \cdot Exp\left(\frac{i(y_1-x_1)y}{2T}\right)\int dx_2\, dy_2 \frac{Exp\left(-\frac{i}{4T}\left(y_2^2 - x_2^2\right)\right)}{T} \cdot Exp\left(\frac{i(y_2-x_2)y}{2T}\right)\right) \times$$

$$\times\left\langle \operatorname{sgn}(y_2-y_1)\operatorname{sgn}(x_2-x_1)\, b^+(y_1)\, b^+(y_2)\, Exp\left(i\pi\int_{x_1}^{y_1}\rho(z)dz\right) Exp\left(i\pi\int_{x_2}^{y_2}\rho(z)dz\right) b(x_1)\, b(x_2)\right\rangle + \ldots..$$

1. Up to $\alpha^3, 1/c^2$
2. ***All expectation values taken w.r.t initial state***
3. ***Valid for any initial state***

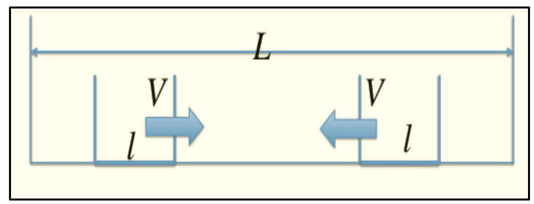
Time evolution- flow chart

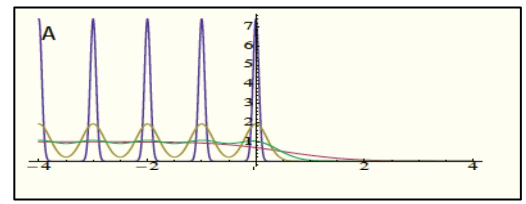
Monster Expression valid for any *time* and any *initial state*

- Some applications:

Initial state translational invariant
example: **Mott insulator**

Initial state not translational invariant
example: **Newton's cradle**, **Domain wall**





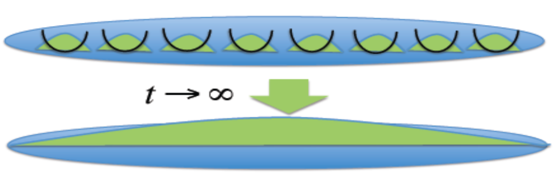
Equilibrates to local GGE (c >0)

Equilibrates but not to local GGE (c < 0)

- Local GGE generally fails when bound states (strings) are present: att-LL, Hubbard, Anderson, Kondo, Sine-Gordon …(XXZ exception? Prosen '15)

Goldstein, NA '13

Universal correlations (if low YY entropy)



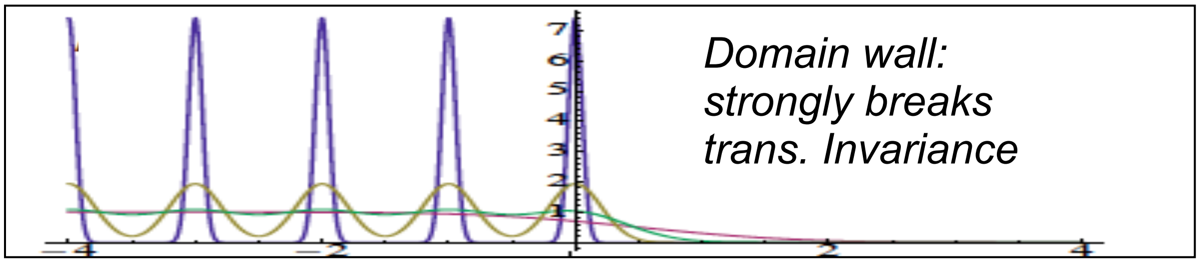
System does not equilibrate:
currents, *local* entropy production

Evolution to NESS: Domain wall

***Example**: time evolution from a non-trans. invariant initial state (no equilibration)*

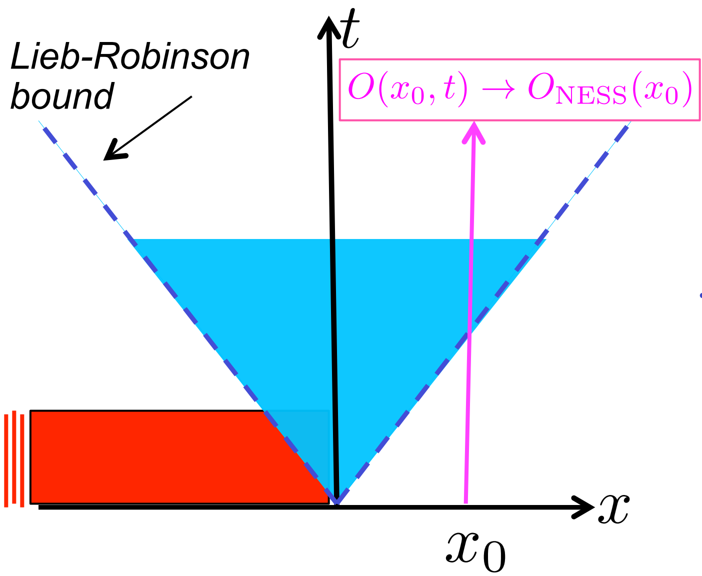
$$|\Psi(t=0)\rangle = \prod_{j=0}^{\infty}\int_{-\infty}^{\infty}\varphi(x+jl)\, b^{\dagger}(x)\,|0\rangle$$

with: $\varphi(x) = \dfrac{e^{-x^2/\sigma}}{(\pi\sigma/2)^{1/4}}$



- System evolves to NESS $\rho(x,t) \to$

(G Goldstein, NA '13)



$$\rho(x,t) \to \begin{cases} \dfrac{1}{l}\left(1+\displaystyle\sum_{s=1}^{\infty} e^{-\pi^2(8t^2/\sigma+\sigma/2)s^2/l^2}\cos(2\pi s x/l)\right) & x \ll -\sqrt{8t^2/\sigma} \\ \dfrac{1}{l}\left(\dfrac{1}{2}-\dfrac{4\pi}{cl}\right) & \sigma \ll |x| \ll t \\ \frac{1}{l}\left(\frac{1}{2}Erfc\left(\frac{x}{\sqrt{A}}\right)+\right. \\ \quad +\frac{16}{\pi c l}e^{-x^2/A}\left(\frac{1}{2}\sqrt{\pi}\frac{x}{\sqrt{A}}Erfc\left(\frac{x}{\sqrt{A}}\right)-\frac{1}{2}e^{-x^2/A}\right) \\ \quad \left.+\frac{16}{\pi c l}\frac{\pi}{2}\left(1-\frac{1}{2}Erfc\left(\frac{x}{\sqrt{A}}\right)\right)Erfc\left(\frac{x}{\sqrt{A}}\right)\right) & x \sim t,\ A \equiv 8t^2/\sigma+\sigma/2 \\ 0 & x \gg \sqrt{8t^2/\sigma} \end{cases}$$

Outside the cone ($x \ll -\sqrt{8t^2/\sigma}$)

Nonequilibrium steady state (NESS)
Independent of x, t; interaction effects ($\sigma \ll |x| \ll t$)

Crossover regime ($x \sim t$, $A \equiv 8t^2/\sigma+\sigma/2$)

Outside the cone ($x \gg \sqrt{8t^2/\sigma}$)

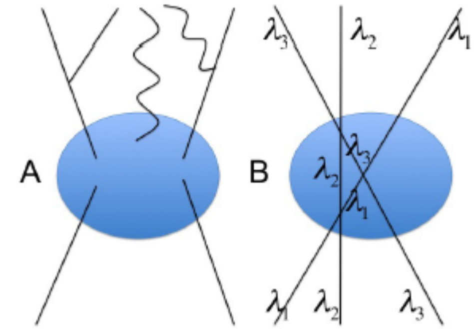
Evolution to GGE: trans. invariance, diagonal ensemble

GGE if: 1. Diagonal ensemble 2. Operator can be expanded

i. For trans. invariant initial states : $\langle\Theta\rangle\,(t\to\infty) = Tr\rho_D\Theta$

$$\rho_D = \sum p_{\{k\}}\,|\{k_i\}\rangle\,\langle\{k_i\}| \quad \text{with} \quad p_k = |\langle\{k\}|\Phi_0\rangle|^2$$

ii. The diagonal element can be Taylor expanded (~ GETH)



$$\langle\{k_i\}|\,\Theta\,|\{k_i\}\rangle = c_0 + c_1\sum k_i + c_{1,1}\sum k_i k_j + c_2\sum k_i^2 + ..$$

iii. Thus:

$$\langle\Theta\rangle \to c_0 + c_1\,\langle I_1\rangle + c_{1,1}\,\langle I_1^2\rangle + c_2\,\langle I_2\rangle + ..$$

with: $\langle I_1\rangle = \sum p_{\{\lambda\}}\sum k_i,\ \ \langle I_1^2\rangle = \sum p_{\{k\}}\sum k_i k_j,\ \ \langle I_2\rangle = \sum p_{\{k\}}\sum k_i^2$ ...

iv. Equivalently:

$$\langle\Theta\rangle = Tr\rho_{GGE}\Theta$$

with $\rho_{GGE} \sim e^{-\sum_n \beta_n I_n}$

Actually need also short correlations among I_n

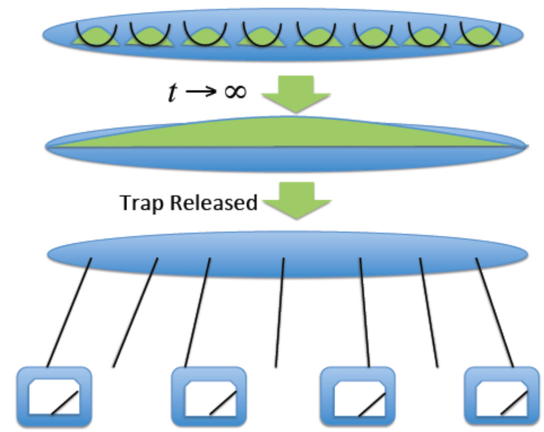
v. For string-momenta Taylor expansion may break down

Time evolution- interaction quench from a Mott state

Example: **Quenching from a Mott insulator to a Lieb-Liniger Liquid (GGE):**

$$t=0 \quad |\Phi_0\rangle = \prod_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(x+jl)\,b^{\dagger}(x)\,|0\rangle$$

with $\varphi(x)=\frac{e^{-x^2/\sigma}}{(\pi\sigma/2)^{1/4}}$



$$t\to\infty \quad |\Phi_0\rangle \to \rho_{GGE}$$ ***How to describe?***

- **GGE** *corresponds to a pure state:* $tr\left[\Theta\rho_{GGE}\right]=\left\langle\vec{k}_0\middle|\Theta\middle|\vec{k}_0\right\rangle$

with the eigenstate $|\vec{k}_0\rangle$ *given by* $\rho_p(k),\rho_t(k)$ *satisfying (Caux) :*

$$\rho_t(k)=\frac{1}{2\pi}+\frac{1}{2\pi}\int dqK(k,q)\,\rho_p(q) \quad \text{with} \quad K(k,q)=\frac{2c}{c^2+(k-q)^2}$$

- But $\langle I_n\rangle_{t=0}=\langle I_n\rangle_{t\to\infty}$

$$L\int dk\rho_p(k)\,k^n=I_n(t=0).=\frac{L}{l}\left(\frac{2}{\sigma}\right)^{\frac{n}{2}}\frac{n!}{2^{\frac{n}{2}}\left(\frac{n}{2}!\right)}$$

$\longrightarrow$ *Final distribution*

$$\begin{cases}\rho_p(k)=\frac{\sigma^{\frac{1}{2}}}{\pi^{\frac{1}{2}}l}\exp\left(-\frac{k^2\sigma}{2}\right)\\ \rho_t(k)\cong\frac{1}{2\pi} \quad \text{for} \quad l\gg\sqrt{\sigma}\end{cases}$$

→ **The occupation probability** $f(k)\equiv\frac{\rho_p(k)}{\rho_t(k)}\cong\frac{2\sqrt{\pi\sigma}}{l}\exp\left(-\frac{k^2\sigma}{2}\right)$

Time evolution – interaction quench from a Mott state

Can compute various correlation functions:

1. $\left\langle b^{\dagger}(0)\, b^{\dagger}(0)\, b(0)\, b(0)\right\rangle \cong 2\int \frac{dk_1}{2\pi}\int \frac{dk_2}{2\pi} f(k_1)\, f(k_2)\, \frac{(k_2-k_1)^2}{(k_2-k_1)^2+c^2} + \ldots..$

$$= \frac{2}{l^2} - \frac{2\sqrt{\pi c^2\sigma}}{l^2}\left[\exp\left(\frac{\sigma c^2}{4}\right) Erfc\left(\sqrt{\frac{\sigma c^2}{4}}\right)\right] \rightarrow \begin{cases} \cong \frac{2}{l^2} & c^2\sigma \ll 1 \\ \cong \frac{1}{l^2 c^4 \sigma^2} & c^2\sigma \gg 1 \end{cases}$$

Suppression of density correlations, measurable by Time of Flight experiments

2. $\left\langle b^{\dagger}(0)\, b^{\dagger}(0)\, b^{\dagger}(0)\, b(0)\, b(0)\, b(0)\right\rangle \cong 6\int dk_1 dk_2 dk_3 f(k_1)\, f(k_2)\, f(k_3)\, \frac{(k_2-k_1)^2}{(k_2-k_1)^2+c^2}\frac{(k_3-k_1)^2}{(k_3-k_1)^2+c^2}\frac{(k_3-k_2)^2}{(k_3-k_2)^2+c^2}$

Strong suppression of three body decay rates, measurable through trap loss or third moment of particle number (Bouchoule'10)

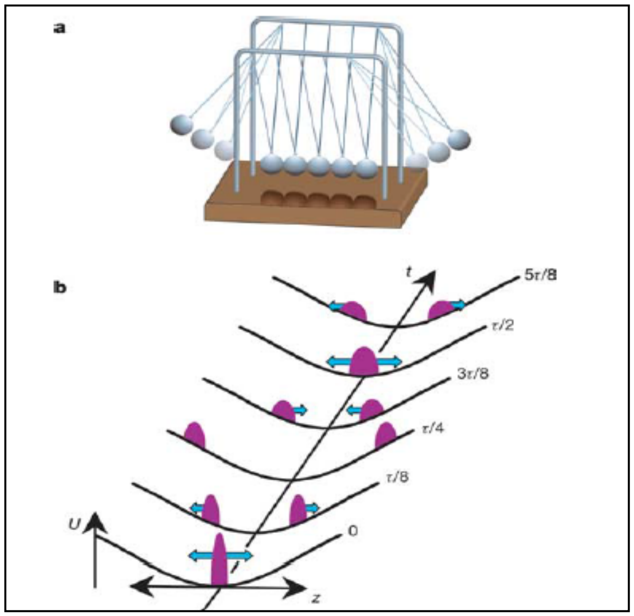
$$\rightarrow \begin{cases} \cong \frac{6}{l^3} & c^2\sigma \ll 1 \\ \cong \frac{9\times 2^{\frac{9}{2}}}{l^3 c^6 \sigma^3} & c^2\sigma \gg 1 \end{cases}$$

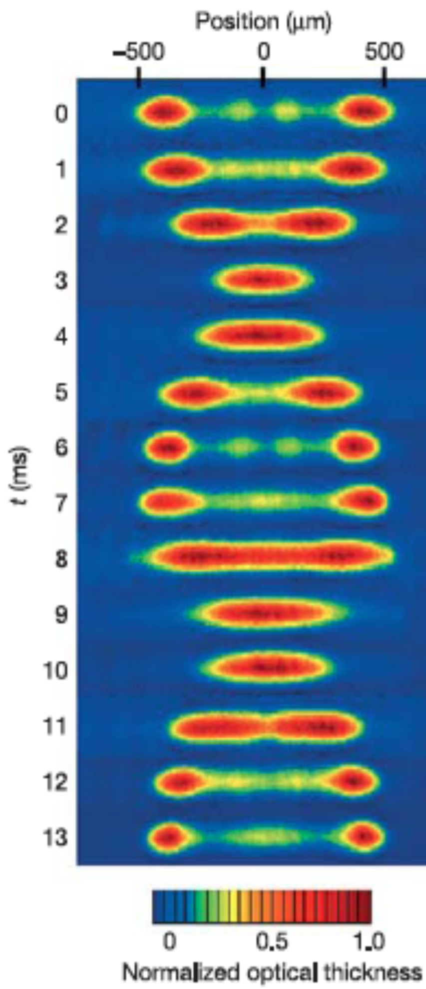
3. $\langle \rho(x)\, \rho(0)\rangle \cong \rho^2 + \frac{1}{4\pi^2 e^2 l^2}\exp\left(-\frac{x^2}{\sigma}\right)$ *for* $l \gg \sqrt{\sigma}$

Gaussian decay of density-density function

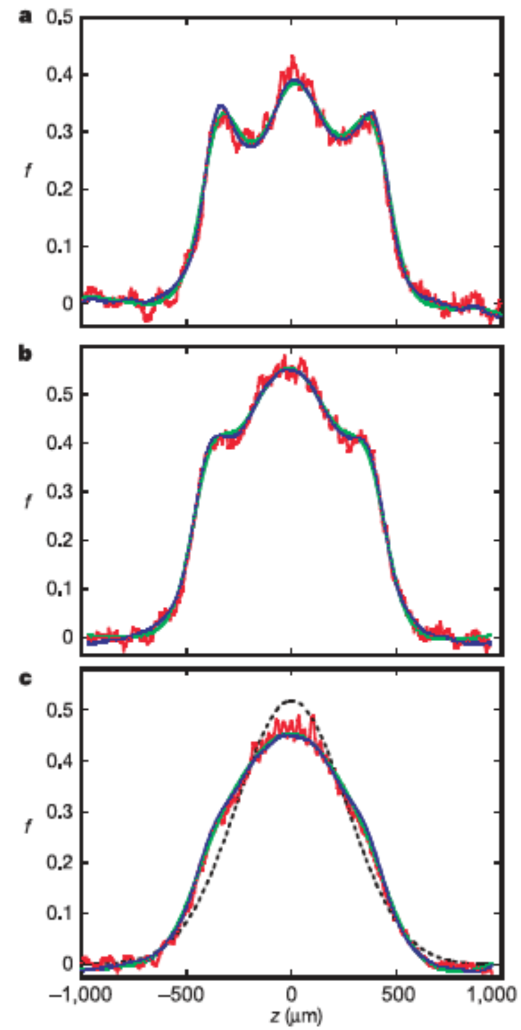
Newton's Cradle - simplified

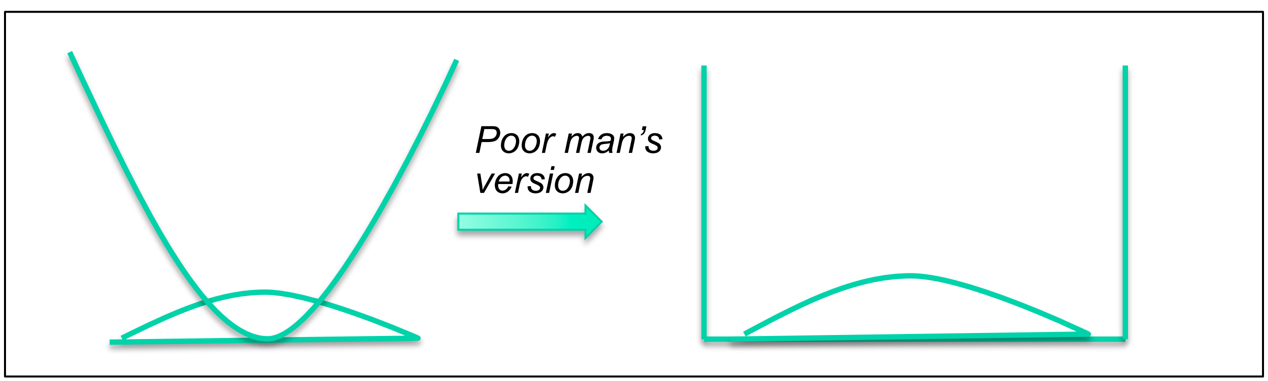
Kinoshita, T. Wenger, D. S. Weiss, Nature (2006)





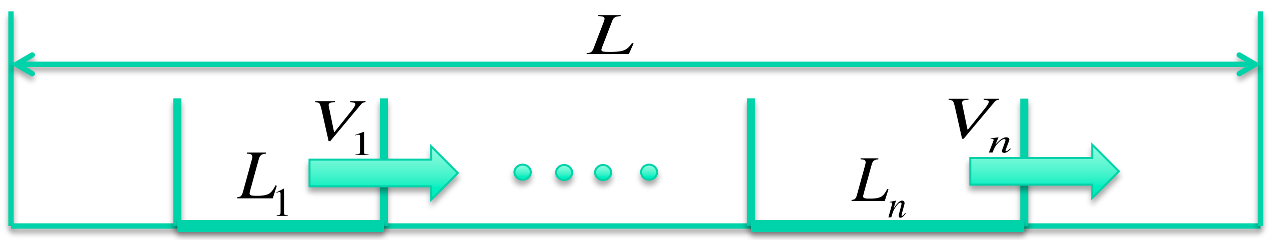
averaged momentum distributions





Newton's Cradle

Initial state: several moving boxes within a larger container
– poor man's version of Weiss' experiment



System does not equilibrate, but long time average is a diagonal ensemble

$$\langle\Theta\rangle_T \equiv \frac{1}{T}\int_0^T dt\langle\Psi|e^{iHt}\Theta e^{-iHt}|\Psi\rangle = \frac{1}{T}\sum_\lambda\sum_\kappa \frac{e^{i(E_\lambda-E_\kappa)T}-1}{i\left(E_\lambda-E_\kappa\right)}\langle\Psi\mid\lambda\rangle\langle\lambda|\Theta|\kappa\rangle\langle\kappa\mid\Psi\rangle$$
$$\cong\sum_\lambda\langle\Psi\mid\lambda\rangle\langle\lambda|\Theta|\lambda\rangle\langle\lambda\mid\Psi\rangle$$

LL in a box *(Gaudin)*

$$\psi\left(|k_1|,..|k_N|\right)=\sum_{\{\varepsilon\}}C\{\varepsilon\}\,\bar{\psi}\left(\varepsilon_1|k_1|,..\varepsilon_N|k_N|\right)$$

GGE for the average - *match* conservation laws:

$$L\int dk\rho_p^f(k)\,k^{2n}=\sum L_i\int\rho_p^i(k)\left(k+\frac{1}{2}V_i\right)^{2n}$$

All even charges are conserved $\{I_{2n}\}$

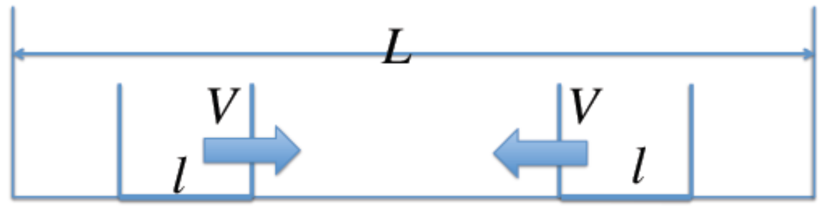
Solution for final quasi-particle density

$$\rho_p^f(k)=\sum\frac{L_i}{2L}\left(\rho_p^i\left(k+\frac{1}{2}V_i\right)+\rho_p^i\left(k-\frac{1}{2}V_i\right)\right)$$

Newton's Cradle in a box

Initial state:

Two boxes of length l each containing N bosons in a given initial state ρ^i moving towards each other at speed V



1. $$\rho^i_{gs}(k) = \theta(-k_F, k_F)\frac{1}{2\pi}\left(1 + \frac{2k_F}{\pi c}\right) + o(\frac{k_F}{c})....$$

2. $$\rho^i_{\mathrm{BEC}}(x) = \frac{\tau \frac{d}{d\tau} a(x,\tau)}{1 + a(x,\tau)} \qquad x = \frac{k}{c}, \tau = \frac{n}{c}$$

Caux et al '12

$$a(x,\tau) = \frac{2\pi\tau}{x \sinh(2\pi x)} J_{1-2ix}(4\sqrt{\tau}) J_{1+2ix}(4\sqrt{\tau})$$

- The velocity distribution (measurable): $$P(v,t) = \int dx e^{-i\frac{v}{2}x} \langle b^\dagger(x) b(0)\rangle_t$$

- *The field-field correlation given in terms of the occupation probability* $$f(k) = \frac{\rho_p(k)}{\rho_t(k)}$$

$$\langle b^\dagger(x) b(0)\rangle_{t\to\infty} = \int \frac{dk}{2\pi} f(k) e^{-ikx} \omega(k) \exp\left(-x \int du\, f(k) p_u(k)\right)$$

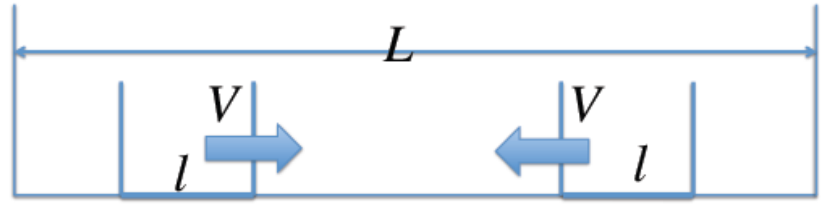
Korepin, Izergin '87

with:

$$2\pi p_u(k) = -\frac{k-u+ic}{u-k+ic}\exp\left(-\int ds\, f(s) K(u,s) p_s(k)\right) - 1 \qquad \omega(k) = \exp\left(-\frac{1}{2\pi}\int dq K(k,q) f(q)\right) \qquad K(k,q) = \frac{2c}{(k-q)^2 + c^2}$$

Newton's Cradle in a box

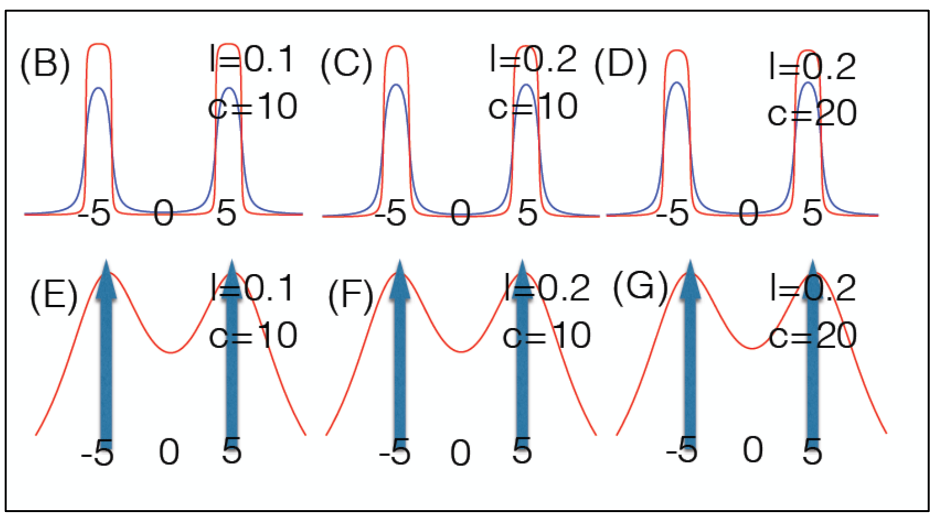
The velocity distribution:



$$P\left(v,t\right)=\int dx e^{-i\frac{v}{2}x}\left\langle b^{\dagger}\left(x\right)b\left(0\right)\right\rangle_t$$

ρ^i - ground state $\longrightarrow$ $P\left(v\right)\sim A_L\frac{\exp\left(-\frac{F_L}{\pi c}\right)}{2\pi}\sum_{i,j=\pm}\left(-1\right)^j\arctan A_{i,j}\left(v\right)$

ρ^i - BEC $\longrightarrow$ $P\left(v\right)\sim nB_L\frac{\exp\left(-\frac{G_L}{\pi c}\right)}{\pi}\left(\frac{H_L}{H_L^2+\frac{1}{4}\left(v-VK_L\right)}+\frac{H_L}{H_L^2+\frac{1}{4}\left(v+VK_L\right)}\right)$



(G Goldstein, NA '15)

$A_{\pm\pm}(v)=C_L\left(\left(1-F_L\frac{\exp\left(-2F_L/\pi c\right)}{\pi c}\right)\left(\pm\frac{V}{2}\pm k_F\right)+\frac{v}{2}\right)$ $F_L=4k_FA_L$ $K_L=\left(1-G_L\frac{\exp\left(-2G_L/\pi c\right)}{\pi c}\right)$ $C_L=\frac{2\pi}{4k_FA_L\left(1+\exp\left(-\frac{2F_L}{\pi c}\right)\right)}$

$A_L=\frac{l}{L}\left(1+\frac{2k_F}{\pi c}\left(1-\frac{2l}{L}\right)\right)$ $B_L=\frac{l}{L}\frac{1}{\frac{1}{2\pi}+\frac{2N}{\pi cL}}$ $G_L=2nB_L$ $H_L=\frac{G_L}{2\pi}\left(1+\exp\left(-\frac{2G_L}{\pi c}\right)\right)+2n\left(1-G_L\frac{\exp\left(-2G_L/\pi c\right)}{\pi c}\right)$

Failure of GGE for models with strings

- *Thus far: Evolution of repulsive* $(c > 0)$ *Lieb-Liniger* $\longrightarrow$ *GGE, GGGE, etc*
- *Not so for* $c < 0$, *attractive Lieb-Liniger*

Failure of GGE for systems with bound states

e.g. *Attractive Lieb - Liniger, Gaudin-Yang, Hubbard, sine-Gordon…*

- *Momenta fall into n-string configurations (bound states):* $k_j = k_0 + \frac{ic}{2}(n-2j)\,,\; j = 1...n$
- *Described by n-string densities,* $\rho_p^n(k)$

- *GGE determine by:* $\langle I_i \rangle_{final} \equiv Tr\left\{\rho_{GGE} I_i\right\} = \langle I_i\,(t=0)\rangle \equiv \langle I_i \rangle_{initial} = I_i^0$

Need to solve:

$$I_i\left\{\rho_p^1, \rho_p^2, ..\right\} = \sum_{n=0}^{\infty}\sum_{l=0}^{i} J_l^n \left(\frac{ic}{2}\right)^{i-l} \sum_{j=0}^{n}(n-2j)^{i-l} = I_i^0$$

Contribution to I^i *of single n-string centered at* k_0 *:* $\sum_{j=0}^{n}\left(k_0 + \frac{ic}{2}(n-2j)\right)^i = \sum_{l=0}^{i} k_0^{i-l}\binom{i}{l}\left(\frac{ic}{2}\right)^i \sum_{j=1}^{n}(n-2j)^i$

Integral over positions : $J_i^n = \int dk \rho_p^n(k)\, k^i$

- ***Claim: There are infinitely many solutions, each corresponding to different correlation functions***

Local GGE fails: *Need full time evolution*

Does GGE require non-local charges? Mazur??

2. The Heisenberg Chain: Theory and Experiment

The XXZ Hamiltonian

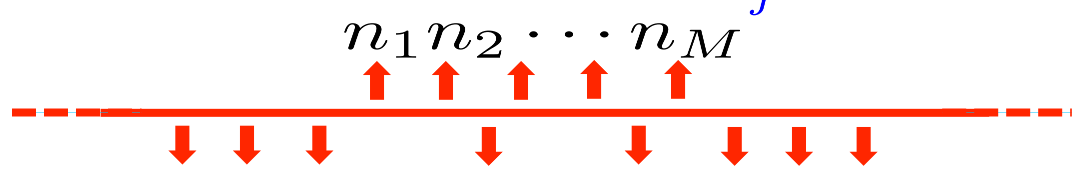
$$H = J\sum_j \sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta(\sigma_j^z\sigma_{j+1}^z - 1)$$

The phase diagram

Quench from initial states: *Recall GGE fails*

$$|\Psi_0\rangle = \theta(n_1 > n_2 > \dots n_N)\prod_j \sigma_{n_j}^+|0\rangle$$

e. g.

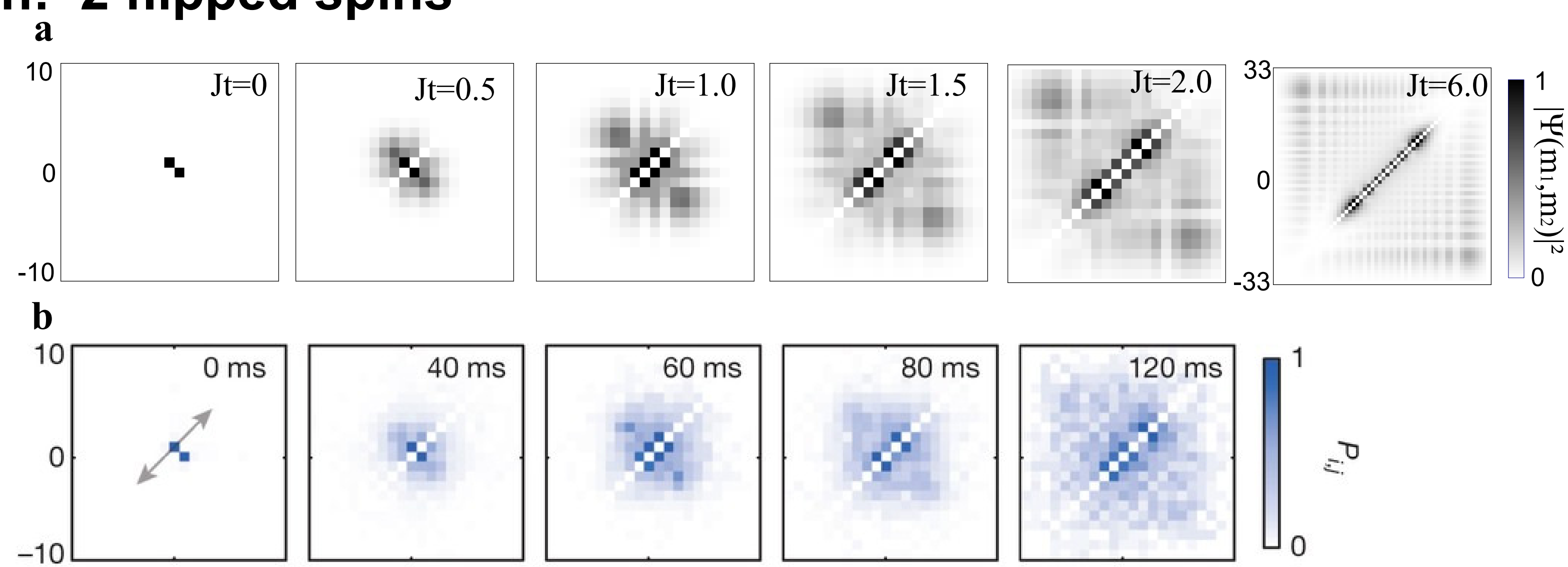


Time evolution: 2 flipped spins

Theory:

Experiment:

Munich group '13



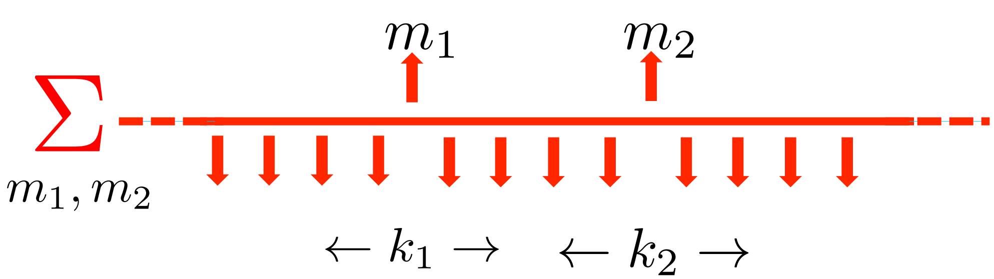
Eigenstates of the Heisenberg Chain

Eigenstates of the XXZ (M flipped spins)

$$|k\rangle = \sum_{\{m_j\}} \mathcal{S} \prod_{i<j} [\theta(m_i - m_j) + s(k_i, k_j)\theta(m_j - m_i)] \prod_j e^{ik_j m_j} \sigma^+_{m_j} |0\rangle$$

$$s(k_i, k_j) = e^{i\phi(k_i,k_j)} = -\frac{1 + e^{ik_i + ik_j} - 2\Delta e^{ik_i}}{1 + e^{ik_i + ik_j} - 2\Delta e^{ik_j}}$$

$$E = 4J \sum_{j=1}^{M} (\Delta - \cos k_j)$$



Time evolution of the XXZ magnet

i. Critical region $-1 < \Delta < 0$ $\quad \Delta = -\cos\mu \quad (0 < \mu < \frac{\pi}{2})$

Reparametrize: $\Delta \to \mu, \quad k \to \alpha$

$$e^{ik} \to \frac{\sinh\frac{i\mu-\alpha}{2}}{\sinh\frac{i\mu+\alpha}{2}} \longrightarrow s(k_1,k_2) \to \frac{\sinh(\frac{\alpha_1-\alpha_2}{2} - i\mu)}{\sinh(\frac{\alpha_1-\alpha_2}{2} + i\mu)}$$

$$E(k) \to E(\alpha) = \frac{4J\sin^2\mu}{\cosh\alpha - \cos\mu}$$

The contour expression of the initial state:

$$|\Psi_0\rangle = \theta(n_1 > n_2 > ...n_N)\prod_j \sigma^+_{n_j}|0\rangle$$

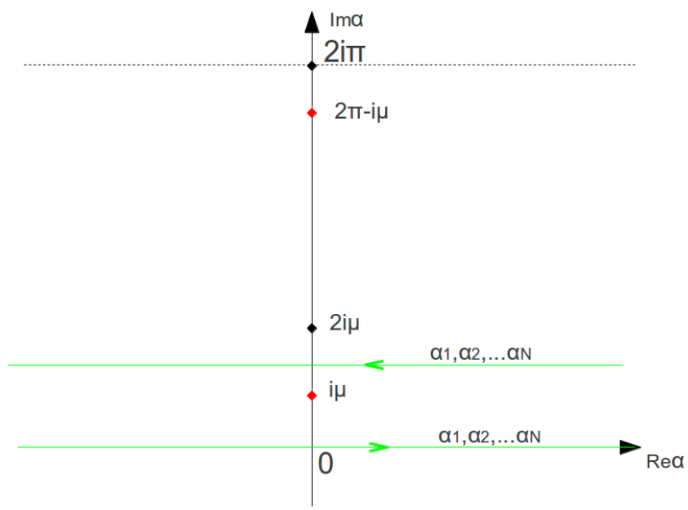
Expanded in terms of eigenstates

$$|\Psi_0\rangle = \sum_{\{m_j\}}\int_{\gamma_j}\prod_j[\frac{d\alpha_j}{2\pi}\frac{\sin\mu}{2\sinh\frac{\alpha_j+i\mu}{2}\sinh\frac{\alpha_j-i\mu}{2}}]\prod_j[\frac{\sinh(\frac{i\mu-\alpha_j}{2})}{\sinh(\frac{i\mu+\alpha_j}{2})}]^{m_j-n_j}$$

$$\times\prod_{i<j}[\theta(m_i-m_j) + \frac{\sinh(\frac{\alpha_i-\alpha_j}{2} - i\mu)}{\sinh(\frac{\alpha_i-\alpha_j}{2} + i\mu)}\theta(m_j-m_i)]\prod_j \sigma^+_{m_j}|0\rangle$$

Evolution of the XXZ magnet

The contour:



The time evolved state:

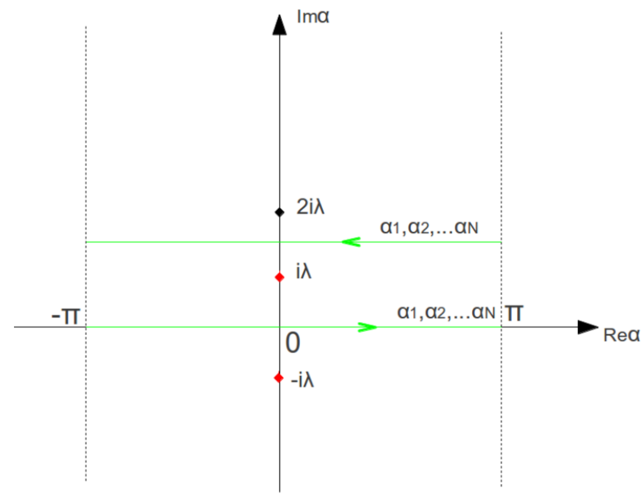
$$|\Psi(t)\rangle = \sum_{\{m_j\}} \int_{\gamma_j} \prod_j \left[\frac{d\alpha_j}{2\pi} \frac{\sinh\lambda}{2\sin\frac{\alpha_j+i\lambda}{2}\sin\frac{\alpha_j-i\lambda}{2}}\right] \prod_{i<j}\left[\theta(m_i-m_j) + \frac{\sin(\frac{\alpha_i-\alpha_j}{2}-i\lambda)}{\sin(\frac{\alpha_i-\alpha_j}{2}+i\lambda)}\theta(m_j-m_i)\right]$$

$$\times \prod_j \left[\frac{\sin(\frac{i\lambda-\alpha_j}{2})}{\sin(\frac{i\lambda+\alpha_j}{2})}\right]^{m_j-n_j} e^{-iE(\alpha_j)t}\sigma^+_{m_j}|0\rangle$$

Evolution of the XXZ magnet

ii. $\Delta < -1$ *Ferromagnetic regime*

The contour:



$$\Delta = -\cosh\lambda \to \lambda > 0$$

Reparametrize:

$$e^{ik} \to \frac{\sin\frac{i\lambda-\alpha}{2}}{\sin\frac{i\lambda+\alpha}{2}}$$

$$s(k_1,k_2) \to \frac{\sin(\frac{\alpha_1-\alpha_2}{2} - i\lambda)}{\sin(\frac{\alpha_1-\alpha_2}{2} + i\lambda)}$$

$$E(k) \to E(\alpha) = -\frac{4J\sinh^2\lambda}{\cos\alpha - \cosh\lambda}$$

The time evolved state

$$|\Psi(t)\rangle = \sum_{\{m_j\}} \int_{\gamma_j} \prod_j [\frac{d\alpha_j}{2\pi} \frac{\sinh\lambda}{2\sin\frac{\alpha_j+i\lambda}{2}\sin\frac{\alpha_j-i\lambda}{2}}] \prod_{i<j} [\theta(m_i-m_j) + \frac{\sin(\frac{\alpha_i-\alpha_j}{2}-i\lambda)}{\sin(\frac{\alpha_i-\alpha_j}{2}+i\lambda)}\theta(m_j-m_i)]$$

$$\times \prod_j [\frac{\sin(\frac{i\lambda-\alpha_j}{2})}{\sin(\frac{i\lambda+\alpha_j}{2})}]^{m_j-n_j} e^{-iE(\alpha_j)t} \sigma^+_{m_j} |0\rangle$$

Evolution of the XXZ magnet

Some results - *local magnetization and bound states*

- *Spin currents*

Start from

$$|\Psi_0\rangle = \sigma^-_{-1}\sigma^-_0\sigma^-_{+1}|\Uparrow\rangle$$

Calculate:

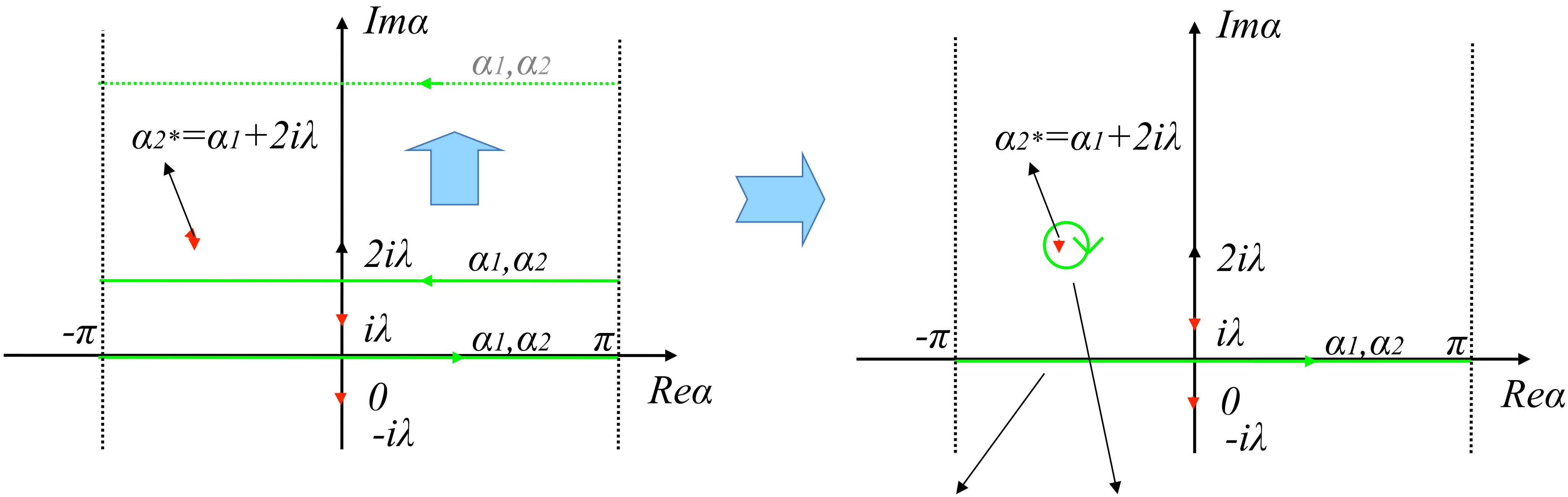
$$M(n,t) = \langle\Psi(t)|\sigma^z_n|\Psi(t)\rangle$$

$$I(n,t) = i\langle\Psi(t)|(\sigma^+_n\sigma^-_{n+1} - \sigma^-_n\sigma^+_{n+1})|\Psi(t)\rangle$$

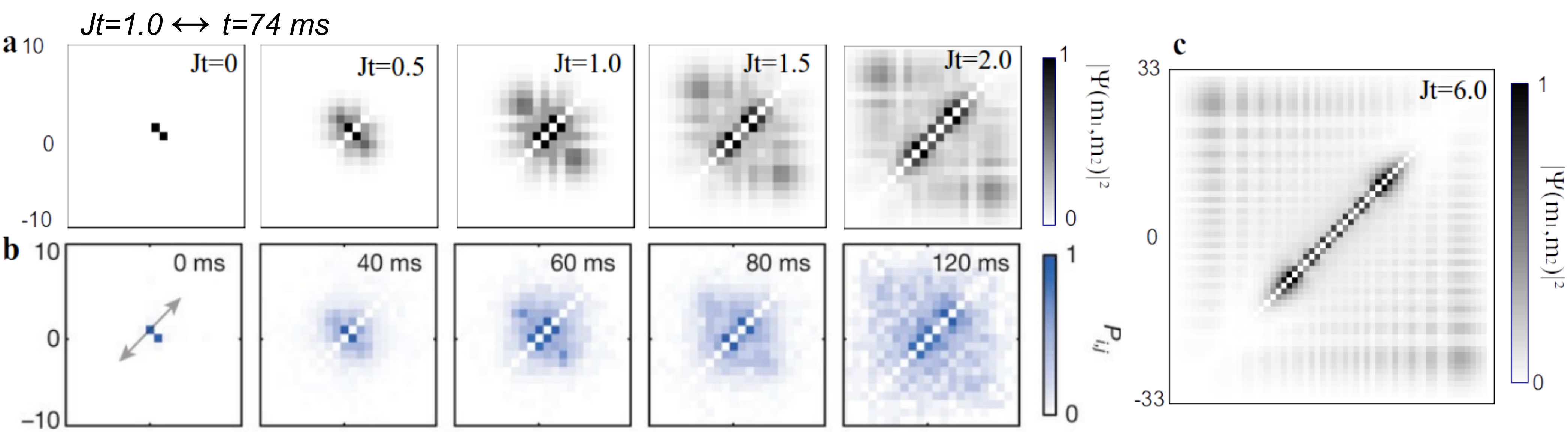
For different values of anisotropy Δ

- as the anisotropy increases the weight of the bound states increases

Contour Shift and Bound States



$$\Psi^{1,0}(m_1, m_2; t) = \Psi_{magn}(m_1, m_2; t) + \Psi_{bound}(m_1, m_2; t)$$

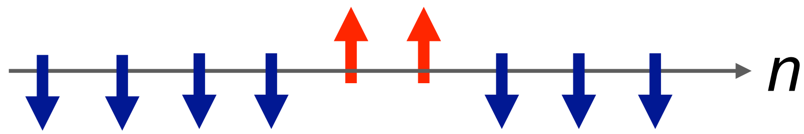


b. T. Fukuhara et al, Nature **502**, 76 (2013)

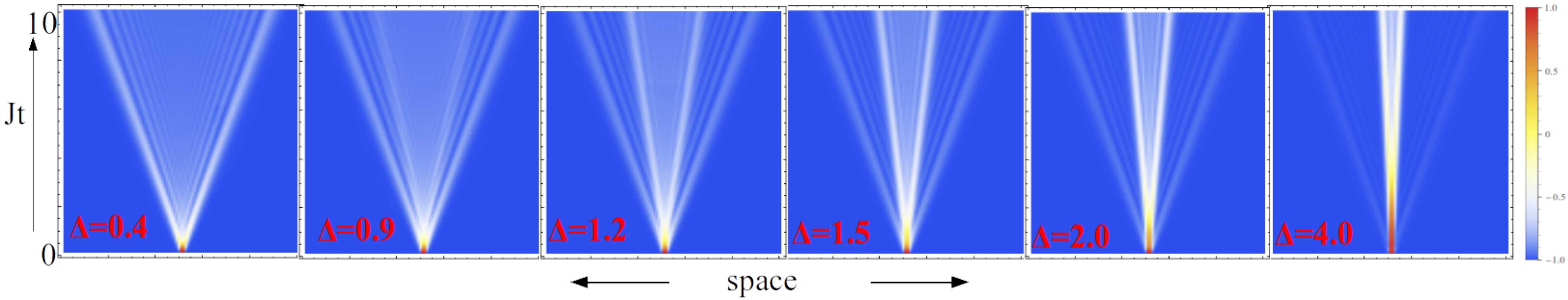
Observables

- **Local Magnetization** $M(n,t) \equiv \langle \Psi(t)|\sigma_n^z|\Psi(t)\rangle$

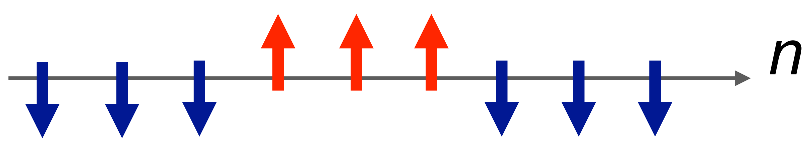
$$|\Psi_0\rangle = \sigma_1^+ \sigma_0^+ |\Downarrow\rangle =$$

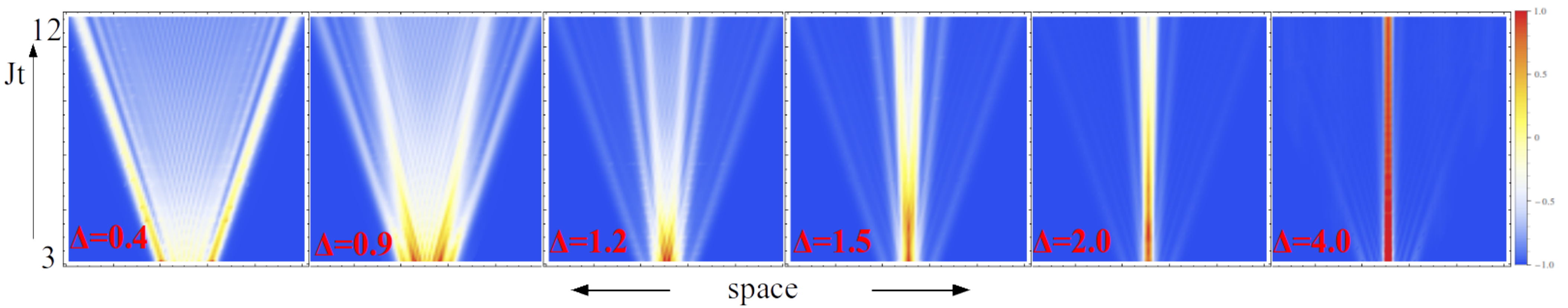


(cf. Ganahl et al. '12)



$$|\Psi_0\rangle = \sigma_1^+ \sigma_0^+ \sigma_{-1}^+ |\Downarrow\rangle =$$

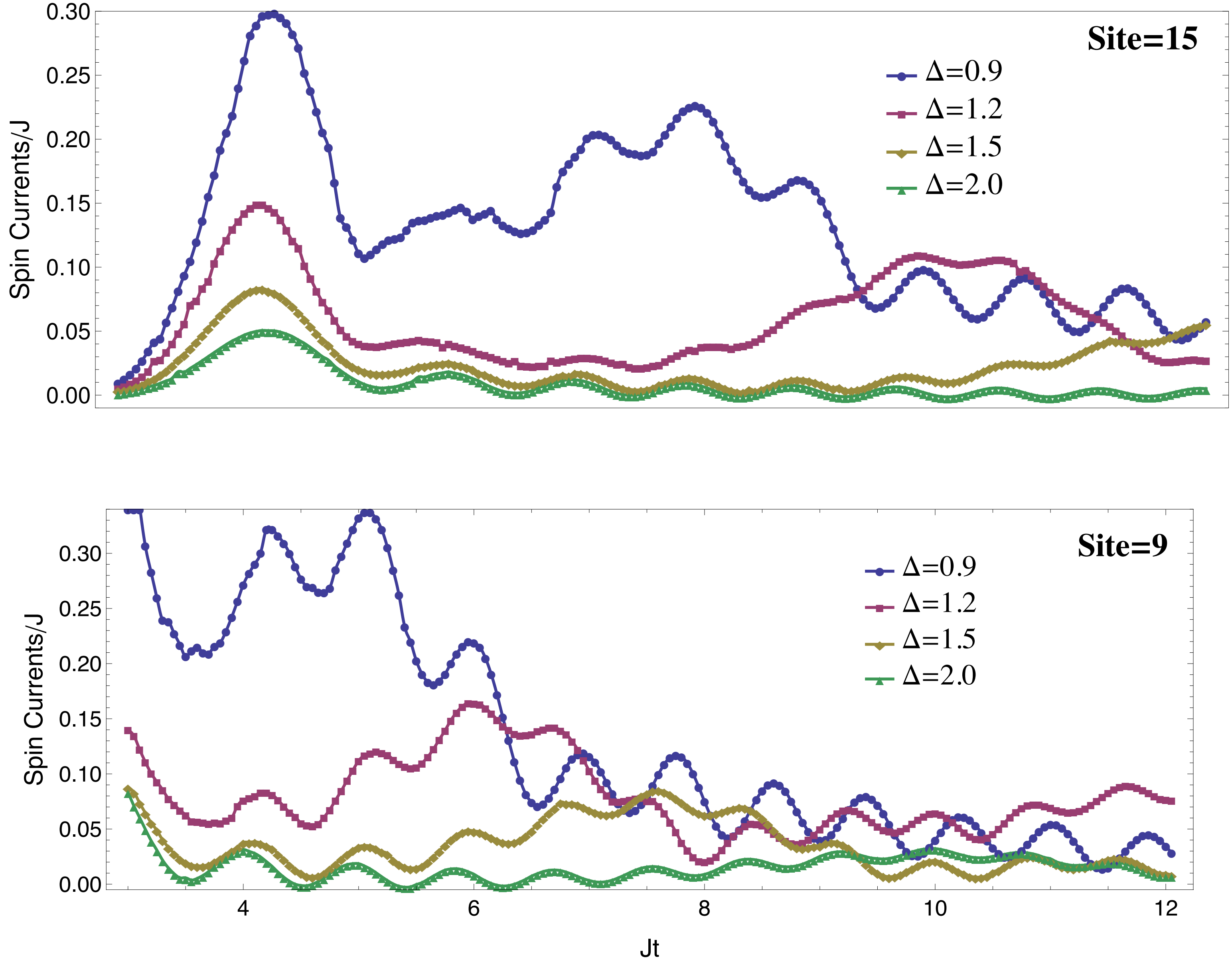




$$\frac{v_{bound}}{v_{magn}} = \frac{\sin\mu}{\sin(n\mu)} (|\Delta| = \cos\mu) \qquad \frac{v_{bound}}{v_{magn}} = \frac{\sinh\lambda}{\sinh(n\lambda)} (|\Delta| = \cosh\lambda)$$

Evolution of the XXZ magnet

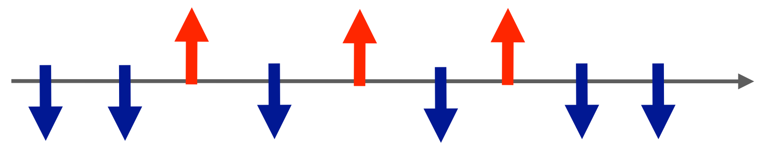
Spin currents - evolution

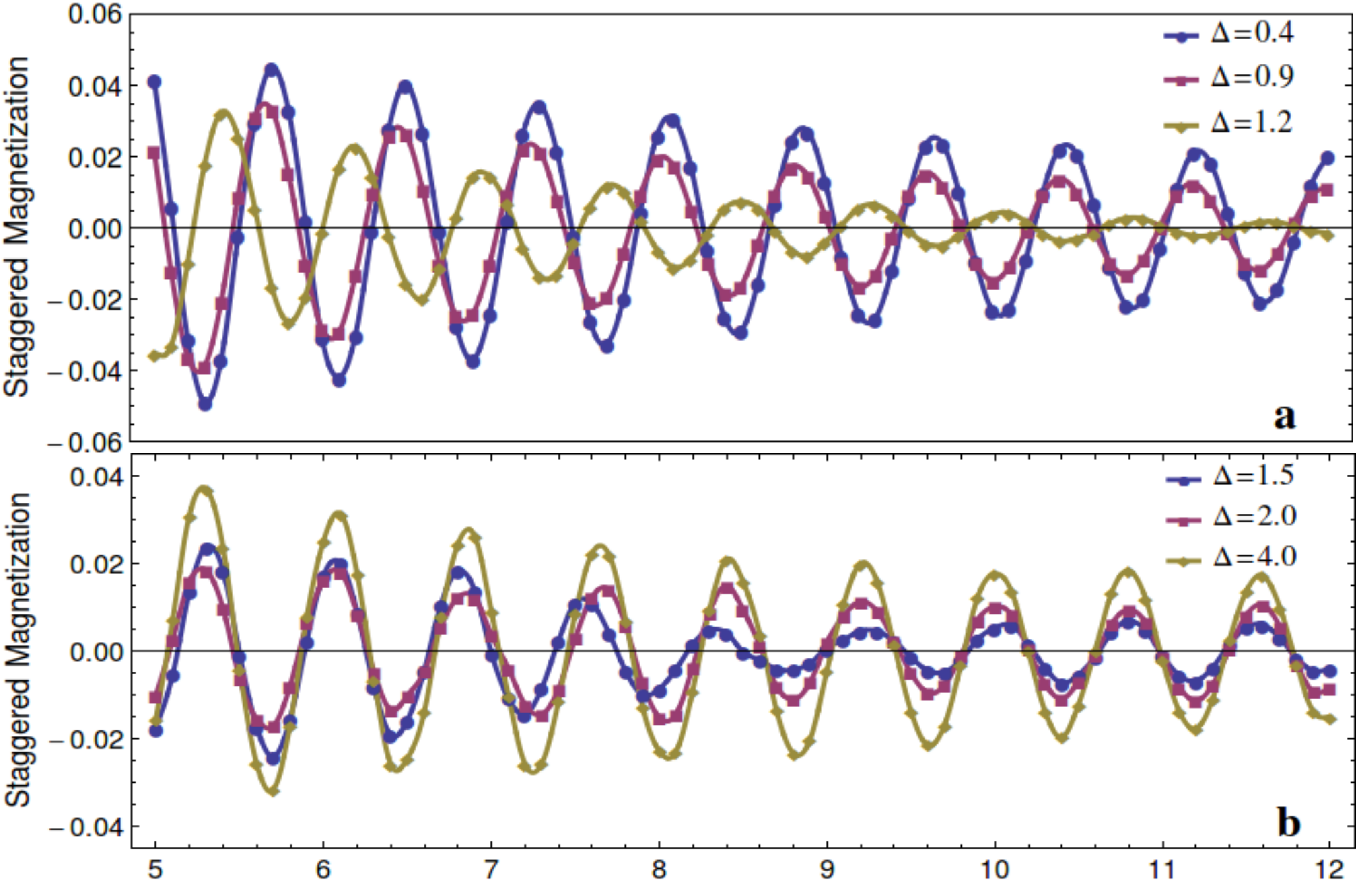


- *Staggerd Magnetization (Order Parameter)* $M_s(t) = \frac{1}{N}\sum_n (-1)^n \langle \sigma_n^z \rangle (t)$

Quench across a QCP $\Delta = \infty \to |\Delta| < 1$

$|\Psi_0\rangle = \sigma \quad \sigma_0^+ \sigma \quad _2 | \Downarrow \rangle =$





Outlook

Conclusions:

- **Time evolution at *infinite* volume; no need for spectrum of Hamiltonian or overlaps**
  - **Takes into account existence of bound states w/o sums over strings**
- **Time evolution at *finite* volume, finite density (need spectrum, no need for overlaps)**
- **Evolution calculable for all coupling regimes, for all initial states** (asymptotic equilibrium or not)

To do list:

- **Generalize to other integrable models:**

Anderson model **(Adrian Culver),** *Lieb-Liniger + impurity, Gaudin-Yang* **(Huijie Guan),**
Sine-Gordon model, spin-boson **(Colin Rylands),** *Kondo Model* **(Roshan Tourani),**
Multi-dot **(Huijie Guan, Chris Munson***)*

- **Approach to nonequilibrium steady state (Adrian Culver)**

- **Correlation functions (Garry Goldstein)**

Big Questions: *(Being Boltzmann?)*

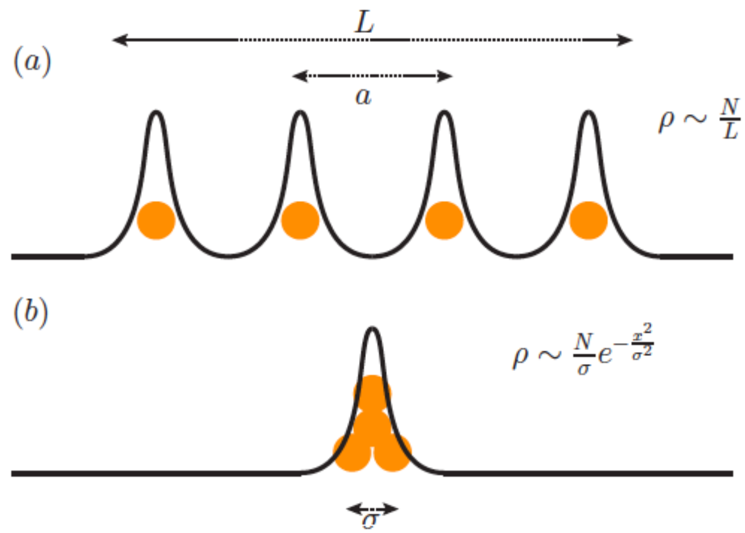
- **What drives thermalization of pure states? Canonical typicality, entanglement entropy (Lebowitz, Tasaki, Short…)**

- **General principles, variational? F-D theorem out-of-equilibrium? Heating? Entanglement?**

- **What is universal? RG Classification?**

Time scales in the Bosonic system

The system



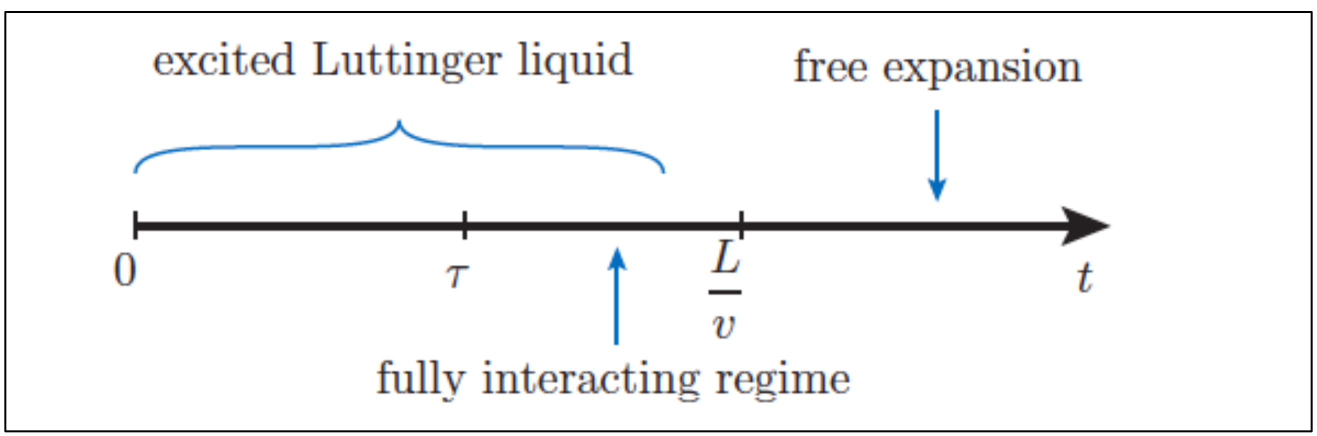
Time scales:

Thermodynamic regime - constant density $t \ll \frac{L}{v}$

$t \gg \frac{L}{v}$ *Expansion of interacting gas - density decreases*

Interaction time

$$\tau \sim \frac{1}{c^2}$$



Evolution of a bosonic system: saddle point app

Corrections to long time asymptotics -

Stationary phase approx at large times (carry out λ - integration)

- **Repulsive** – only stationary phase contributions (on real line); (cf. Lamacraft 2011)

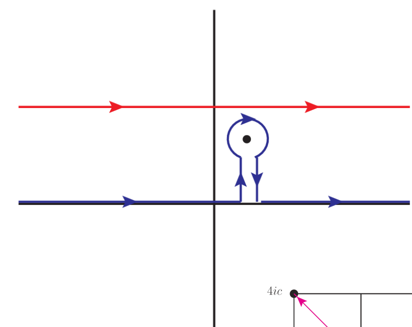
$$\phi\left(\xi \equiv \frac{y}{2t}, x, t\right) = S_\xi \frac{1}{(4\pi i t)^{\frac{N}{2}}} \prod_{i<j} \frac{\xi_i - \xi_j - ic\, sgn(\xi_i - \xi_j)}{\xi_i - \xi_j - ic} e^{\sum_j i\xi_j^2 t - i\xi_j x_j}$$

- **Attractive** – contributions from stationary phases and poles.

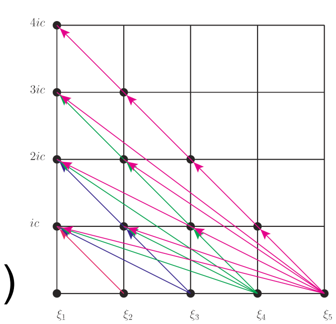
e. g for two particles:

Pole contributions from deformation of contours – formation of bound states

$$\phi(\xi, x, t) = S_\xi \Big[\frac{1}{4\pi i t} \frac{\xi_1 - \xi_2 - ic\, sgn(\xi_1 - \xi_2)}{\xi_1 - \xi_2 - ic} e^{\sum_j i\xi_j^2 t - i\xi_j x_j} + \\ + \frac{2c\theta(\xi_2 - \xi_1)}{\sqrt{4\pi i t}} e^{i\xi_1^2 t - i\xi_1 x_1 - i(\xi_1 - ic)^2 t + i(\xi_1 - ic)(2t\xi_2 - x_2)}\Big]$$



Bound states (string solutions) appear naturally

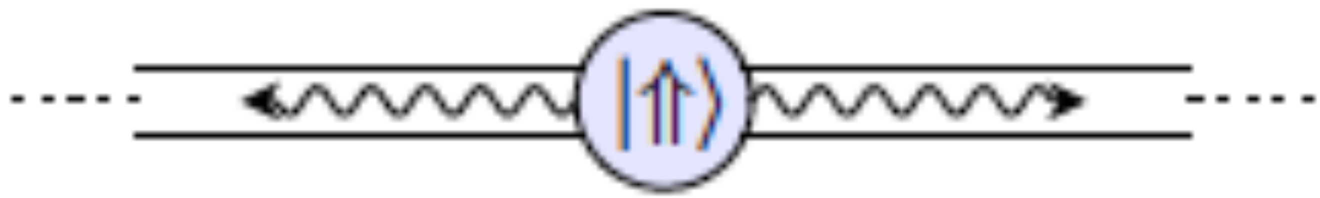


- repulsive correlations depend on $\xi = \frac{y}{2t}$ only (light cone propagation)
- attractive correlations maintain also t dependence (bd. states provide scales)

Evolution of Super-radiance (Dicke model)

M 2-level atoms located at x=0 in a waveguide: $s_i^{\pm},\ i = 1...M$

$$H_{\rm nc} = \int \Big(ib_L^\dagger(x)\partial_x b_L(x) - ib_R^\dagger(x)\partial_x b_R(x)\Big) - \sqrt{c/2}\,\Big(S^+(b_L(0)+b_R(0)) + S^-(b_L^\dagger(0)+b_R^\dagger(0)\Big)$$



Unfold*:*

Jaynes-Cummings, Tavis-Cummings model

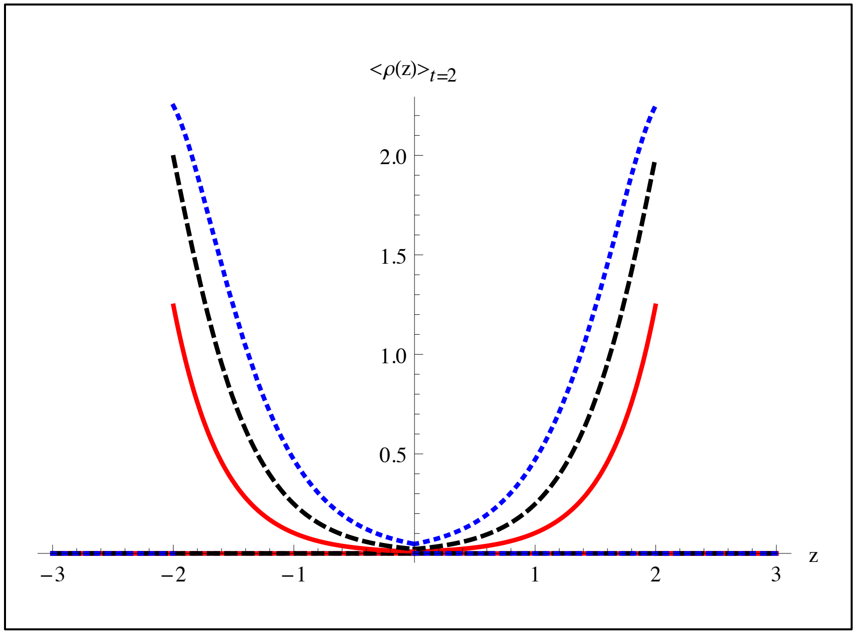
$$H = -i\int dx\, b^\dagger(x)\partial_x b(x) - \sqrt{c}\,\big(S^+ b(0) + S^- b^\dagger(0)\big)\,.$$

$$S^{\pm} = \sum_{i=1}^{M} s_i^{\pm}$$

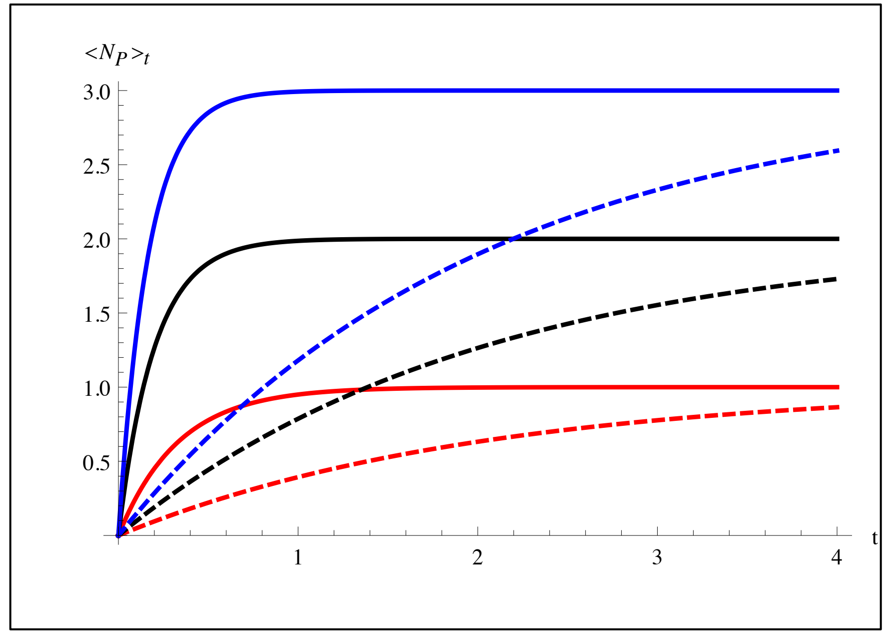
Prepare system in an excited state *e.g.* Excite $N \leq M$ atoms, no photons

$$|\Phi_0\rangle = \left(\frac{(M-N)!}{M!N!}\right)^{1/2} (S^+)^N |0\rangle$$

Evolution of Super-radiance (Dicke model)



$\langle j(z, t=2) \rangle$ at t = 2 for
M = 6, N = 3 (blue, dotted),
N = 2 (dashed) and N = 1 (red)
for the non-chiral model



The total photon number $\langle N_p \rangle_t$
for c = 1, M = 6, N = 3 (blue), N = 2
(black) and N = 1
also corresponding expressions
ignoring cooperative eects,

Note:

$$\partial_t \left\langle \hat{N}_{\rm p} \right\rangle_t = cN^2(1+M-N)e^{-cN(1+M-N)t}$$

Dicke cooperative effect: decay rate $\sim N^2$ (rather than $\sim N$ for incoherent decay)

Evolution of Super-radiance (Dicke model)

Time evolution of the photon current:

$$\langle j(z)\rangle_t = \langle \Phi_0 | e^{-iHt} \rho(z) e^{iHt} | \Phi_0 \rangle \quad \textit{with} \quad \rho(z) = b^\dagger(z) b(z)$$

Time evolution of photon number

$$\left\langle \hat{N}_{\mathrm{p}} \right\rangle_t = \int dz \, \langle \rho(z) \rangle_t$$

Expand in eigenstates (Rupasov and Yudson) and use Yudson representation

$$|\vec{\lambda}\rangle = \frac{1}{(2\pi)^{\frac{N}{2}} N!^{\frac{1}{2}}} \int d^N x \prod_{i<j} \left(1 - \frac{2ic\theta(x_i - x_j)}{\lambda_i - \lambda_j + ic} \right) \prod_{j=1}^{N} e^{i\lambda_j x_j} f(\lambda_j, x_j) r^\dagger(\lambda_j, x_j) |0\rangle$$

with

$$r^\dagger(\lambda_j, x_j) = b^\dagger(x_j) - \frac{\sqrt{c}}{\lambda_j} S^+$$ *photon-atom creation operator*

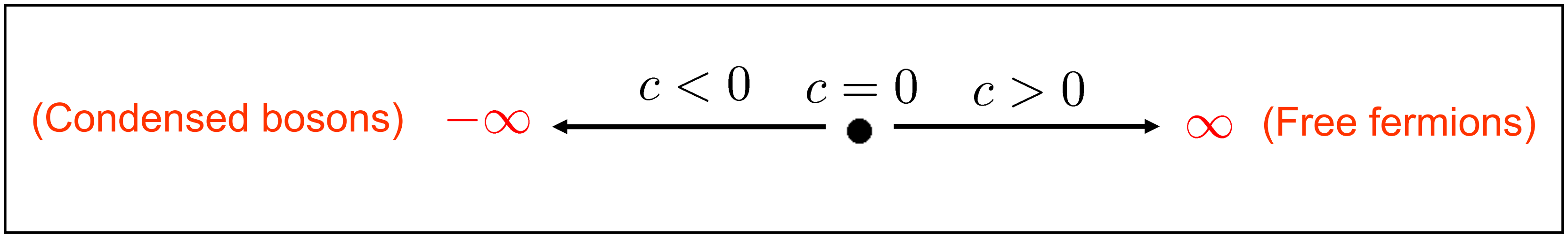
$$f(\lambda_j, x_j) = \frac{\lambda_j - icM/2 \, \mathrm{sgn}(x_j)}{\lambda_j + icM/2}$$ *photon-atom scattering*

Time evolution "Renormalization Group"

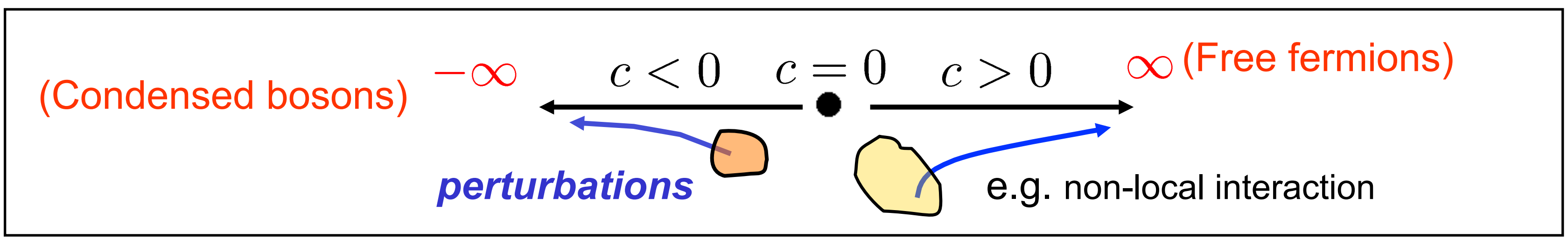
"Dynamic" RG interpretation

- **Universality out of equilibrium**
- **Can view time evolution as RG flow** $t \sim \ln(D_0/D)$

- As time evolves the weight of eigenstate contributions varies, time successively "integrates out" high energy states

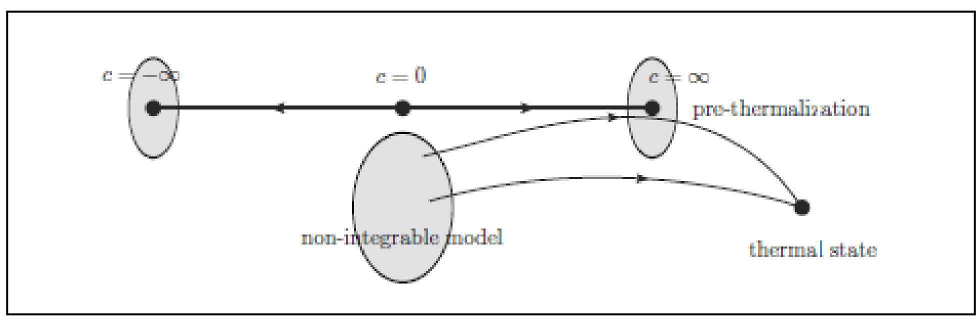


- **Are there "basins of attraction" for perturbations flowing to dynamic fixed points**



What is beyond $c = \infty$ **?**

Thermalization?



Quench action for time averaged quantities

$$\langle\Theta\rangle_T \equiv \frac{1}{T}\int_0^T dt\,\langle\Psi|\,e^{iH_{LL}t}\Theta e^{-iH_{LL}t}\,|\Psi\rangle =$$
$$= \frac{1}{T}\sum_\lambda\sum_\kappa \frac{e^{i(E_\lambda-E_\kappa)T}-1}{i(E_\lambda-E_\kappa)}\langle\Psi\mid\lambda\rangle\langle\lambda|\,\Theta\,|\kappa\rangle\langle\kappa\mid\Psi\rangle$$
$$\cong \sum_\lambda\langle\Psi\mid\lambda\rangle\langle\lambda|\,\Theta\,|\lambda\rangle\langle\lambda\mid\Psi\rangle\,,$$

"Time average diagonal ensemble"

<u>Quench Action</u>

$$\langle\Theta\rangle_{T\to\infty} = \int D\left(\frac{\rho_t}{\rho_p}\right) e^{2L\cdot S_{Quench}}\langle\{k\}|\,\Theta\,|\{k\}\rangle$$

$$S_{Quench} = \frac{2}{2L}\log\left(|\langle\Phi\mid\{k\}\rangle|\right) + \frac{1}{2}S\left(\{\rho\left(\{k\}\right)\}\right)$$

$$\frac{2}{2L}\log\left(|\langle\Phi\mid\{k\}\rangle|\right) = \int f\left(k\right)\rho\left(k\right)$$

Caux et al.

What do we want to do?

Want to calculate $\langle \Phi | Exp(\alpha Q_{xy}(T)) | \Psi \rangle$

Here $Q_{xy} \equiv \int_x^y b^+(z) b(z) dz$

$$Exp(\alpha Q_{xy}(T)) \text{ Generating function}$$

We can compute:

$$\langle b^+(x) b(x) \rangle = -\frac{\partial}{\partial \alpha} \frac{\partial}{\partial x} \langle Exp(\alpha Q_{xy}) \rangle_{\alpha=0}$$

$$\langle b^+(x) b(x) b^+(y) b(y) \rangle = -\frac{1}{2} \frac{\partial^2}{\partial \alpha^2} \frac{\partial^2}{\partial x \partial y} \langle Exp(\alpha Q_{xy}) \rangle_{\alpha=0}$$

Time evolution – low entropy intial state,

Generically:

If a system equilibrates and reaches GGE – what are the experimental signatures?

i. *GGE can be reduced to a pure state* $\rho_{GGE} \cong \left|\vec{k}_0\right\rangle \left\langle \vec{k}_0\right|$

ii. Define $\Theta_\beta(x) \equiv \exp\left(\beta \int_0^x b^\dagger(z)\, b(z)\, dz\right)$

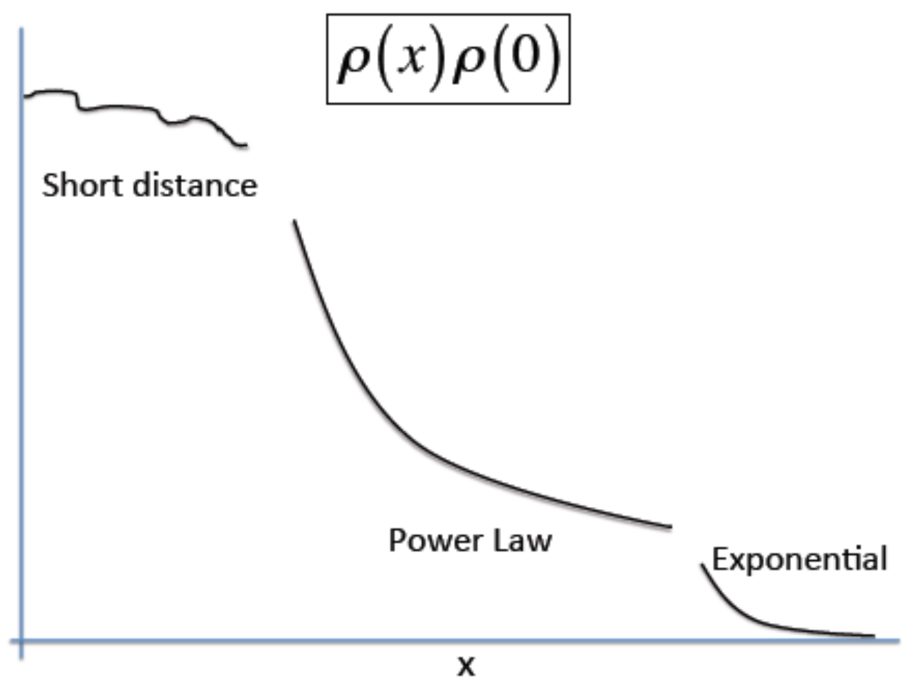
- for hard core bosons (Tonks-Girardeau gas)

$$tr\rho_{GGE}\Theta_\beta(x) = \left\langle \vec{k}_0 \right| \Theta_\beta(x) \left| \vec{k}_0 \right\rangle = \det\left(I + \frac{e^\beta - 1}{\pi} \frac{1}{\sqrt{1+e^{\varepsilon(k)}}} \frac{\sin(k-q)\frac{x}{2}}{k-q} \frac{1}{\sqrt{1+e^{\varepsilon(q)}}}\right)$$

Slavnov '10

iii. Expand in β, x :

$$\rightarrow \quad tr\hat{\rho}_{GGE}\rho(x)\rho(0)$$



iv. *Generic initial state low YY entropy*

Note: *exponential decay: finite YY entropy of initial state ~ finite T*

Keldysh

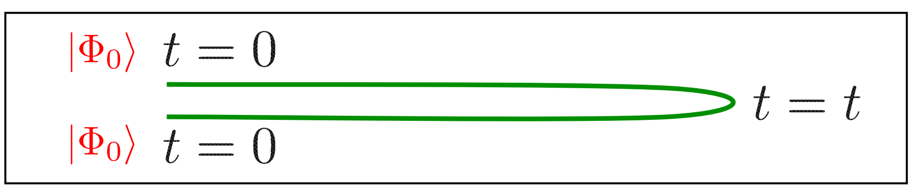
- **Time evolution of expectation values:**

$$O_{\Phi_0}(t) = \langle \Phi_0 | e^{iHt}\, \hat{O}\, e^{-iHt} | \Phi_0 \rangle = \langle \Phi_0, t |\, \hat{O}\, | \Phi_0, t \rangle$$

Non-perturbative Keldysh:

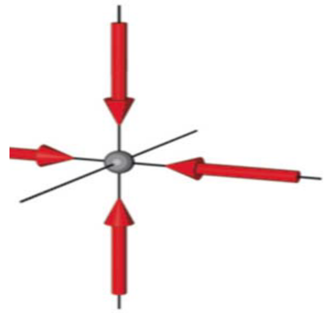
$$= \int \mathcal{D}\, b^* \mathcal{D}\, b\, \hat{O}\; e^{-i \int_C [S_0(b,b^*) + S_I(b,b^*)]\, dt}$$

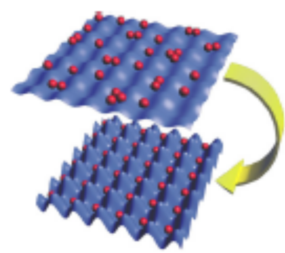
carried out on the Keldysh contour C *, with separate fields for the top and bottom lines:*



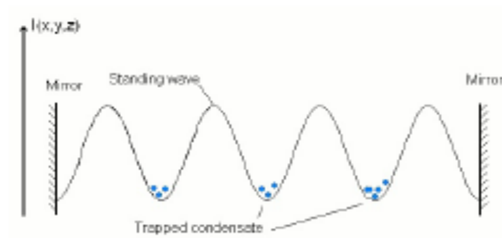
1. Boson Systems - experiments

Bosons in optical traps

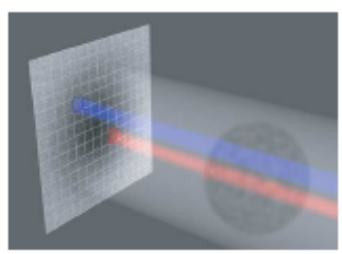




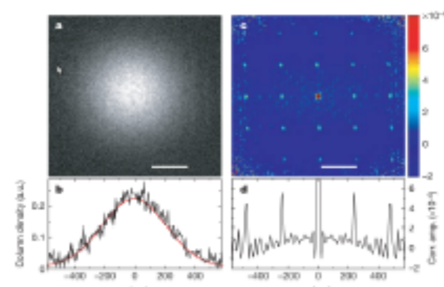
Superfluid Mott insulator transition



Mott insulator – initial condition



Imaging of density cloud using a CCD



Density and noise correlation functions

Bloch et al (Nature 2005, Rev Mod Phys 2008)

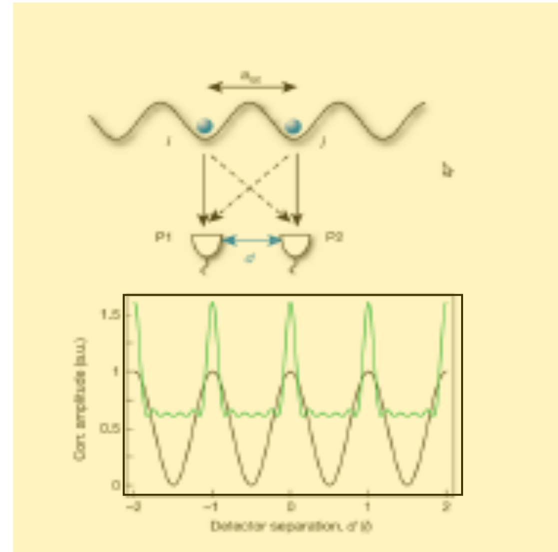
What to calculate?

- **We shall study:**

1. Evolution of the density

$$C_1(x,t) = \langle \rho(x,t) \rangle$$ Time of Flight experiment

competition between quantum broadening and attraction



Hanbury-Brown Twiss effect

Measure: $C_2(x_1, x_2, t)$

- two sources: originally stars

Free bosons $C_2(x,-x) \sim \cos x$

Free Fermions $C_2(x,-x) \sim -\cos x$

- Two bosons:

Similar, but time dependent

- Many bosons:

More structure: main peaks, sub peaks

Effects of interactions?

2. Evolution of noise correlation

$$C_2(x_1, x_2; t) = \frac{\langle \rho(x_1,t)\rho(x_2,t) \rangle}{\langle \rho(x_1,t) \rangle \langle \rho(x_2,t) \rangle} - 1$$

time dependent Hanbury Brown - Twiss effect

- repulsive bosons evolve into fermions
- attractive bosons evolve to a condensate

Evolution of a bosonic system: density

Density evolution:
(Time of flight experiment)

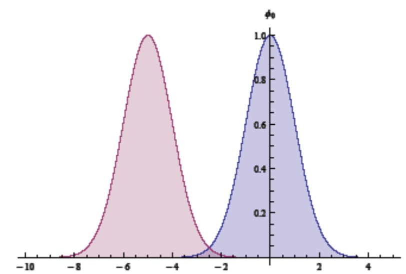
$$\langle \rho(x_0,t)\rangle = \langle \Phi_0(t)|b^\dagger(x_0)b(x_0)|\Phi_0(t)\rangle$$

- Example: Two bosons $$|\Phi_0\rangle_2 = \int_x \Phi_0(x_1,x_2)b^\dagger(x_1)b^\dagger(x_2)|0\rangle$$ →

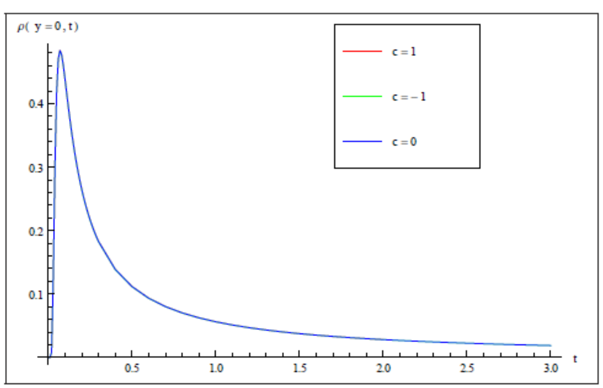
$$|\Phi_0(t)\rangle_2 = \int_y \int_c \Phi(x_1,x_2)\frac{e^{i\frac{(y_1-x_1)^2}{4t}+i\frac{(y_2-x_2)^2}{4t}}}{4\pi i t}\left(1 - c\sqrt{\pi i t}\,\theta(y_2-y_1)e^{\frac{i}{8t}\alpha^2(t)}\mathrm{erfc}\left(\frac{i-1}{4}\frac{i\alpha(t)}{\sqrt{t}}\right)\right)b^\dagger(y_1)b^\dagger(y_1)|0\rangle$$

with $$\alpha(t) = 2ct - i(y_1-x_1) - i(y_2-x_2)$$

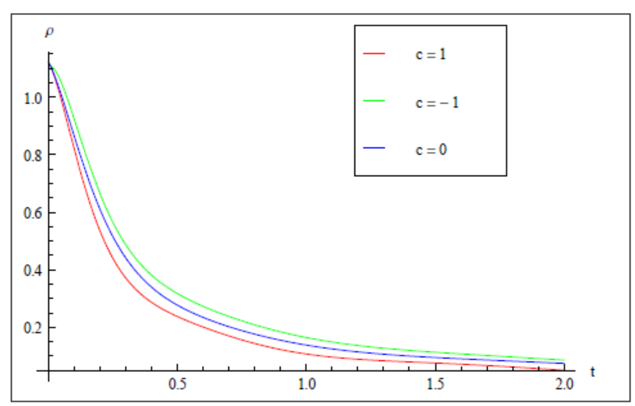
Initial state $$|\Phi_0\rangle = \frac{1}{(\pi\sigma^2)^{\frac{1}{2}}}\int_x e^{-\frac{(x_1)^2}{2\sigma^2}}e^{-\frac{(x_2+a)^2}{2\sigma^2}}b^\dagger(x_1)b^\dagger(x_2)|0\rangle$$



i. **Initial condition:** $a \gg \sigma$



ii. **Initial condition:** $a \ll \sigma$



Strong dependence on initial state

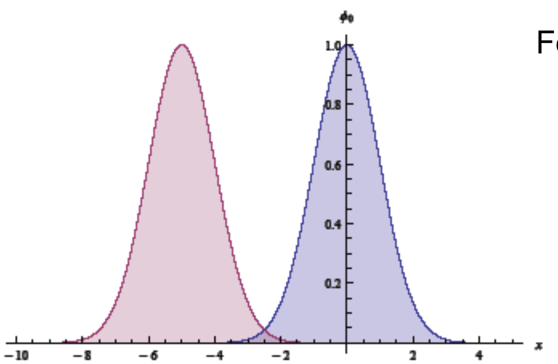
Evolution of repulsive bosons into fermions: HBT

Density - Density correlation: *long time asymptotics - repulsive:*

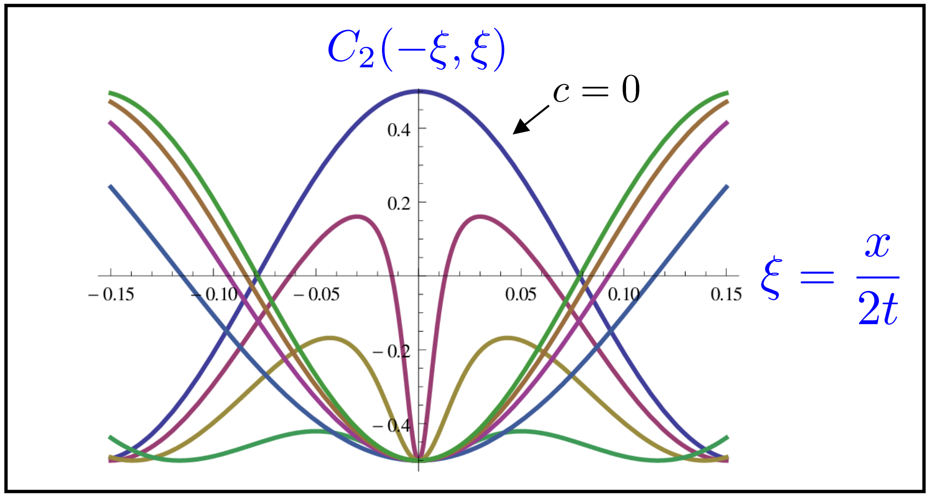
- **Bosons turn into fermions as time evolves (for any $c > 0$)**
- **Can be observed in the noise correlations: (dependence on t only via $\xi = x/2t$)**

$$C_2(x_1, x_2, t) \rightarrow C_2(\xi_1, \xi_2) = \frac{\langle \rho(\xi_1)\rho(\xi_2)\rangle}{\langle \rho(\xi_1)\rangle\langle \rho(\xi_2)\rangle} - 1,$$

$c/a = 0, .3, \cdots, 4$



Fermionic correlations evolve



- **Fermionic dip develops for any repulsive interaction on time scale set by $1/c^2$**

Emergence of an asymptotic Hamiltonian

Long time asymptotics - repulsive:

- **Bosons turn into fermions as time evolves (for any $c>0$)** (cf. Buljan et al. '08)

$$
\begin{aligned}
|\Phi_0,t\rangle &= \int_x\int_y\int_\lambda \theta(\vec{x})\Phi_0(\vec{x})\prod_{i<j}\frac{\lambda_i-\lambda_j-ic\,sgn(y_i-y_j)}{\lambda_i-\lambda_j-ic}\prod_j e^{-i\Sigma_j\lambda_j^2 t-\lambda_j(y_j-x_j)}\prod_j b^\dagger(y_j)|0\rangle \\
&= \int_x\int_y\int_\lambda \theta(\vec{x})\Phi_0(\vec{x})\prod_{i<j}\frac{\lambda_i-\lambda_j-ic\sqrt{t}\,sgn(y_i-y_j)}{\lambda_i-\lambda_j-ic\sqrt{t}}e^{-i\Sigma_j\lambda_j^2-\lambda_j(y_j-x_j)/\sqrt{t}}\prod_j b^\dagger(y_j)|0\rangle \\
&\to \int_x\int_y\int_\lambda \theta(\vec{x})\Phi_0(\vec{x})e^{-i\Sigma_j\lambda_j^2-\lambda_j(y_j-x_j)/\sqrt{t}}\prod_{i<j}sgn(y_i-y_j)\prod_j b^\dagger(y_j)|0\rangle \\
&= e^{-iH_0^f t}\int_{x,k}\mathcal{A}_x\,\theta(\vec{x})\Phi_0(\vec{x})\prod_j c^\dagger(x_j)|0\rangle.
\end{aligned}
$$

$\mathcal{A}_x$ *antisymmetrizer*

where

$H_0^f = -\int_x c^\dagger(x)\partial^2 c(c)$

- In the long time limit repulsive bosons for any $c>0$ propagate under the influence of **Tonks – Girardeau** ***Hamiltonian (hard core bosons=free fermions)***

- The state equilibrates, does not thermalize

- Argument valid for any initial state Φ_0

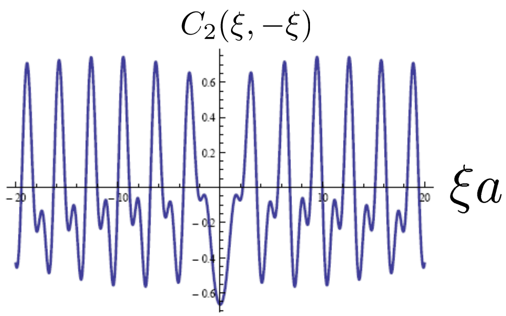
- **Scaling argument fails for attractive bosons** (instead, they form bound states)

Evolution of a bosonic system: noise correlations

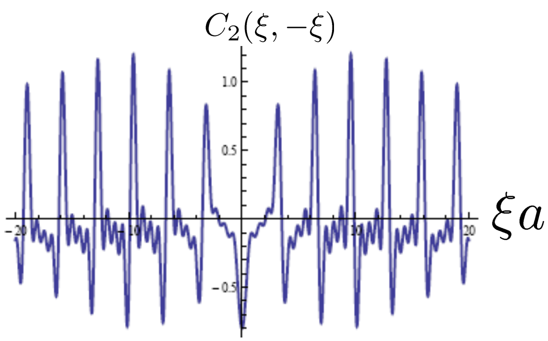
Noise correlations – starting from a lattice

Repulsive bosons

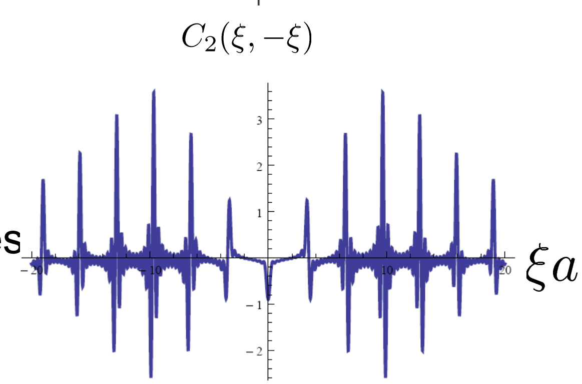
3 particles



5 particles



10 particles

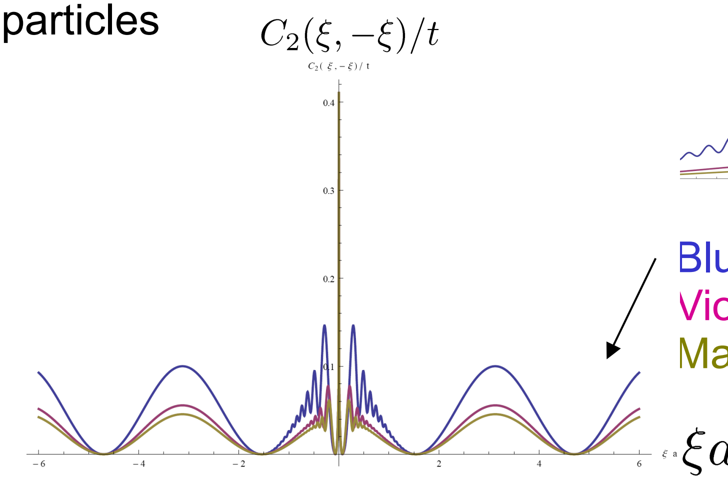


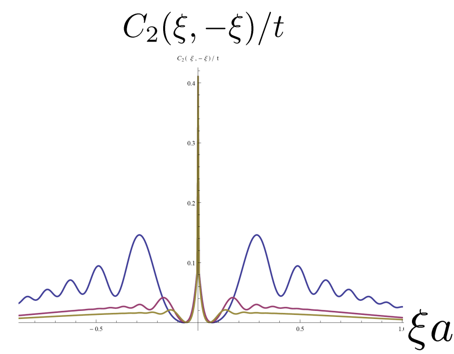
Fermionic dip as $\xi \to 0$

Structure emerges at $\xi a = \sigma$

Attractive bosons

2 particles

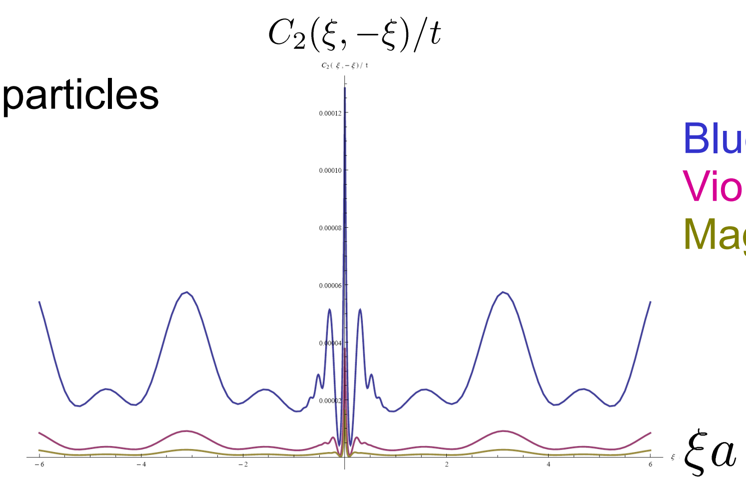




Blue - short times
Violet - longer
Magenta - longest

central peaks increase with time
- weight in the bound states increases

3 particles



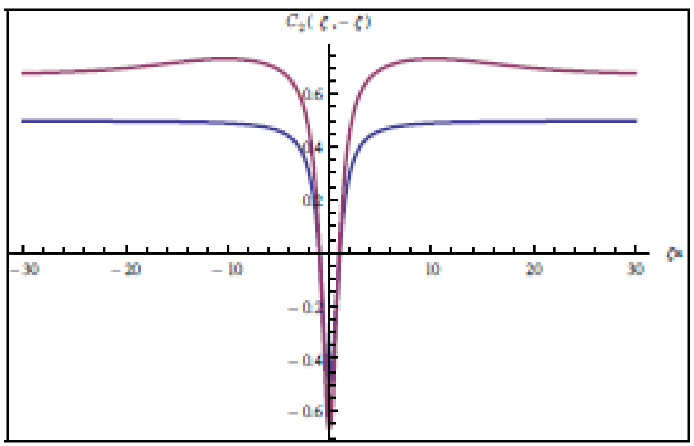
Blue - short times
Violet - longer
Magenta - longest

peaks diffuse – momenta redistribute

Evolution of a bosonic system: noise correlation

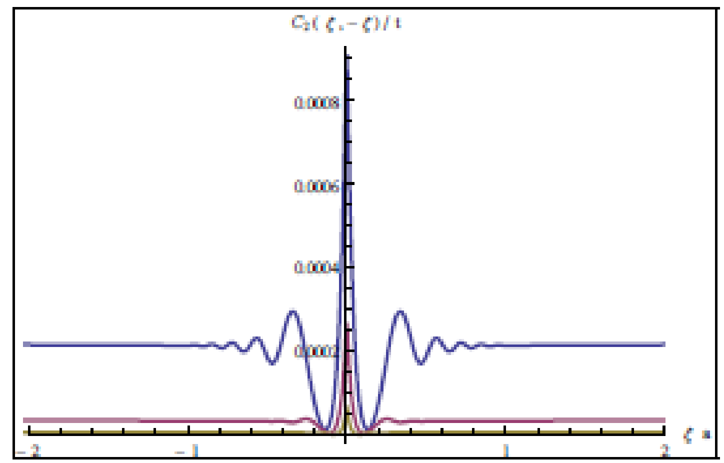
Noise correlations – starting from a condensate

Repulsive bosons



Two (blue) and three bosons,

Attractive bosons

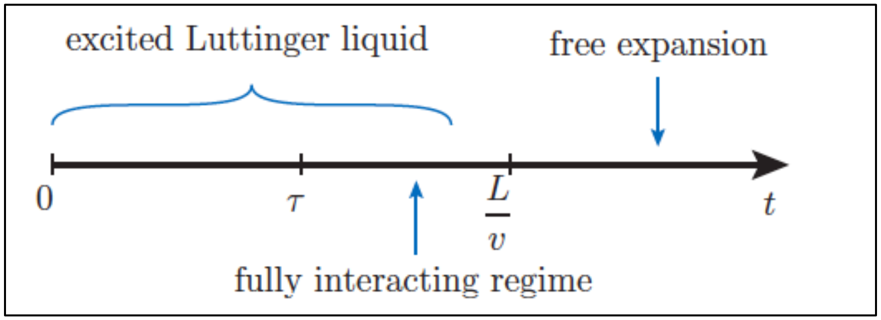


Three bosons, at times: $tc^2 = 20; 40; 60$

Evolution of the LL bosonic system – thermodynamic limit

Wish to study the system in the limit:

$$N, L \to \infty, \quad n = N/L \ fixed, \quad t \ll L/v_{typ}$$



Finite size Yudson representation :

$$|\Phi_0\rangle = \sum_{n_1=-\infty}^{\infty} ... \sum_{n_N=-\infty}^{\infty} \mathcal{N}\left(\{k_n\}\right)^{-1} |k_{n_1}....k_{n_N}\rangle \left(k_{n_1}...k_{n_N}\right| |\Phi_0\rangle \, .$$

Allows the computation of the time evolution of an observable - $\hat{\Theta}$

$$G(\Theta, t; x_1..x_N; y_1..y_N) = \langle 0|b(y_1)..b(y_N)\ e^{iHt}\Theta e^{-iHt}\ b^\dagger(x_1)..b^\dagger(x_N)|0\rangle$$

Time evolution – Diaginal ensemble, GGGE and GGE

- ***For translationally invariant initial states,*** $c > 0$

i. The system equlibrates, the limit $t \to \infty$ *is well defined*

ii. The system equilibrates to a ***diagonal*** *ensemble*

$$\rho_D = \sum_\lambda |\langle \Phi(t=0)|\lambda\rangle|^2 \, |\{\lambda_i\}\rangle \, \langle \{\lambda_i\}|$$

iia. The system obeys GGE (if no long range correlations present in initial state)

$$\hat{\rho}_{GGE} = Z^{-1} exp\left[-\sum_m \alpha_m I_m\right]$$

The conserved charges: $I_m|\lambda\rangle = \sum_i \lambda_i^m |\lambda\rangle$

with $Tr\left[I_m \widehat{\rho}\right] = \langle I_m \rangle (t=0)$

No long range correlations: $\langle I_{m_1} I_{m_2} \ldots \rangle = \langle I_{m_1}\rangle \langle I_{m_2}\rangle \ldots$

iib. The system obeys "generalized" GGE (if long range correlations present)

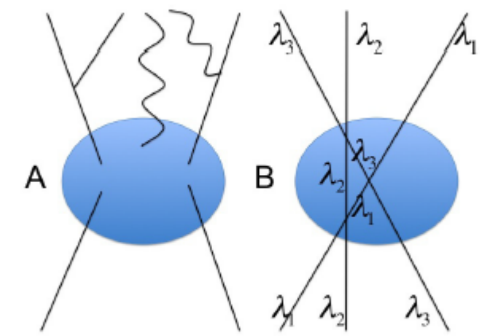
$$\widehat{\rho}_{GGGE} = \widetilde{Z}^{-1} \exp\left[-\sum_{m_1 m_2 \ldots} \alpha_{m_1 m_2 \ldots} I_{m_1} I_{m_2} \ldots\right]$$

- ***For sufficiently non-translationally invariant initial states (e.g. domain wall)***

iii. System does not equilibrate, does not reach diagonal ensemble

Time evolution – Diagonal ensemble and GGE

GGE if: 1. Diagonal ensemble 2. Operator can be expanded



i. The diagonal element can be Taylor expanded (~ ETH)

$$\langle\{k_i\}|\,\Theta\,|\{k_i\}\rangle = c_0 + c_1\sum k_i + c_{1,1}\sum k_i k_j + c_2 \sum k_i^2 + ..$$

ii. So for trans. invariant initial states : $\langle\Theta\rangle\,(t\to\infty) = Tr\rho_D\Theta$

$$\rho_D = \sum p_{\{k\}}\,|\{k_i\}\rangle\,\langle\{k_i\}| \quad \textit{with} \qquad p_k = |\langle\{k\}|\Phi_0\rangle|^2$$

iii. Thus: $\langle\Theta\rangle \to c_0 + c_1\langle I_1\rangle + c_{1,1}\left\langle I_1^2\right\rangle + c_2\langle I_2\rangle + ..$

with: $\langle I_1\rangle = \sum p_{\{\lambda\}}\sum k_i,\ \ \langle I_1^2\rangle = \sum p_{\{k\}}\sum k_i k_j,\ \ \langle I_2\rangle = \sum p_{\{k\}}\sum k_i^2$ ...

iv. Equivalently:

$$\rho_D = \hat{\rho}_{GGGE} = \widetilde{Z}^{-1}\exp\left[-\sum_{m_1 m_2\ldots}\alpha_{m_1 m_2\ldots} I_{m_1} I_{m_2}\ldots.\right]$$

with $\{\alpha_{m_1 m_2\ldots}\}$ *determined from* $Tr\left[I_{m_1} I_{m_2}\ldots\rho_{GGGE}\right] = \langle I_{m_1} I_{m_2}\ldots\rangle\,(t=0)$

v. GGGE ⟶ *GGE for short range correlations in initial state*

$$\langle I_{m_1} I_{m_2}\ldots.\rangle = \langle I_{m_1}\rangle\,\langle I_{m_2}\rangle\ldots$$

Interacting Bosons in a box

The Lieb-Liniger model with hard boundary conditions: (Gaudin '71)

$$H_{LL} = \int_0^L dx \left\{ \partial_x b^\dagger(x)\, \partial_x b(x) + c \left(b^\dagger(x)\, b(x) \right)^2 \right\} \qquad \begin{array}{l} \psi(x_1 = 0, x_2, \ldots x_N) = 0 \\ \psi(x_1, x_2, \ldots x_N = L) = 0 \end{array}$$

The wave functions are of the form:

$$\psi(|k_1|, \ldots |k_N|) = \sum_{\{\varepsilon\}} C\{\varepsilon\}\, \bar{\psi}(\varepsilon_1 |k_1|, \ldots \varepsilon_N |k_N|) \qquad \varepsilon_j = \pm 1$$

with:

$$\bar{\psi}(k_1, \ldots k_N) = \sum_P A(P)\, e^{i \sum k_{P_i} x_i}, \quad x_1 < x_2 < \ldots < x_N$$

$$A(P) = \prod_{i<j} \left(1 + \frac{ic}{k_{P_i} - k_{P_j}} \right)$$

$$C(\varepsilon_1, \ldots \varepsilon_N) = \prod \varepsilon_j \prod_{i<j} \left(1 - \frac{ic}{\varepsilon_i |k_i| + \varepsilon_j |k_j|} \right)$$

satisfying the BA equations

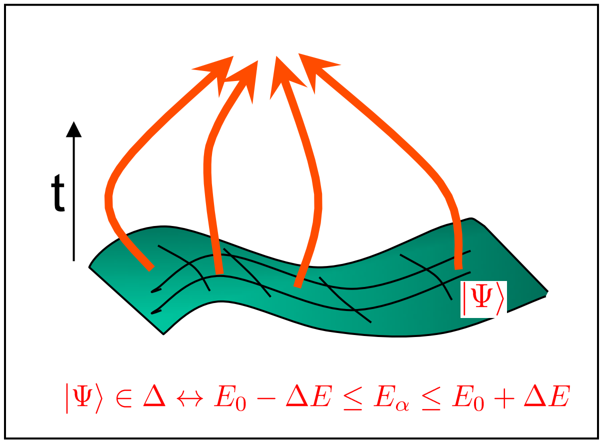
$$k_i L = \pi n_i + \sum_{j \neq i} \left(\arctan\left(\frac{c}{k_i - k_j} \right) + \arctan\left(\frac{c}{k_i + k_j} \right) \right) \qquad k_i = \grave{\varepsilon}_i |k_i|$$

Quenching – long time limit, thermalization

Time evolution and statistical mechanics:

$$\langle A(t)\rangle = \langle\Phi_0|e^{iHt}Ae^{-iHt}|\Phi_0\rangle = \sum_{\alpha,\beta}\langle\Phi_0|\alpha\rangle A_{\alpha\beta}\langle\beta|\Phi_0\rangle e^{i(E_\alpha - E_\beta)t}$$

- **Long time limit and thermalization:** **(Gibbs ensemble - GE)**
  - is there a limit for local op. $\bar{A} = \lim_{t\to\infty}\langle A(t)\rangle$? (equilibration)
  - is there a density operator ρ such that $\bar{A} = Tr(\rho A)$?
  - does it depend only on $E_0 = \langle\Phi_0|H|\Phi_0\rangle$, not on $|\Phi_0\rangle$? (ETH)



$$\bar{A} = \sum_\alpha |C_\alpha|^2 A_{\alpha\alpha} \stackrel{?}{=} \langle A\rangle_{\text{microcan}}(E_0) \equiv \frac{1}{N_{E_0,\Delta E}}\sum_{\alpha\in\Delta} A_{\alpha\alpha}$$

- **Scenarios of thermalization (ETH** and others)
  - Diagonal matrix elements of physical operators $A_{\alpha\alpha} = \langle\psi_\alpha|A|\psi_\alpha\rangle$ do not fluctuate much around constant energy surface (**ETH-*eigenstate thermalization hypothesis***, Deutsch 92, Srednicki 94)
  - Overlaps $|C_\alpha|^2 = |\langle\psi_\alpha|\Phi_0\rangle|^2$ do not fluctuate on the energy surface for reasonable IC
  - Both fluctuate but are uncorrelated

- **Thermalization:** in translationally inv. Systems, $t\to\infty$ leads to $|\alpha\rangle = |\beta\rangle$, **diagonal ensemble**
  - Assume ETH (for non integrable models): $\langle\alpha|A|\alpha\rangle = f(E_\alpha)$, with f smooth function
- → Gibbs ensemble: $\langle A(t\to\infty)\rangle = Tr(e^{-\beta H}A)Z^{-1}$ with $\langle H\rangle_{t=0} = Tr(e^{-\beta H}H)Z^{-1}$

Newton's Cradle: BEC initial state

BEC described by (Caux):

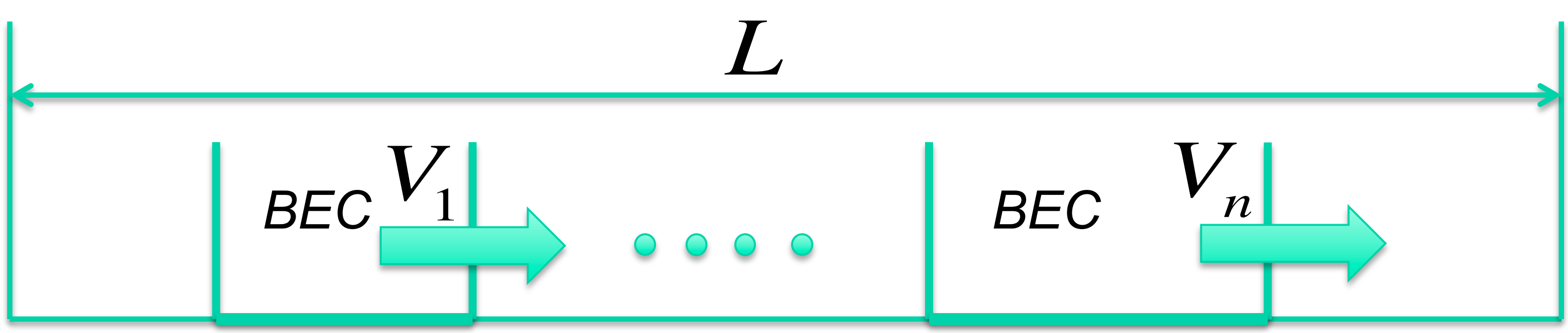
$$\rho_p^{BEC_i}(x) = \frac{\tau_i \frac{a}{d\tau_i} a(x,\tau)}{1 + a(x,\tau)} \qquad x = \frac{k}{c},\ \tau_i = \frac{n_i}{c}$$

$$a(x,\tau) = \frac{2\pi\tau}{x \sinh(2\pi x)} J_{1-2ix}\left(4\sqrt{\tau}\right) J_{1+2ix}\left(4\sqrt{\tau}\right)$$

Combine with $$L \int dk \rho_p^f(k) k^{2n} = \sum L_i \int \rho_p^i(k) \left(k + \frac{1}{2} V_i\right)^{2n}$$

We find:

$$\rho_p^f(k) = \sum \frac{L_i}{2L} \left(\rho_p^{BEC_i}\left(k + \frac{1}{2} V_i\right) + \rho_p^{BEC_i}\left(k - \frac{1}{2} V_i\right) \right)$$



Time evolution of an observable

Where:

$$F_{\alpha,xy}(X,Y,t) \equiv i\frac{e^{\alpha}-1}{2\pi}\frac{\exp\left(-\frac{i}{4t}\left(Y^2-X^2\right)\right)}{Y-X}\times\left[\exp\left(\frac{i\left(Y-X\right)\cdot x}{2t}\right)-\exp\left(\frac{i\left(Y-X\right)\cdot y}{2t}\right)\right]$$

$$G_{\alpha,x}(X,Y,t) \equiv \exp\left(i\frac{(Y-X)\,x}{2t}\right)\frac{\exp\left(-i\left(Y^2-X^2\right)/2t\right)}{t}$$

$$G_{\alpha,y}(X,Y,t) \equiv \exp\left(i\frac{(Y-X)\,y}{2t}\right)\frac{\exp\left(-i\left(Y^2-X^2\right)/2t\right)}{t}$$

$$F_{\alpha,x}(X,Y,t) \equiv \frac{\exp\left(-\frac{i}{4t}\left(Y^2-X^2\right)\right)}{Y-X}\exp\left(\frac{i\left(Y-X\right)\cdot x}{2t}\right)$$

$$F_{\alpha,y}(X,Y,t) \equiv \frac{\exp\left(-\frac{i}{4t}\left(Y^2-X^2\right)\right)}{Y-X}\exp\left(\frac{i\left(Y-X\right)\cdot y}{2t}\right)$$

- ***The time evolution is expressed:***

i. in terms of these time dependent functions

ii. correlation functions in initial states

- ***Valid for any initial state***

Time evolution of an observable

Consider: $Q_{xy} \equiv \int_x^y b^\dagger(z)\, b(z)\, dz$ *Charge between x and y.*

- *Thermodynamic limit* $L, N \to \infty, \quad t \to \infty \ll L/v_{typ}$

- After a long long calculation ….. a monster expression:

(G Goldstein, NA '13)

$$
\begin{aligned}
&\langle \exp(\alpha Q_{xy}(t)) \rangle = 1 + \int dX dY F_{\alpha,xy}(X,Y,t) \times \\
&\quad \times \left\langle b^\dagger(Y) \exp\left[i \int_X^Y dz \pi b^\dagger(z) b(z) \right] b(X) \right\rangle + \\
&+ \int dX_1 dX_2 dY_1 dY_2 \times F_{\alpha,xy}(X_1,Y_1,t) F_{\alpha,xy}(X_2,Y_2,t) \times \\
&\times \left\langle sgn(Y_2 - Y_1)\, b^\dagger(Y_1)\, b^\dagger(Y_2)\, e^{i\int_{X_1}^{Y_1} dz \pi b^\dagger(z) b(z)} \times \right. \\
&\left. \times\ sgn(X_2 - X_1)\, e^{i\int_{X_2}^{Y_2} dz \pi b^\dagger(z) b(z)} b(X_1)\, b(X_2) \right\rangle - \\
&- \tfrac{i\alpha}{\pi^2 c} \int dX_1 dY_1 dX_2 dY_2 \{ F_{\alpha,x}(X_1,Y_1,t) G_{\alpha,x}(X_2,Y_2,t) - \\
&- F_{\alpha,y}(X_1,Y_1,t) G_{\alpha,y}(X_2,Y_2,t) \} \langle sgn(y_2 - y_1) \\
&\cdot sgn(x_2 - x_1) \cdot b^\dagger(y_1)\, b^\dagger(y_2)\, e^{i\int_{X_1}^{Y_1} dz \pi b^\dagger(z) b(z)} \cdot \\
&\cdot e^{i\int_{X_2}^{Y_2} dz \pi b^\dagger(z) b(z)} b(X_1)\, b(X_2) \rangle - \\
&- \tfrac{i\alpha}{2\pi^2 c} \int dX dY G_{\alpha,X}(X,Y,t) \left\langle b^\dagger(Y)\, e^{i\int_X^Y dz \pi b^\dagger(z) b(z)} \cdot \right. \\
&\left. \cdot \int_{-\infty}^{\infty} dv\, sgn(x - v)\, \rho(v)\, sgn(v - Y)\, sgn(v - X) \right\rangle + \\
&+ \tfrac{i\alpha}{2\pi^2 c} \int dX dY G_{\alpha,y}(X,Y,t) \left\langle b^\dagger(Y)\, e^{i\int_X^Y dz \pi b^\dagger(z) b(z)} \cdot \right. \\
&\left. \cdot \int_{-\infty}^{\infty} dv\, sgn(y - v)\, \rho(v)\, sgn(v - Y)\, sgn(v - X) \right\rangle + \ldots.
\end{aligned}
$$

All expectation values taken w.r.t initial state

up to: $\alpha^3, 1/c^2$

Non-thermalization: Integrable systems, Nonequilibrium systems

- **Not all system thermalize in the long time limit:** ***Integrability, GETH, GGE***

Rigol, Cazallila. Mussardo..

- Integrable models possess an infinite number of non trivial conserved charges I_n

- For example, in the case of the Lieb-Liniger model: $I_n|\{k\}\rangle = \sum_j (k_j)^n|\{k\}\rangle$

- The *generalized* ETH: $\langle\alpha|A|\alpha\rangle = f(I_{1,\alpha}, I_{2,\alpha}, \ldots)$ - smooth function, together with assumption of diagonal ensemble, leads to:

- GGE, generalized Gibbs ensemble:

$$\langle A(t\to\infty)\rangle = Tr(\hat{\rho}A) \qquad \text{with} \qquad \hat{\rho} = Z^{-1}\exp(-\sum_n \beta_n I_n)$$

- Inverse temperatures β_n determined by initial conditions: $Tr(I_n\hat{\rho}) = \langle I_n\rangle_{t=0}$

- If GGE is valid it provides a short cut to time evolution (see later)

- But is GGE always valid for integrable systems? **Usually not!** (see later)

- **Not all systems equilibrate in the long time limit :** ***breaking tranlational inv.***

If initial state sufficiently non-translational invariant → no diagonal ensemble

- NESS? nonequilibrium steady state (*point-wise!*) in the long time limit? How to describe ρ_s?
- Another scenario: Oscillatory motion : average may exist, or not
- Entropy production, currents