Introduction to modeling turbulence in oceanic flows

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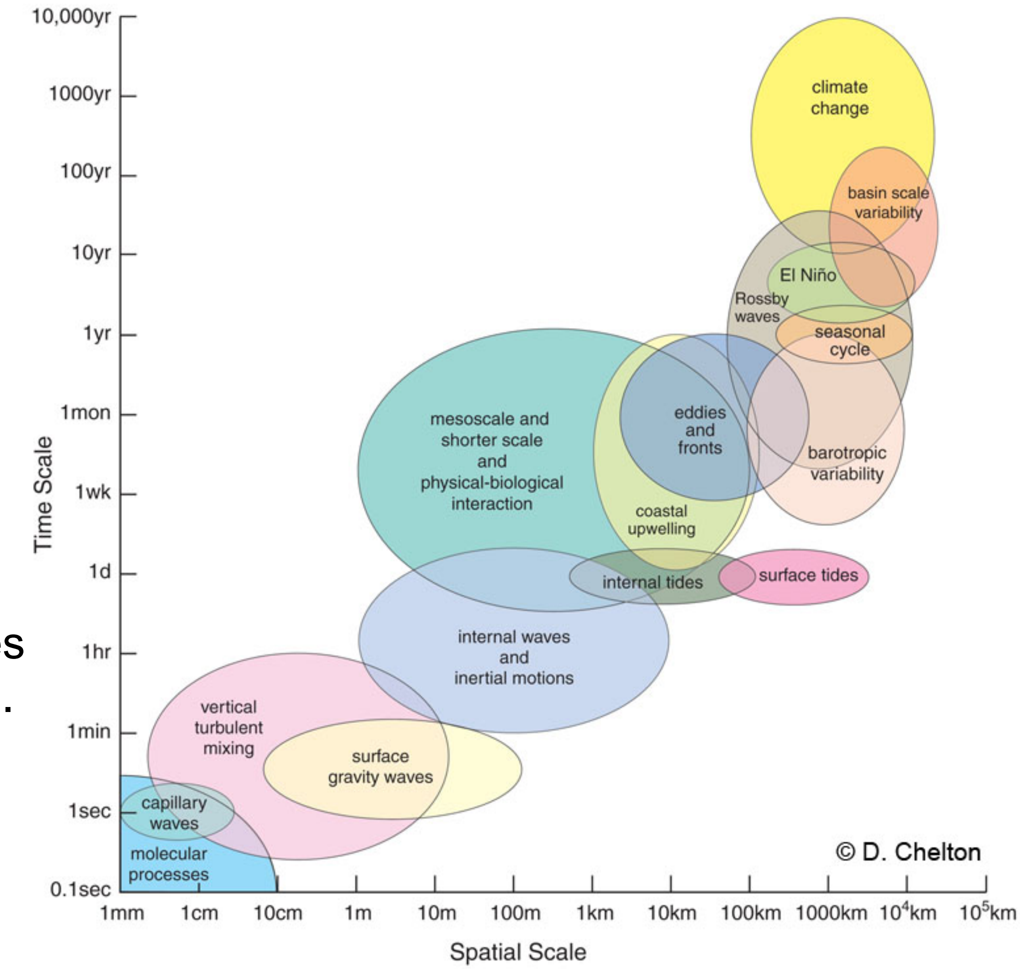
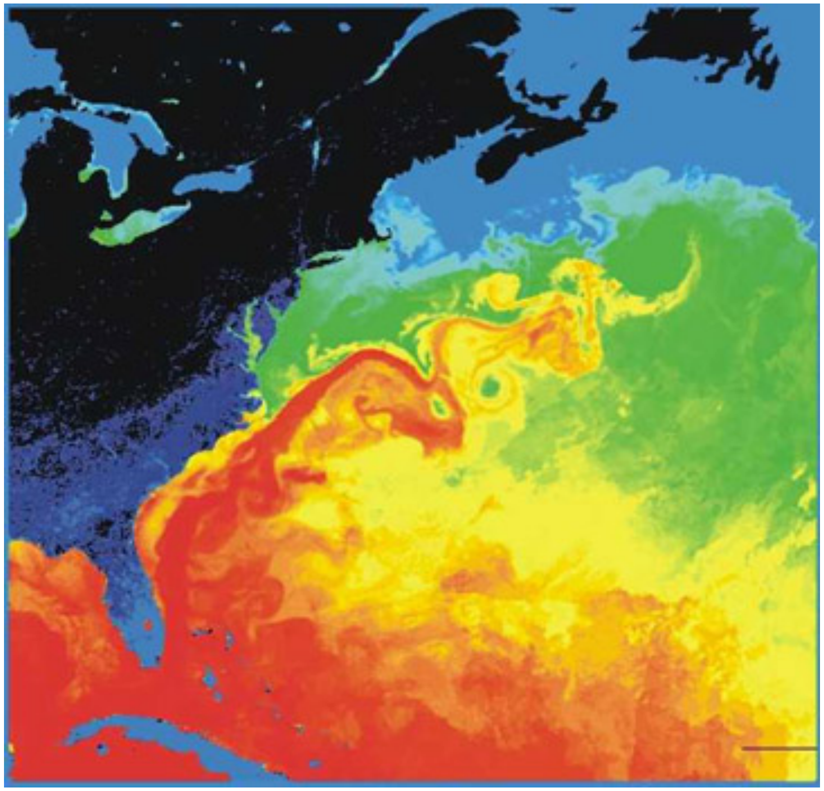
The biggest challenge in geophysical flows

- Large scale motion, L ~ O(10-1000) km.
- High Reynolds number, Re = UL/ν ~ O (10^{12}).
- Sampling/observation is a challenge:
  - Lab. experiments are limited.
  - Difficult to get χ(x,y,z,t) in the real ocean/atmosphere.
- Rotation of Earth.
- Stratification: effect of density differences
  - Vertical displacement are restricted.
  - Mixing between different density layers.
  - Presence of internal waves.



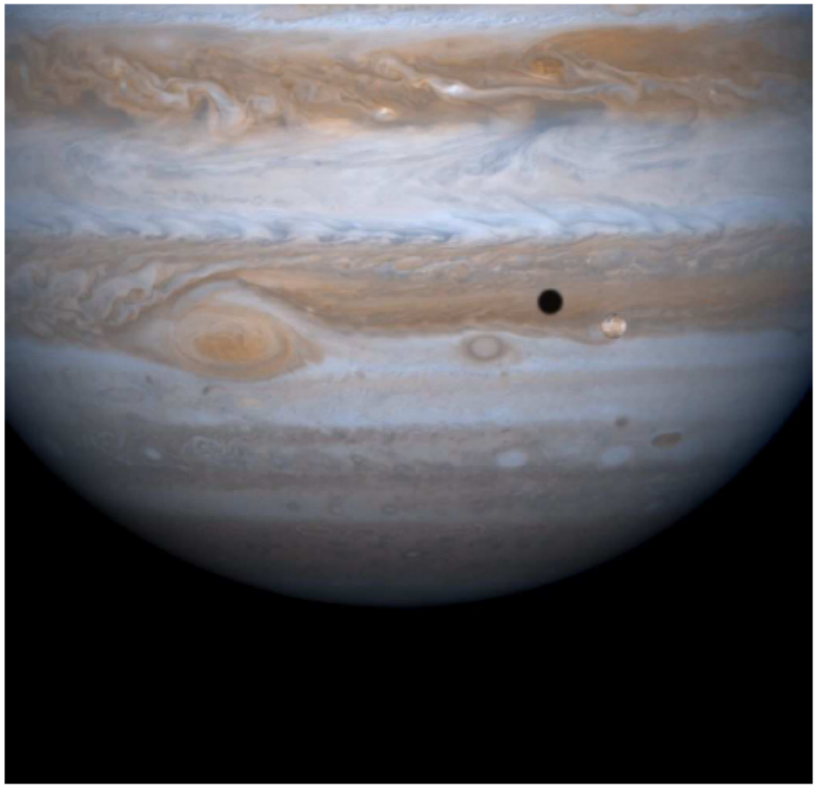
Geophysical flows

Gulf Stream



http://www.tidetech.org/system/files/01/gulfstream%20image.gif

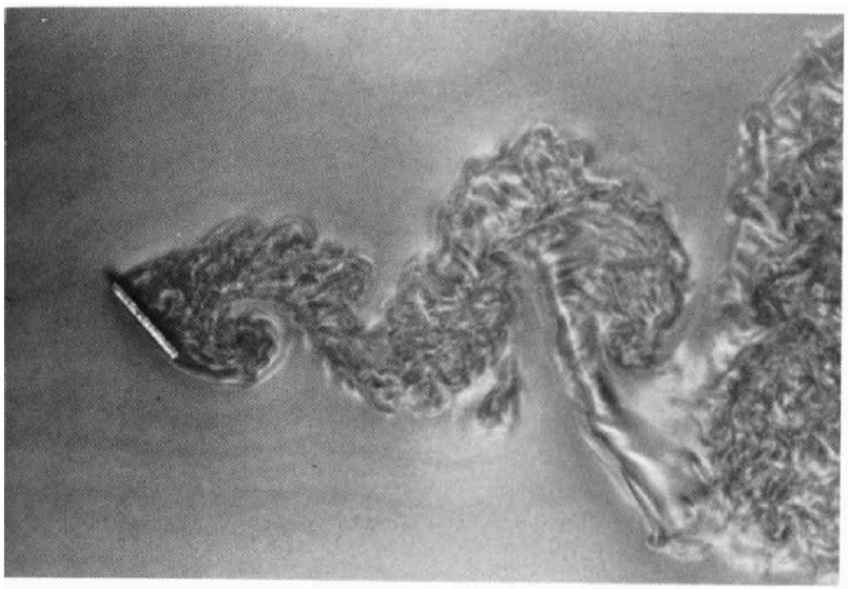
Great Red Spot



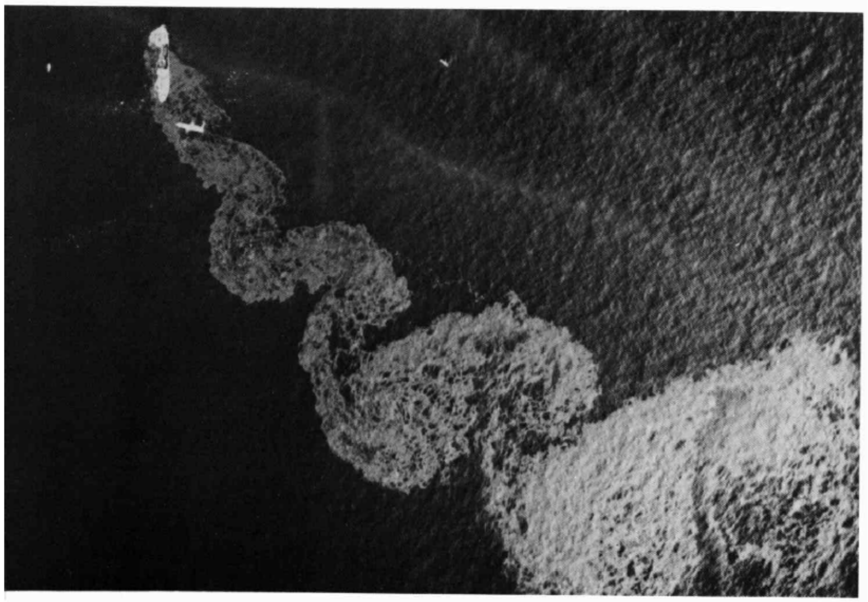
http://photojournal.jpl.nasa.gov/target/Jupiter

Turbulent coherent structures

Lab experiment: Re=4300



Oil discharge from a ship: $Re \sim 10^7$



from p. 100 of Van Dyke (1982)

Navier-Stokes equations

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{\partial p}{\rho_0\partial x}+\frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu\frac{\partial u}{\partial z}\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{\partial p}{\rho_0\partial y}+\frac{\partial}{\partial x}\left(\nu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu\frac{\partial v}{\partial z}\right)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\rho_0\partial z}-g\frac{\rho'}{\rho_0}+\frac{\partial}{\partial x}\left(\nu\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu\frac{\partial w}{\partial z}\right)$$

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=-\frac{\partial p}{\rho_0\partial x_i}-g_i\frac{\rho'}{\rho_0}+\frac{\partial}{\partial x_j}\left(\nu\frac{\partial u_i}{\partial x_j}\right)$$

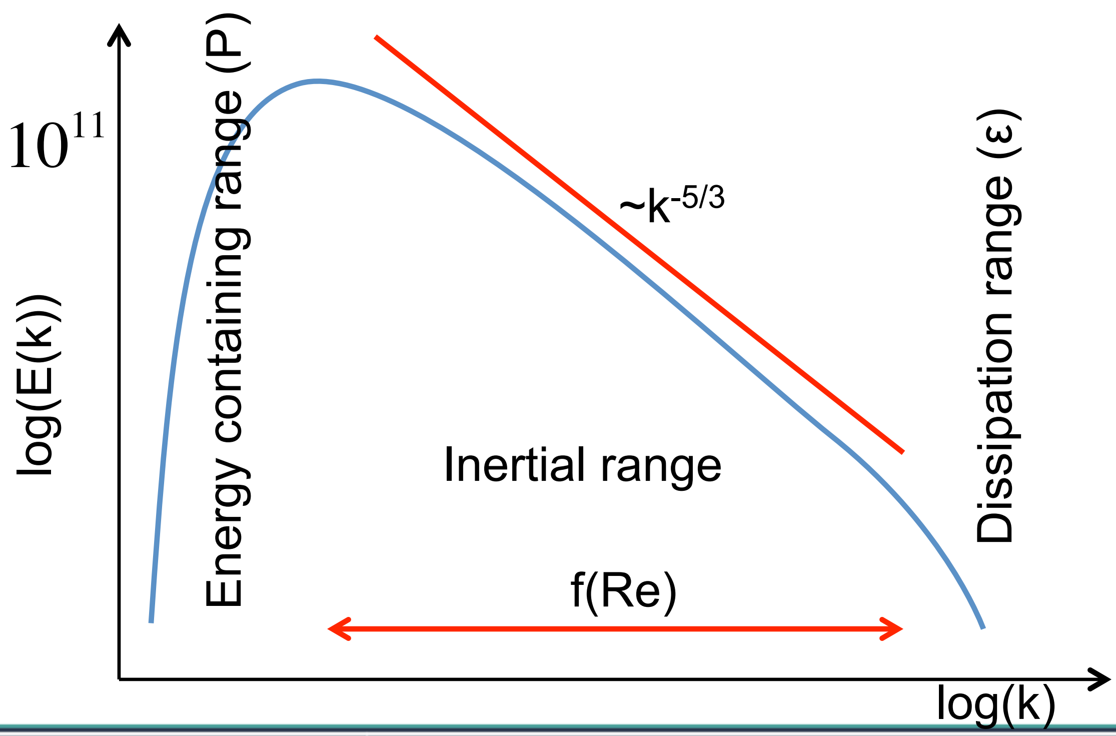
$$i,j=1,2,3$$

Modeling turbulent spectrum

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\rho_0 \partial x_i} - g_i \frac{\rho'}{\rho_0} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial u_i}{\partial x_j}\right)$$

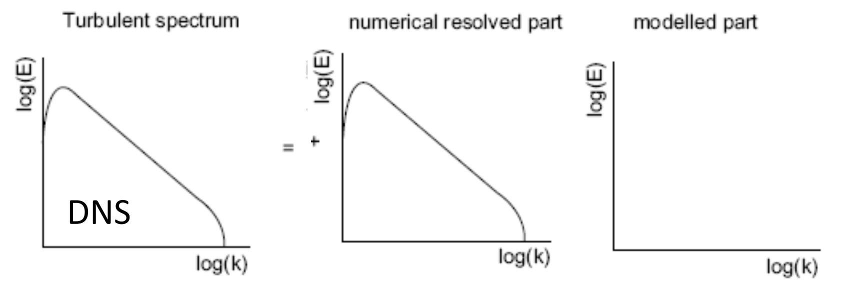
$$\mathrm{Re} = \frac{UL}{\nu}$$

$$\mathrm{Re} = \frac{1[m/s] \times 100 \times 10^3 [m]}{10^{-6}[m^2/s]} = 10^{11}$$



Possible modeling approaches

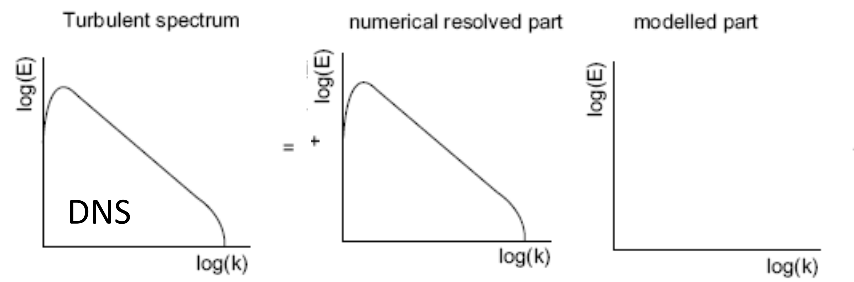
- Direct Numerical Simulations
    - Resolved all scales of motion (i.e. molecular viscosity), but how?



$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\rho_0 \partial x_i} - g_i \frac{\rho'}{\rho_0} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial u_i}{\partial x_j} \right)$$

Possible modeling approaches

- DNS
  - Resolved all scales



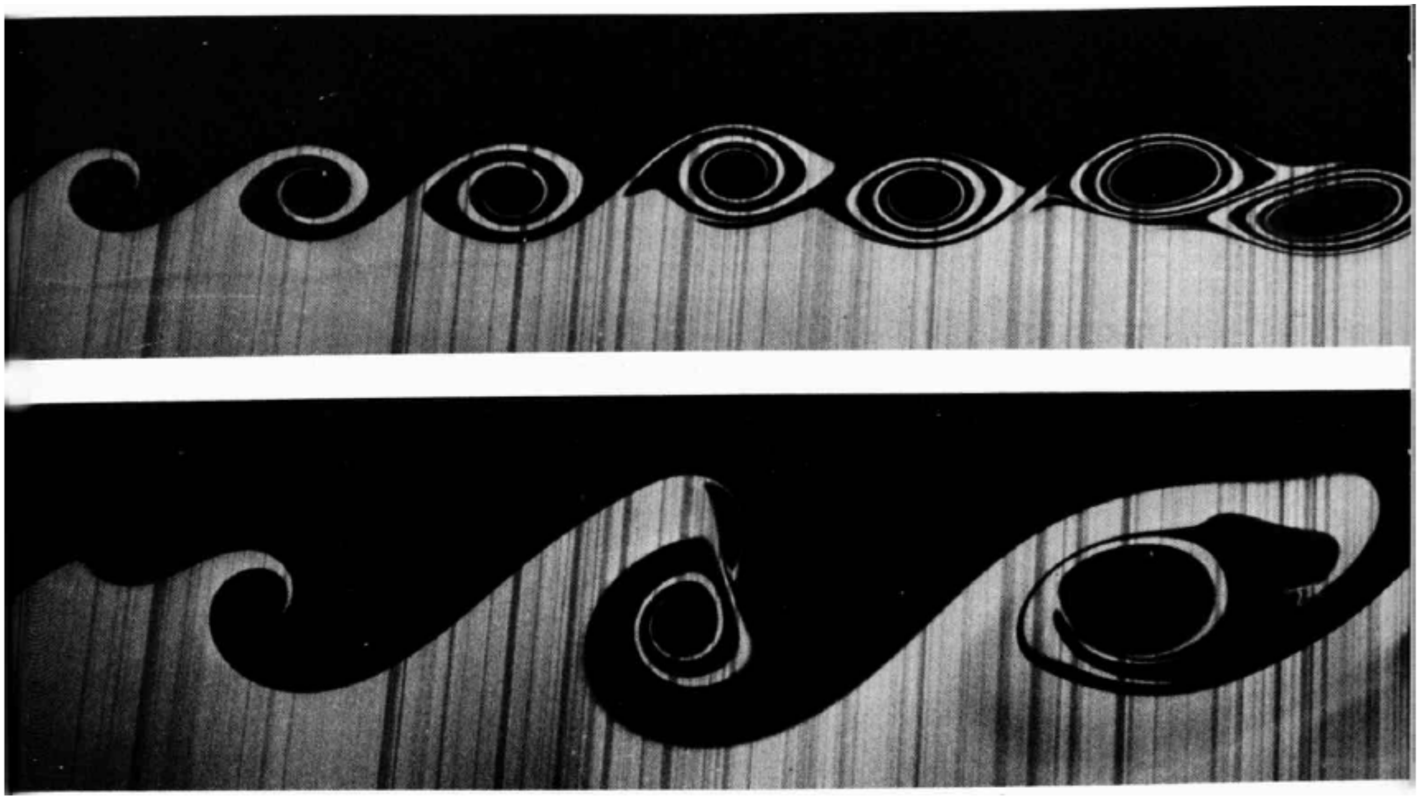
$$N^3 \sim \mathrm{Re}^{9/4}$$

$$\mathrm{Re}_\Delta = \frac{U\Delta x}{\nu} = 2$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\rho_0 \partial x_i} - g_i \frac{\rho'}{\rho_0} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial u_i}{\partial x_j}\right)$$

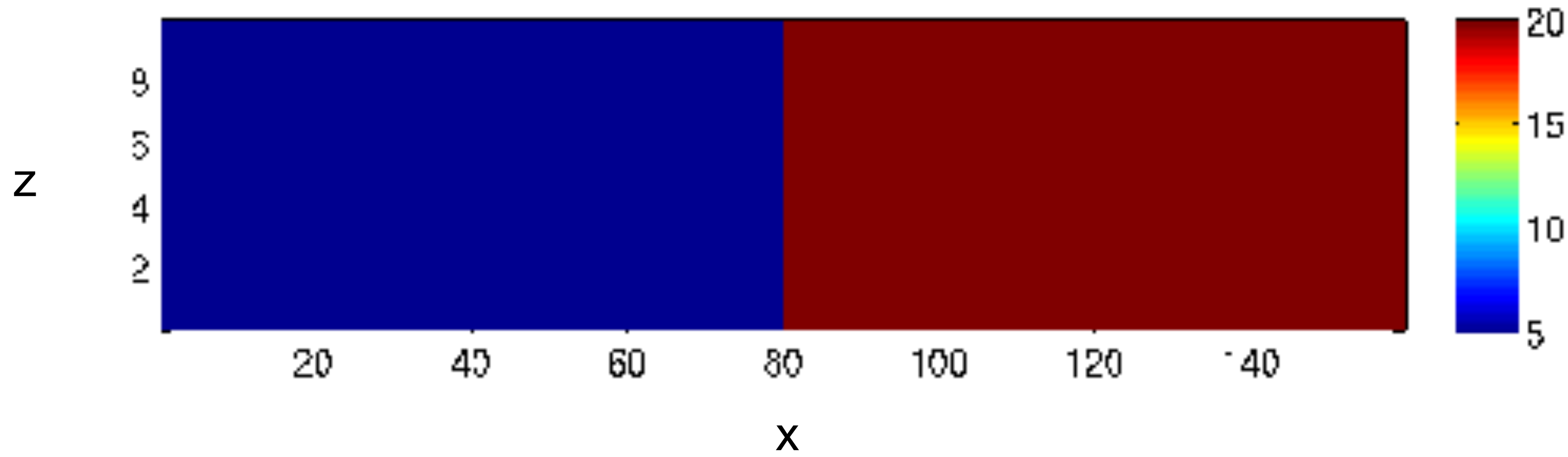
Kelvin-Helmholtz instability

from p. 89 of Van Dyke (1982)



Upper fluid column is moving to the right faster then the lower one.

Direct numerical simulation (DNS)



- 3D Non-hydrostatic model
- $4096 \times 64 \times 256 = 67.1 \times 10^6$ grid points in x,y and z directions
- Run in 128 processors

Reynolds Averaging Navier-Stokes (RANS)

$$u_i = U_i + u_i'$$

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + \underline{\frac{\partial}{\partial x_j}\left(\overline{u_j' u_i'}\right)} = -\frac{\partial p}{\rho_0 \partial x_i} - g_i \frac{\rho}{\rho_0} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial U_i}{\partial x_j}\right)$$

$$\frac{\partial \overline{u_i' u_j'}}{\partial t} + U_k \frac{\partial \overline{u_i' u_j'}}{\partial x_k} + \underline{\frac{\partial \overline{u_k' u_i' u_j'}}{\partial x_k}} = P_{ij} + \prod_{ij} - \varepsilon_{ij} + \frac{\partial}{\partial x_k}\left(\nu \frac{\partial \overline{u_i' u_j'}}{\partial x_k}\right)$$

unknowns > # equations → System is NOT closed

The "famous" Turbulence Closure Problem

Turbulence closure models

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\overline{u_j' u_i'}\right) = -\frac{\partial p}{\rho_0 \partial x_i} - g_i \frac{\rho}{\rho_0} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial U_i}{\partial x_j}\right)$$

Eddy viscosity approach

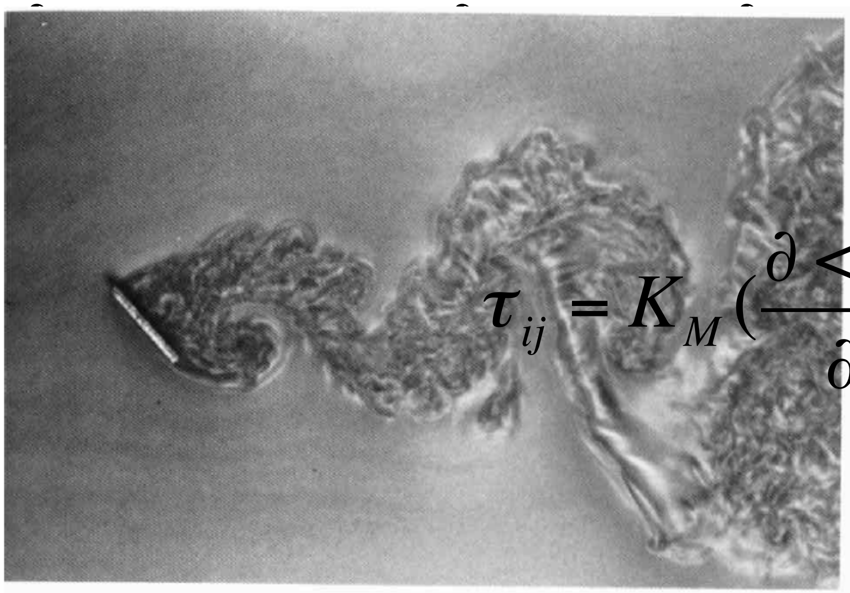
$$\overline{u'w'} = -K_M \frac{\partial U}{\partial z}$$

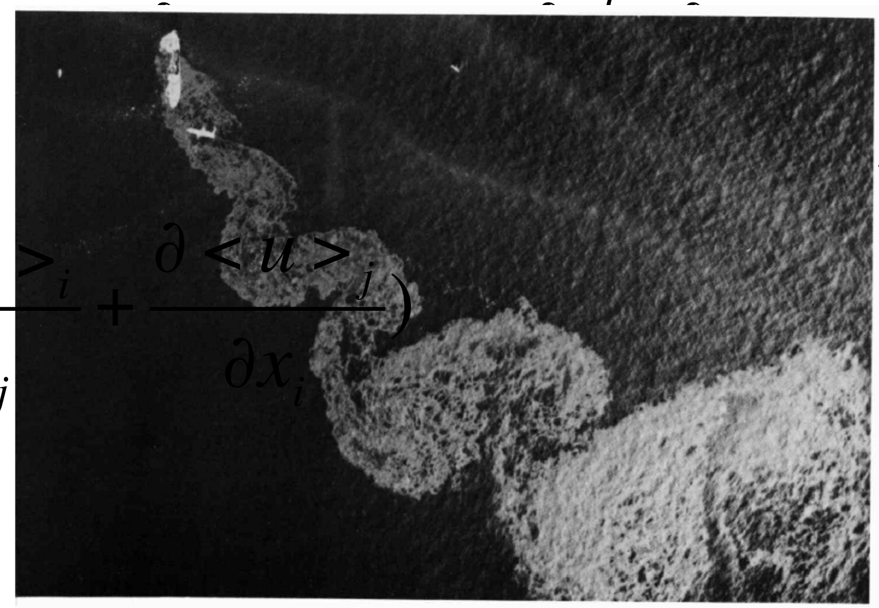
We need to find a way to compute eddy viscosity/diffusivity.
Ex: KPP, k-epsilon, k-omega, MY, standard Smagorinsky.

Large Eddy Simulations (LES)

Large Eddy Simulation (LES): Mixing by the large, energy-containing, eddies is handled through computation, while the effect of small, dissipative, turbulent eddies is modeled analytically.

$$u_i = <u>_i + u_i'$$

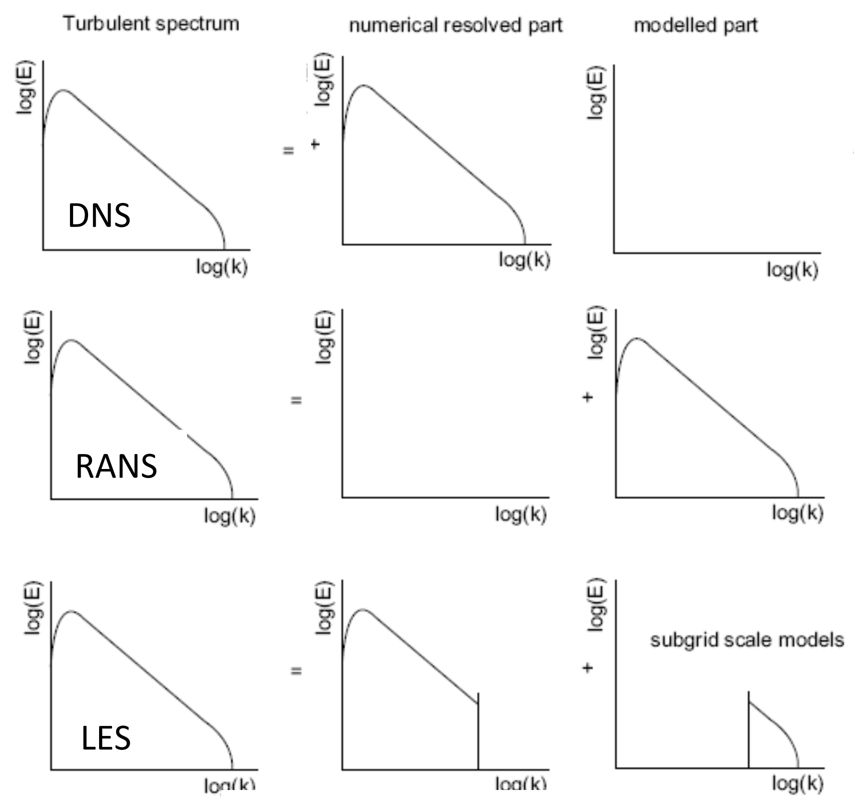




$$\tau_{ij} = K_M\left(\frac{\partial <u>_i}{\partial x_j} + \frac{\partial <u>_j}{\partial x_i}\right)$$

Possible modeling approaches

- DNS $(N^3 \sim \mathrm{Re}^{9/4})$
  - Resolved all scales
- RANS
  - Resolved mean state
- LES
  - Resolved large eddies



Ilıcak et al. (2007)

Possible modeling closures

- SGS turbulence closures; $\overline{u'w'} = -K_M \frac{\partial U}{\partial z}$
  - RANS
    - Constant eddy viscosity/diffusivity
    - Prandtl's (1925) 1 eq. model $K_M = \sqrt{k}l$
    - KPP (Large et al. 1994)
    - 2 eq. turbulence closures (k-epsilon, k-omega, k-kl, MY)
  - LES
    - Constant eddy viscosity/diffusivity
    - Standard Smagorinsky $K_M = C_1 \Delta^2 \sqrt{\frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)^2}$
    - Dynamical Smagorinsky

Ocean model equations

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{\partial p}{\rho_0 \partial x}+\frac{\partial}{\partial x}\left(A_M\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(A_M\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_M\frac{\partial u}{\partial z}\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{\partial p}{\rho_0 \partial y}+\frac{\partial}{\partial x}\left(A_M\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(A_M\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_M\frac{\partial v}{\partial z}\right)$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=\frac{\partial}{\partial x}\left(A_H\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(A_H\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_H\frac{\partial T}{\partial z}\right)$$

- **Smagorinsky**
- **Biharmonic viscosity**
- **Leith**
- **Numerical (i.e. ROMS)**

- **Constant eddy viscosity**
- **Ad-hoc (i.e. Price-Turner, KPP)**
- **One/two eq. turbulence schemes.**

One-equation turbulence closure

$$K_M = \sqrt{k}\, l$$

$$\frac{\partial k}{\partial t} + U_i \frac{\partial k}{\partial x_i} = \frac{\partial}{\partial z}\left(\frac{K_\mathrm{M}}{\sigma_k}\frac{\partial k}{\partial z}\right) + P + B - \varepsilon$$

$$P = -\langle u'w' \rangle \frac{\partial u}{\partial z} - \langle v'w' \rangle \frac{\partial v}{\partial z} = K_\mathrm{M} M^2, \qquad M^2 = \left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2$$

$$B = -\frac{g}{\rho_0}\langle \rho' w' \rangle = -K_\mathrm{H} N^2, \qquad N^2 = -\frac{g}{\rho_0}\frac{\partial \rho}{\partial z}$$

$$\varepsilon = k^{3/2} l$$

where l is prescribed.

Two-equation turbulence closures

$$\frac{\partial k}{\partial t} + U_i \frac{\partial k}{\partial x_i} = \frac{\partial}{\partial z}\left(\frac{K_\mathrm{M}}{\sigma_k}\frac{\partial k}{\partial z}\right) + P + B - \varepsilon$$

$$\frac{\partial \psi}{\partial t} + U_i \frac{\partial \psi}{\partial x_i} = \frac{\partial}{\partial z}\left(\frac{K_\mathrm{M}}{\sigma_\psi}\frac{\partial \psi}{\partial z}\right) + \frac{\psi}{k}(c_1 P + c_3 B - c_2 \varepsilon F_\mathrm{wall})$$

$$P = -\langle u'w'\rangle \frac{\partial u}{\partial z} - \langle v'w'\rangle \frac{\partial v}{\partial z} = K_\mathrm{M} M^2, \qquad M^2 = \left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2$$

$$B = -\frac{g}{\rho_0}\langle \rho' w'\rangle = -K_\mathrm{H} N^2, \qquad N^2 = -\frac{g}{\rho_0}\frac{\partial \rho}{\partial z}$$

$$\varepsilon = (c_\mu^0)^{3+p/n} k^{3/2+m/n} \psi^{-1/n}$$

$$\psi = (c_\mu^0)^p k^m l^n$$

$$l = (c_\mu^0)^3 k^{3/2} \varepsilon^{-1}$$

$$\omega = \frac{\varepsilon}{(c_\mu^0)^4 k}$$

$$K_M = c\sqrt{2k}\, l S_M$$

$$K_H = c\sqrt{2k}\, l S_H$$

Dynamical Smagorinsky

$$\bar{\mathbf{u}}_t + \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} + \nabla \bar{p} - Re^{-1} \nabla^2 \bar{\mathbf{u}} + Fr^{-2} \overline{\rho'} \mathbf{k} = -\nabla \cdot \boldsymbol{\tau},$$
$$\nabla \cdot \bar{\mathbf{u}} = 0,$$
$$\overline{\rho'}_t + \bar{\mathbf{u}} \cdot \nabla \overline{\rho'} - (Re\,Pr)^{-1} \nabla^2 \overline{\rho'} = -\nabla \cdot \boldsymbol{\sigma},$$

$$\boldsymbol{\tau} = -2 \left(c_{ds} \delta \right)^2 \left| \nabla^s \bar{\mathbf{u}} \right| \nabla^s \bar{\mathbf{u}},$$

where $\nabla^s \bar{\mathbf{u}} := (\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T)/2$

$$c_{ds}^2 = \frac{1}{2} \frac{< M_{ij} L_{ij} >}{< M_{kl} M_{kl} >}$$

$$M_{ij} = \tilde{\delta}^2 \left| \nabla^s \tilde{\bar{u}} \right| \nabla^s \tilde{\bar{u}}_{ij} - \delta^2 \widetilde{\left| \nabla^s \bar{u} \right| \nabla^s \bar{u}}_{ij} = \delta^2 \left(\alpha^2 \left| \nabla^s \tilde{\bar{u}} \right| \nabla^s \tilde{\bar{u}}_{ij} - \widetilde{\left| \nabla^s \bar{u} \right| \nabla}^{s} \bar{u}^2 \right)$$

$$\widehat{M}_{ij} \equiv M_{ij} / \delta^2$$

$$c_{ds}^2 = \frac{1}{2} \frac{< \widehat{M}_{ij} L_{ij} >}{< \widehat{M}_{kl} \widehat{M}_{kl} >}$$

Eddy viscosity & diffusivity

$$\overline{u'w'} = -K_M \frac{\partial U}{\partial z}$$

$$\overline{T'w'} = -K_H \frac{\partial T}{\partial z}$$

$$\left.\begin{aligned} K_M &= c\sqrt{2k}\, l S_M \\ K_H &= c\sqrt{2k}\, l S_H \end{aligned}\right\} \Rightarrow \mathrm{Pr}_t = \frac{K_M}{K_H} = \frac{S_M}{S_H}$$

K_M=Eddy viscosity

K_H=Eddy diffusivity

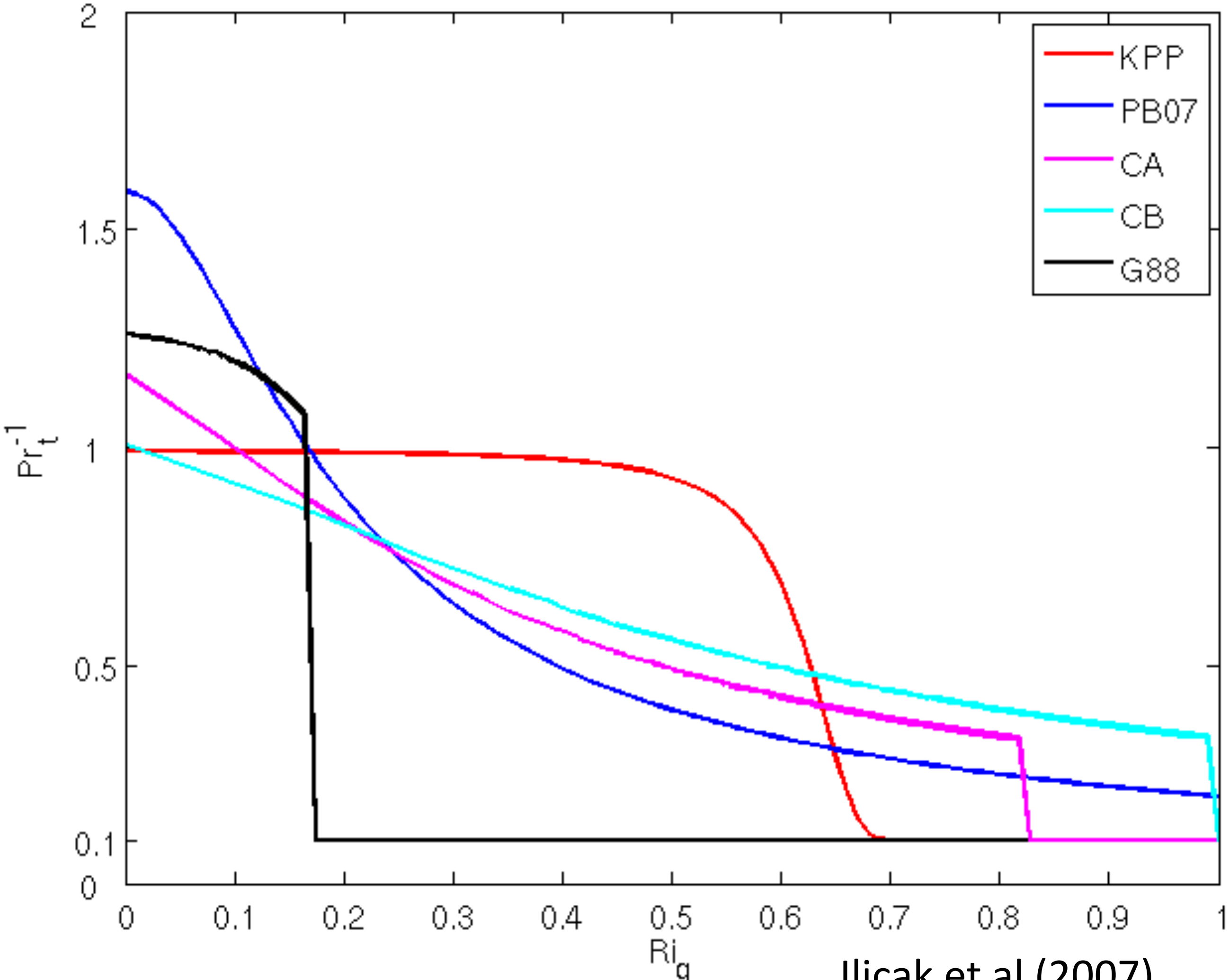
k=Turbulent kinetic energy

l=Turbulent length scale

Pr_t=Turbulent Prandtl Number

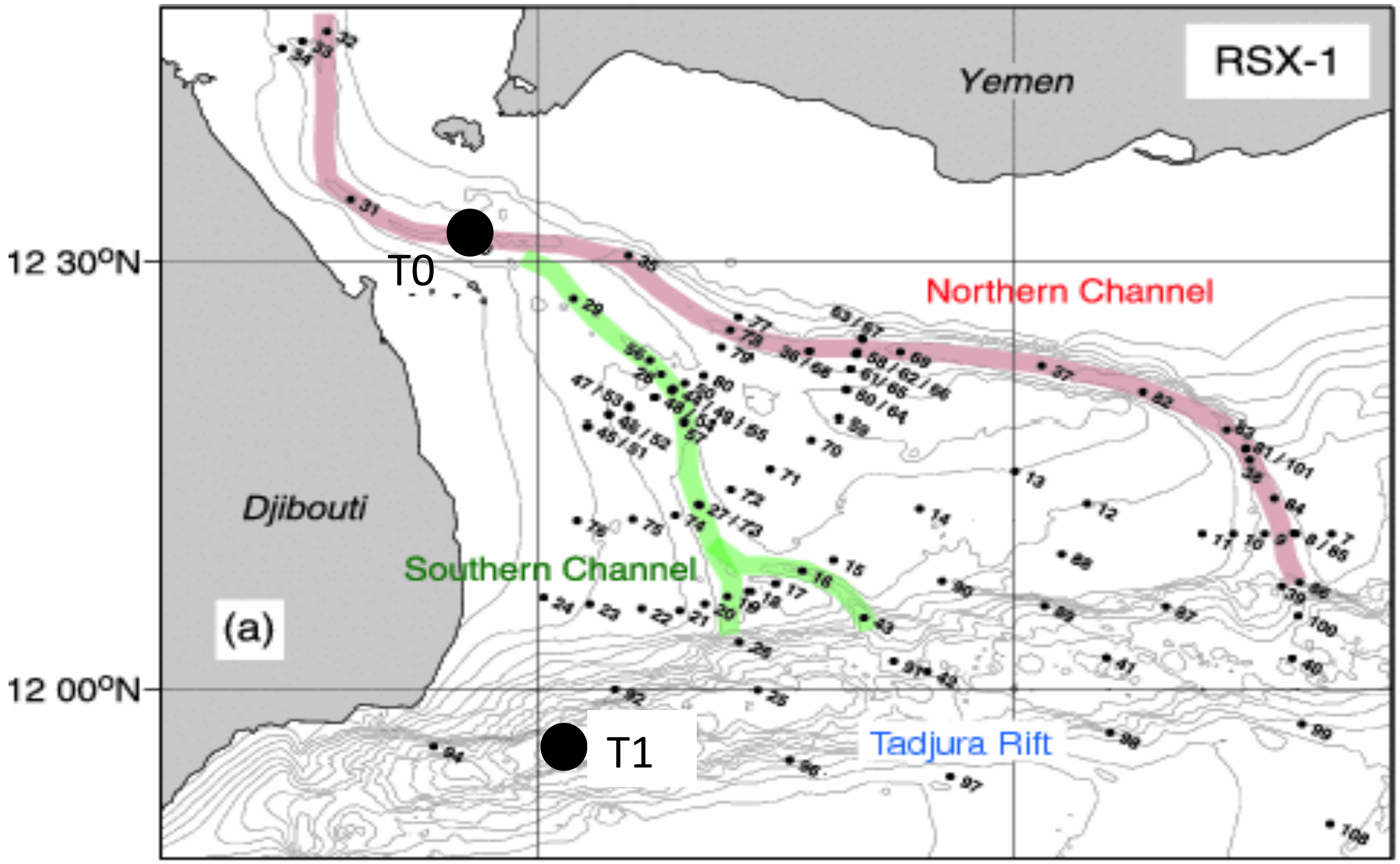
S_M and S_H= Very complex stability functions

$Pr_t(Ri_g)$



Ilicak et al (2007)

Red Sea overflow

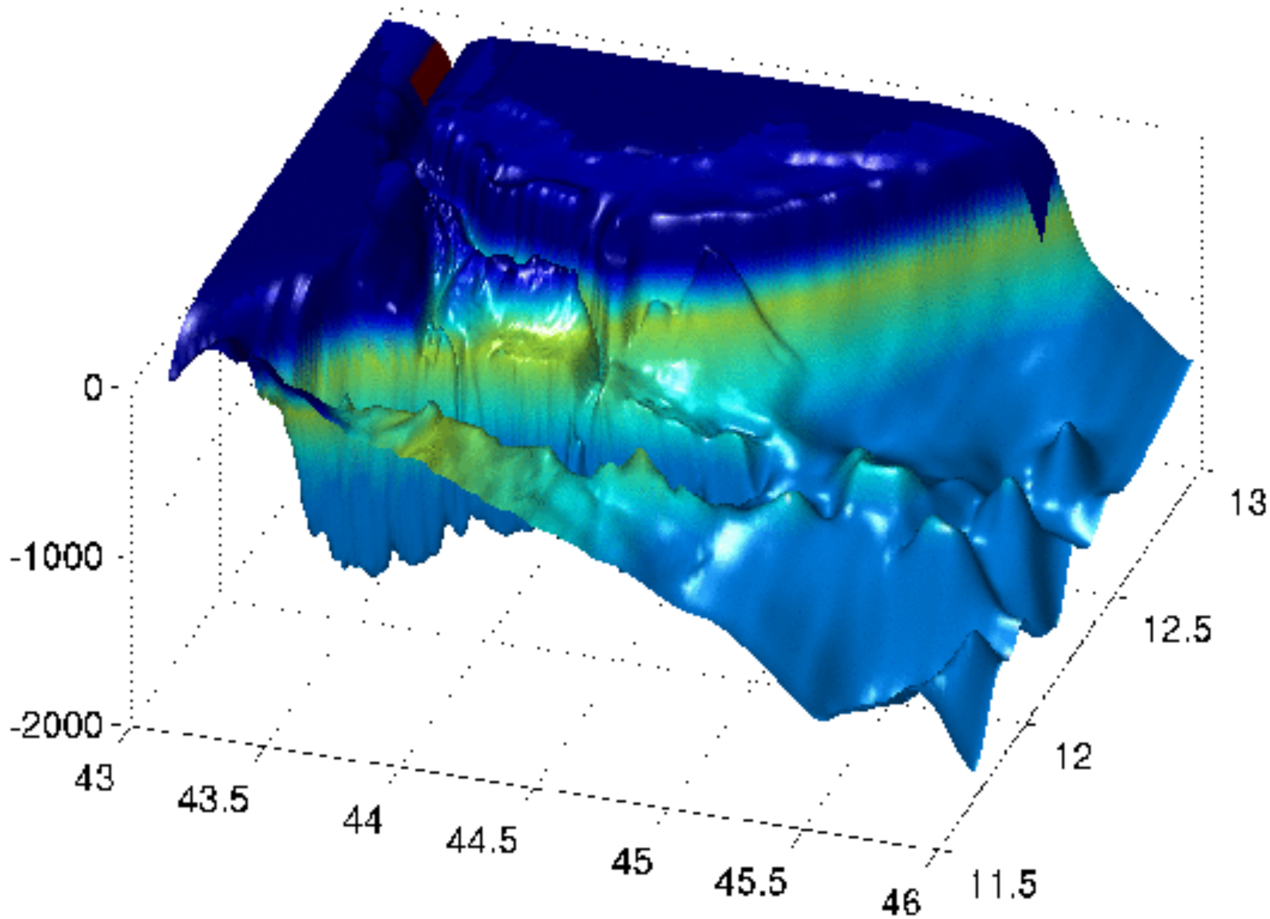


Experiments

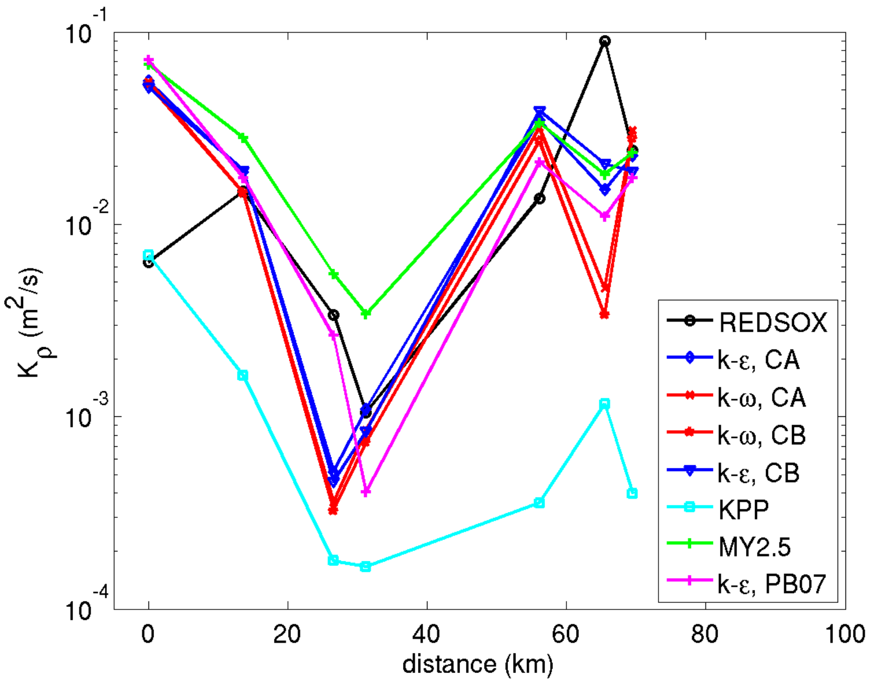
Experiment	Turbulence closure	Stability function
Exp1	k–ε	CA
Exp2	k–ω	CA
Exp3	k–ω	CB
Exp4	k–ε	CB
Exp5	KPP	none
Exp6	MY2.5	G88
Exp7	k–ε	PB07
Exp8	$K_M = K_H = 0$	None
Exp9	MY2.5	KC

- KPP = K-Profile Parameterization
- PB07 = Peters and Baumert (2007)
- CA = Canuto-A, CB = Canuto-B
- G88 = Galperin et al. (1988) , KC = Kantha-Clayson

Propagation of overflow



Eddy diffusivity

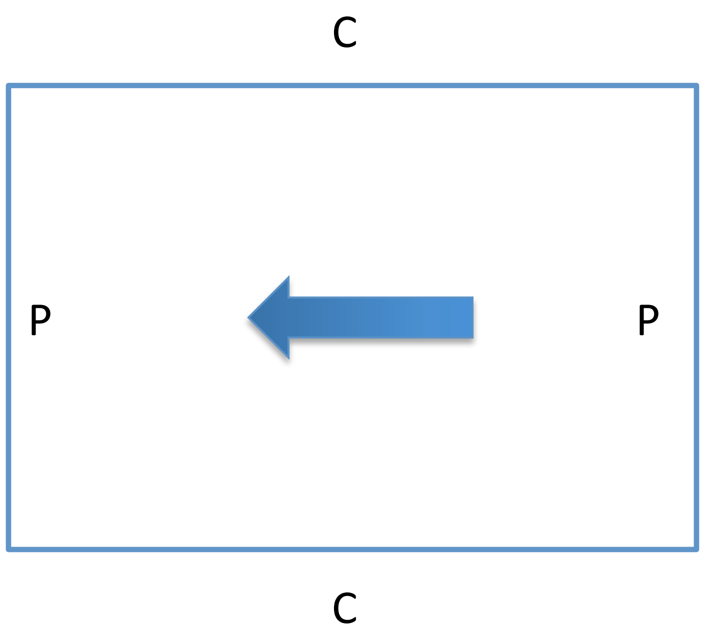


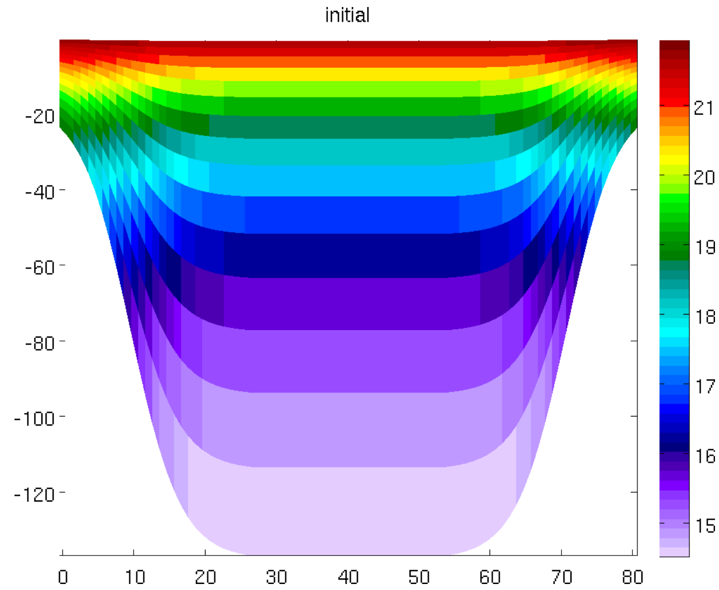
Northern Channel

Summary

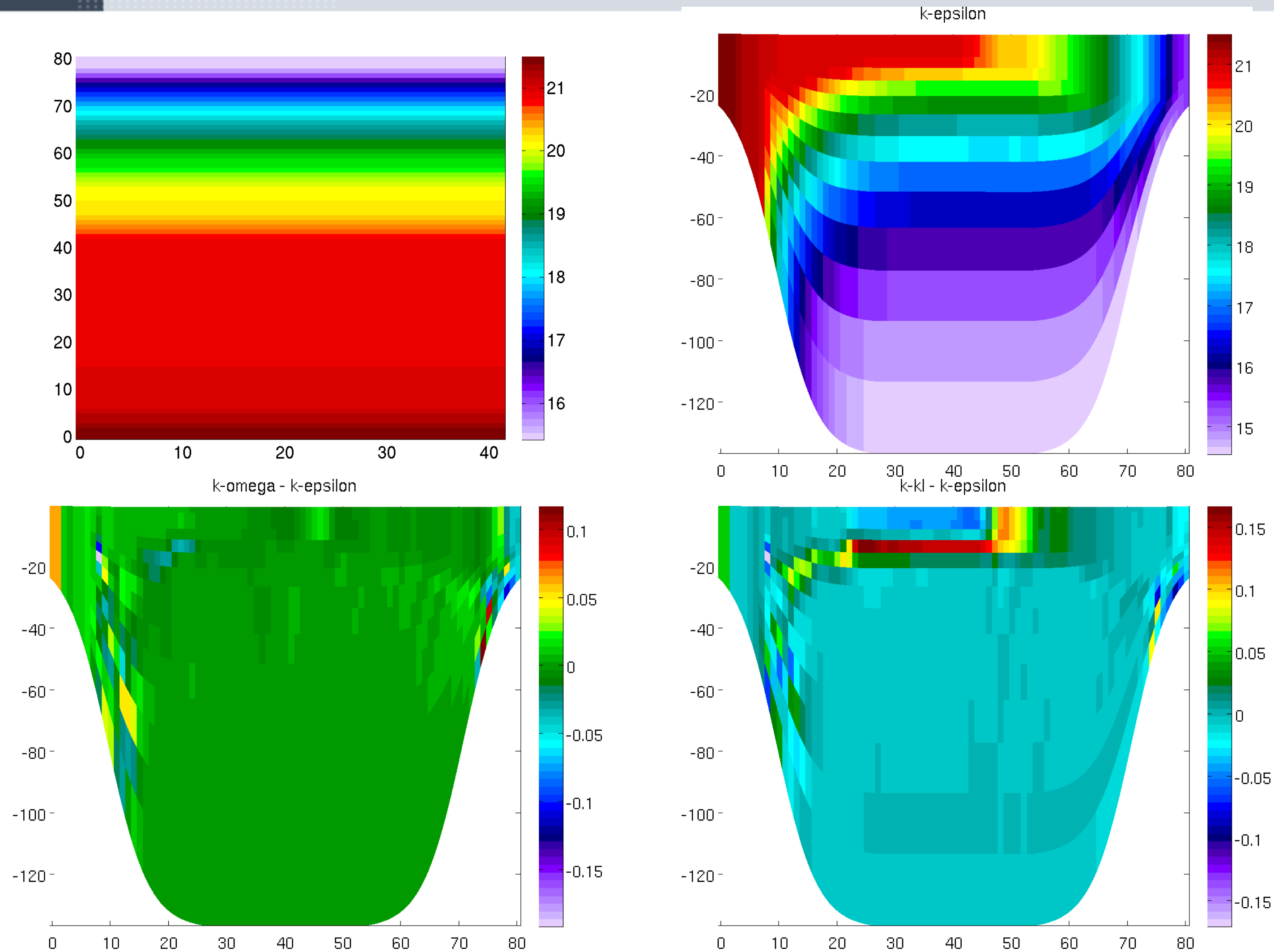
- There are different approaches to tackle turbulence in ocean models; DNS, RANS, LES
- Eddy viscosity/diffusivity approach is a modelling strategy.
- Different complexity of parameterizations out there. Be aware of shortcomings and advantages of these models.

Upwelling test case

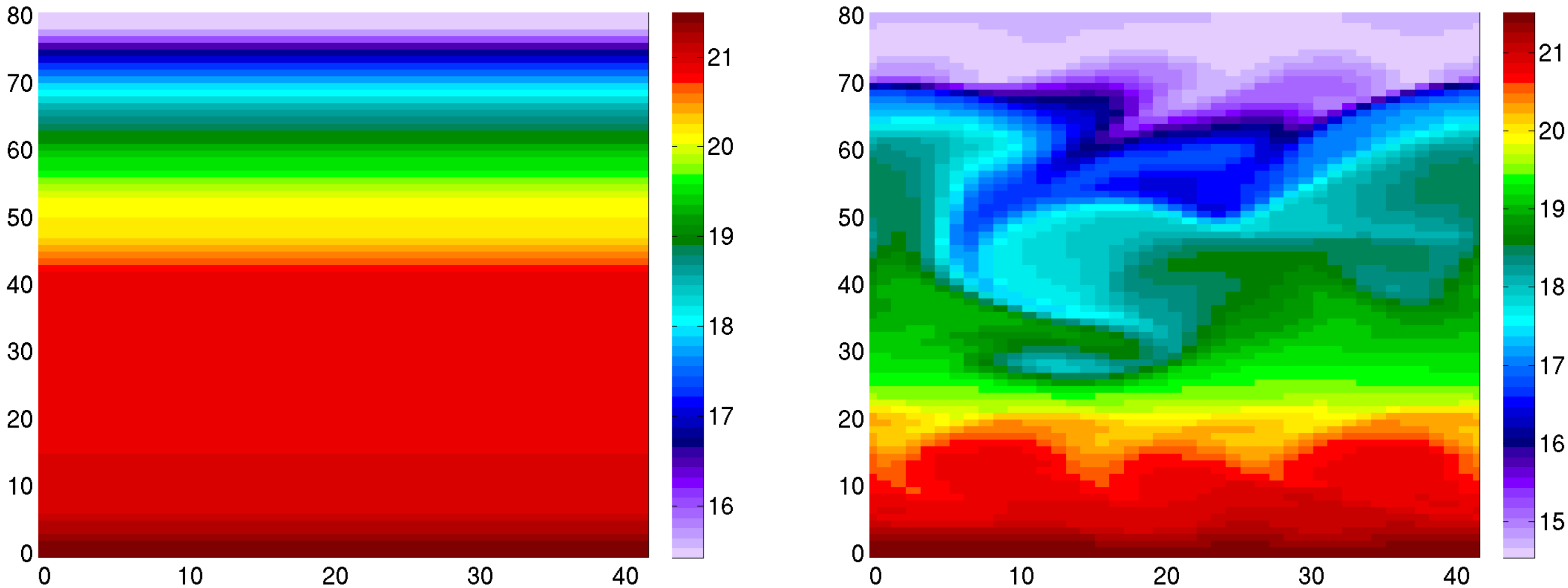




Upwelling test case



Upwelling test case



Stability functions

$$s_0 = (3/2)\lambda_1\lambda_5^2;\ s_1 = -\lambda_4(\lambda_6 + \lambda_7) + 2\lambda_4\lambda_5(\lambda_1 - (1/3)\lambda_2 - \lambda_3) + (3/2)\lambda_1\lambda_5\lambda_8$$

$$s_2 = -(3/8)\lambda_1(\lambda_6^2 - \lambda_7^2);\ s_4 = 2\lambda_5;\ s_5 = 2\lambda_4$$

$$s_6 = (2/3)\lambda_5(3\lambda_3^2 - \lambda_2^2) - (1/2)\lambda_1\lambda_5(3\lambda_3 - \lambda_2) + (3/4)\lambda_1(\lambda_6 - \lambda_7)$$

$$b_0 = 3\lambda_5^2;\ b_1 = \lambda_5(-7\lambda_4 + 3\lambda_8);\ b_2 = \lambda_5^2(3\lambda_3^2 - \lambda_2^2) - (3/4)(\lambda_6^2 - \lambda_7^2)$$

$$b_3 = \lambda_4(4\lambda_4 + 3\lambda_8);\ b_5 = (1/4)(\lambda_2^2 - 3\lambda_3^2)(\lambda_6^2 - \lambda_7^2)$$

$$b_4 = \lambda_4(\lambda_2\lambda_6 - 3\lambda_3\lambda_7 - \lambda_5(\lambda_2^2 - \lambda_3^2)) + \lambda_5\lambda_8(3\lambda_3^2 - \lambda_2^2)$$

$$f_6 = 8/(c_\mu^0)^6$$

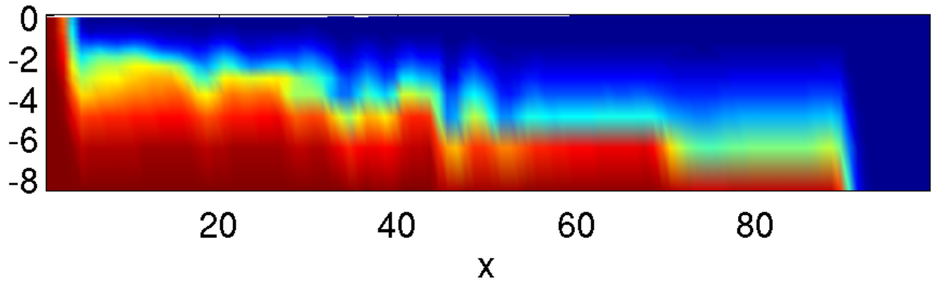
Canuto A and B (2001)

$$S_H = (s_0 - s_1 f_6 G_h + s_2 f_6 G_m)/cff;\ S_M = (s_4 - s_5 f_6 G_h + s_6 f_6 G_m)/cff$$

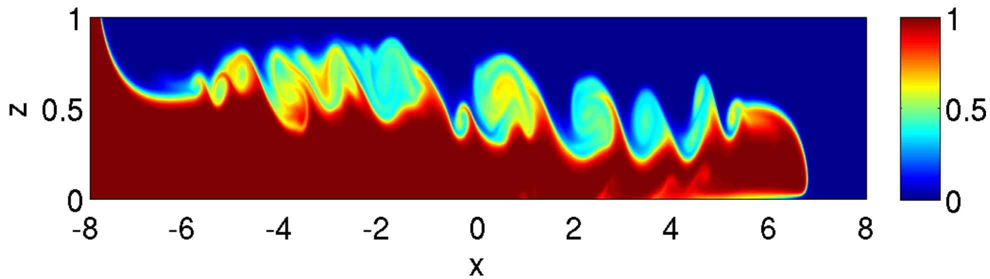
$$cff = b_0 - b_1 f_6 G_h + b_2 f_6 G_m + b_3 f_6^2 G_h^2 - b_4 f_6^2 G_h G_m + b_5 f_6^2 G_m^2$$

Hydrostatic vs. Non-hydrostatic

Hydrostatic model



Non-hydrostatic model



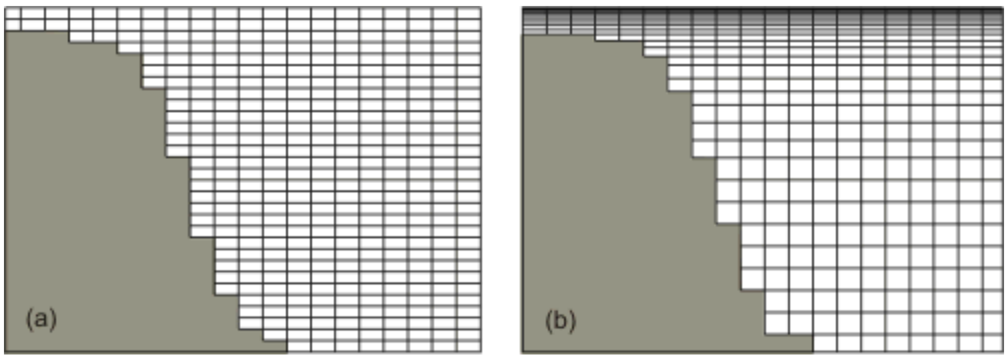
- Cannot resolve shear instabilities such as KH rolls.
- Depend on full closures such as
  - 2nd order turbulence models
  - Algebraic models

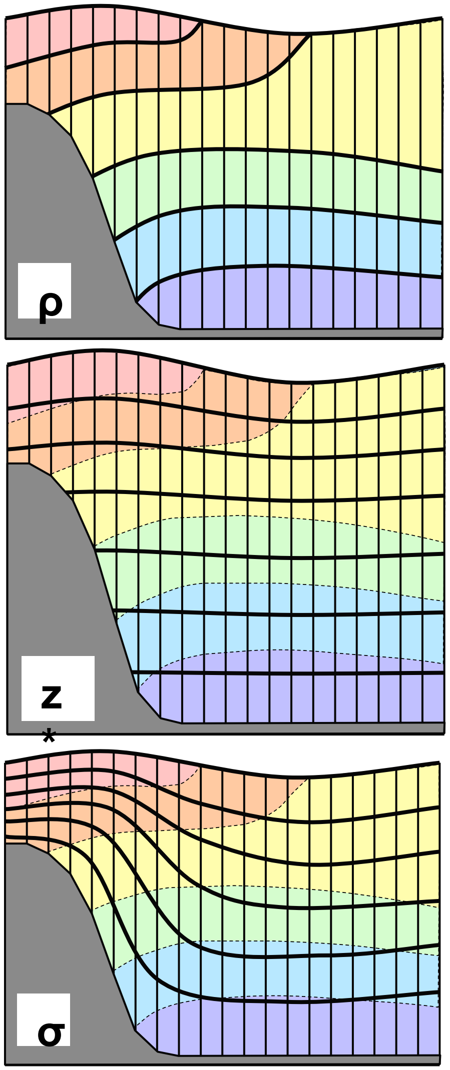
- Can resolve shear instabilities.
- Depend on resolution and sub-grid scale parameterization
  - LES
  - VLES

$$\cancel{\frac{dw}{dt}} = -\frac{\partial P}{\partial z} - \frac{\rho g}{\rho_0} + \cancel{\kappa \nabla^2 w}$$

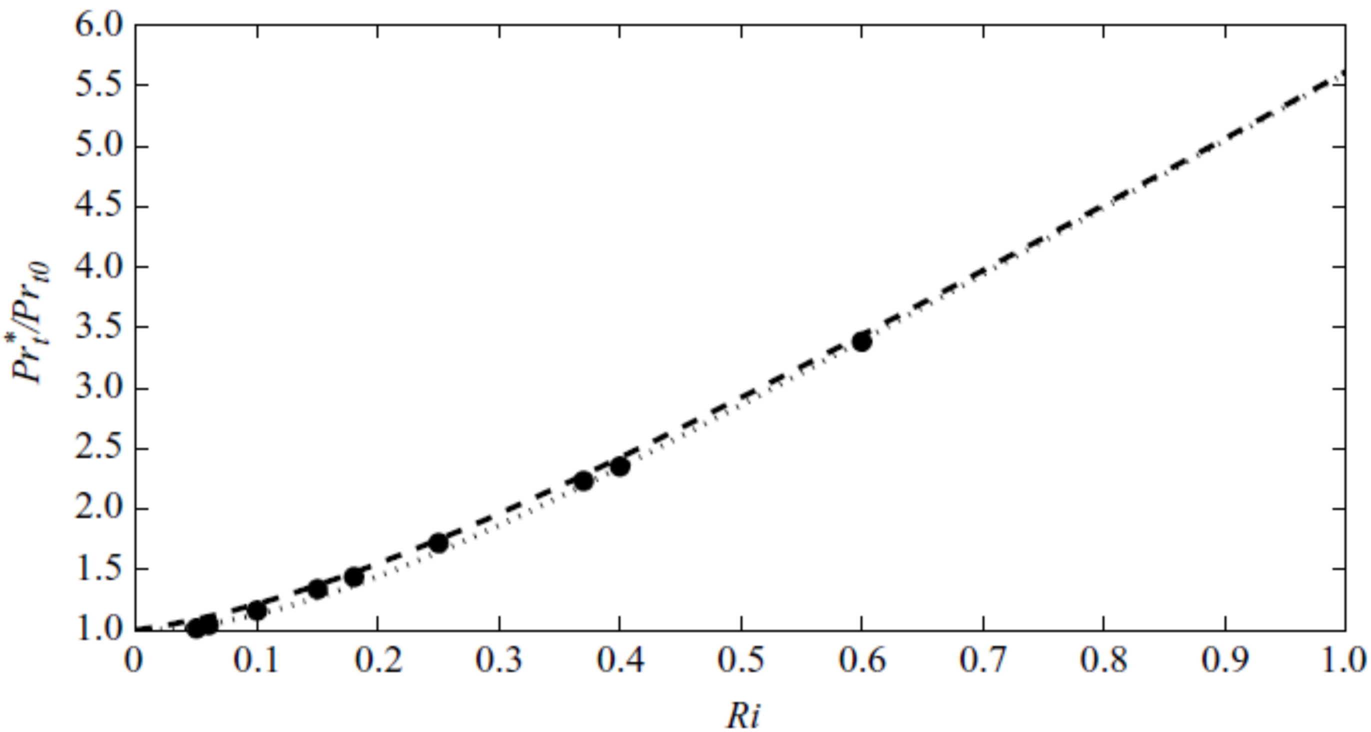
$$\frac{\partial P}{\partial z} = -\frac{\rho g}{\rho_0}$$

Vertical coordinate in OGCMs





The turbulent Prandtl number



$$\frac{Pr_t^*}{Pr_{t0}} = \exp\left(-\frac{Ri}{Pr_{t0}\Gamma_\infty}\right) + \frac{Ri}{R_{f\infty}Pr_{t0}},$$

Venayagamoorthy and Stretch (2010)