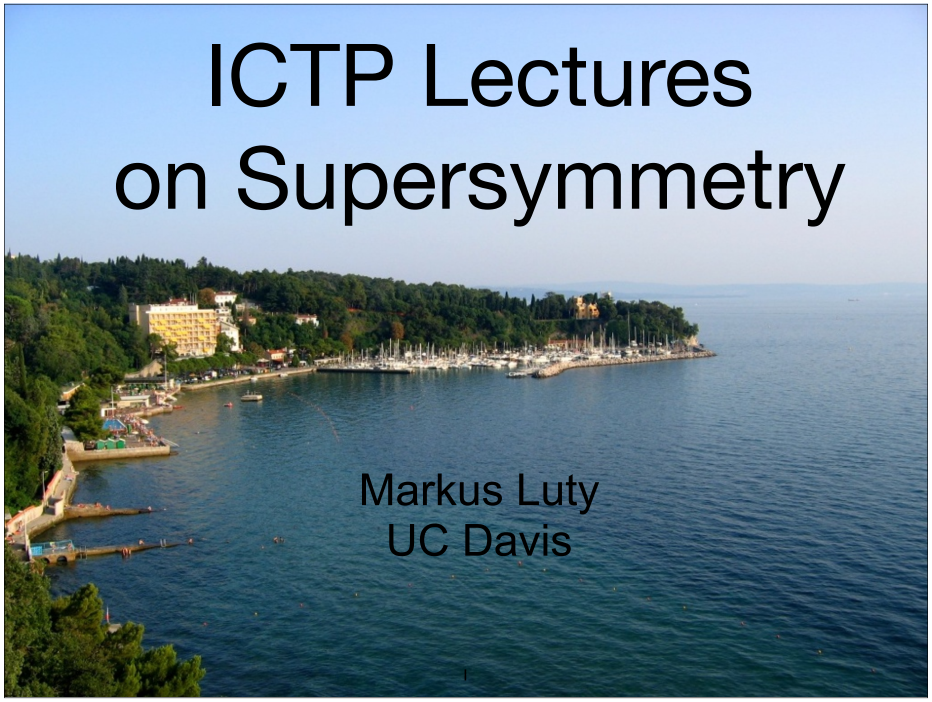
ICTP Lectures on Supersymmetry



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Outline

- The hierarchy problem
- Bose-Fermi symmetry in quantum mechanics
- Weyl fermions
- Supersymmetry in free quantum field theory
- The Supersymmetry Algebra
- Superspace
- Chiral superfields
- Gauge superfields

The Hierarchy Problem

Effective Field Theory

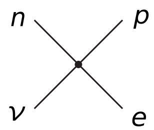
Effective theory = approximate description of physics valid in a limited dynamical range.

Example: the Navier-Stokes equations describe fluids on length scales large compared to atomic distances

Is the Standard Model an effective field theory? If so, at what scale does it break down?

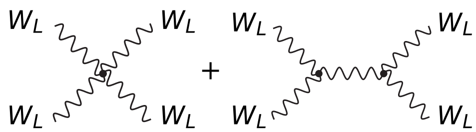
Until very recently, the theoretical description of weak interactions required new physics at the TeV scale:

1930s: Fermi theory



$\sim G_F E^2$ $\Rightarrow$ breaks down for $E \gtrsim G_F^{-1/2} \sim \text{TeV}$

1980s: W, Z bosons, no Higgs



$$\sim \frac{g^2 E^2}{m_W^2} \sim G_F E^2$$

2012: Standard Model with Higgs

Can be consistently extrapolated all the way to the Planck scale. *No guarantee of new physics!*

The Standard Model many important phenomena unexplained $\Rightarrow$ new physics beyond the Standard Model.

Experimental facts:

- Neutrino masses
- Dark matter
- Cosmological density perturbations
- Baryogenesis

Theoretically motivated:

- Grand unification
- Origin of fermion masses and mixing
- Naturalness of the electroweak scale

Only naturalness requires new physics at the TeV scale.

Consider a coupling constant λ with mass dimenson n

$$[\lambda] = n \qquad \lambda = M^n \qquad M = \text{mass scale}$$

Treat λ as a perturbation:

$$\mathcal{A}(E) \sim \underbrace{\mathcal{A}_0(E)}_{=O(\lambda^0)} \left[1 + \underbrace{\left(\frac{M}{E}\right)^n}_{=O(\lambda^1)} + \cdots \right] \qquad E = \text{physical energy scale}$$

$n > 0$: perturbation theory breaks down at small E
relevant coupling

$n < 0$: perturbation theory breaks down at large E
irrelevant coupling

$n = 0$: perturbation theory good at all E?
marginal coupling

There are an infinite number of irrelevant couplings:

$$[\phi] = [A_\mu] = 1, \qquad [\psi] = \tfrac{3}{2}$$

$$\Delta\mathcal{L} = \frac{1}{M^{64}} (\bar{\psi}\psi)^{12} \Box^9 \phi^{14} + \cdots$$

Assume $M \gg \text{TeV} \Rightarrow$ effects of irrelevant operators suppressed at low energies.

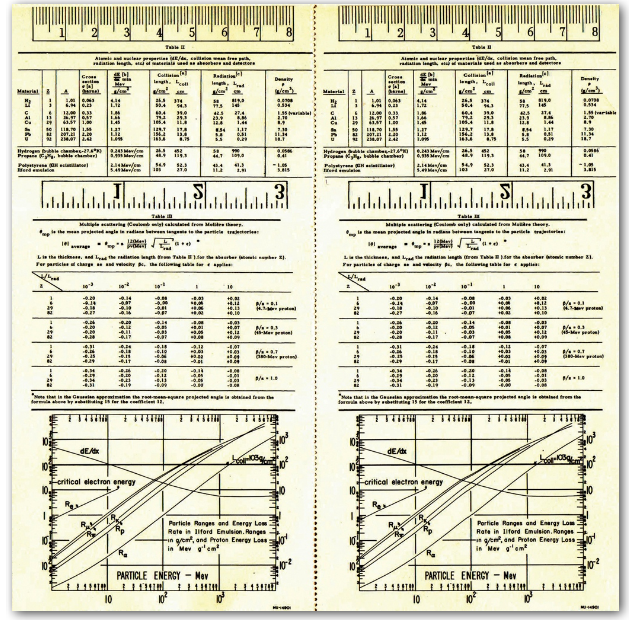
This naturally occurs if these operators are generated by integrating out new physics (particles) with mass scale $M \gg$ TeV.

Effective theory at low energies parameterized by a finite number of marginal and relevant couplings. [K. Wilson]

The Standard Model is the most general effective Lagrangian containing all relevant and marginal couplings of the experimentally observed elementary particles compatible with Lorentz symmetry and $\underbrace{\text{gauge invariance}}_{\text{needed for spin 1}}$.

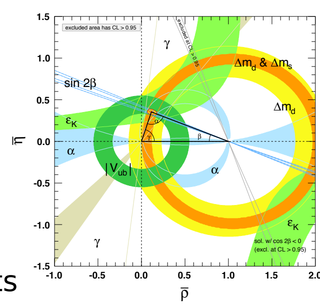
	$SU(3)_C$	$SU(2)_W$	$U(1)_Y$
q_L	3	2	$\frac{1}{6}$
u_R	3	1	$\frac{2}{3}$
d_R	3	1	$-\frac{1}{3}$
ℓ_L	1	2	$-\frac{1}{2}$
e_R	1	1	-1
h	1	2	$\frac{1}{2}$

$\times 3$



First Particle Data Group wallet card (1958)

This effective field theory has an amazing amount of predictive power, and agrees with all experiments performed to date.



- Weak decays
- Quark mixing, *CP* violation
- No flavor-changing neutral currents
- Baryon and lepton number symmetry

Is the standard model the perfect effective field theory?

The Standard Model contains one relevant coupling:

$$\mathcal{L}_{\rm SM} = -m_H^2 H^\dagger H + \cdots \qquad m_H^2 < 0$$

$$V = m_H^2 H^\dagger H + \lambda (H^\dagger H)^2$$

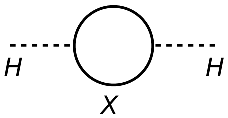
$$v^2 = -\frac{m_H^2}{\lambda}$$

$$m_h^2 = -2m_H^2 = \text{physical Higgs mass}$$

Dimensional analysis suggests that $m_H^2 \sim M^2 \gg$ TeV.

Is this really a problem?

X = any particle with mass $M_X \gg$ TeV



$$\Rightarrow \ \underbrace{\Delta m_H^2 \sim \frac{y^2}{16\pi^2} M_X^2}$$

finite large correction to Higgs mass

Expect new physics at scales $M \gg$ TeV:

$M_{\nu_R} \sim 10^{14}$ GeV

$M_{\rm GUT} \sim 10^{16}$ GeV

$M_{\rm Pl} \sim 10^{19}$ GeV

$m_H^2 \simeq -(88 \text{ GeV})^2$ requires large unexplained cancellation

"hierarchy problem"

Possible explanations:

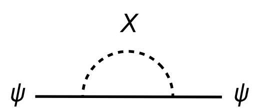
- Δm_H^2 forbidden by symmetry
- No new particles above TeV scale coupling to Higgs
  but: quantum gravity!
- Higgs compositeness
- Antrhopic selection
- Relaxation models

  Graham, Kaplan, Rajendran arXiv:1504.07551

⋮

?

Fermion masses do not suffer from this problem because there is an additional chiral symmetry as $m_\psi \to 0$:



$$\Rightarrow \ \Delta m_\psi \sim \underbrace{\frac{y^2}{16\pi^2} m_\psi}_{\lesssim m_\psi}$$

But scalar mass term $H^\dagger H$ is invariant under all symmetries

. . . except SUSY!

$H \longleftrightarrow \tilde{H}$ = Higgsino

= fermion partner of the Higgs

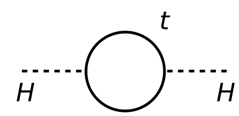
SUSY $\Rightarrow m_H = m_{\tilde{H}}$

Chiral symmetry $\Rightarrow m_{\tilde{H}} = 0$

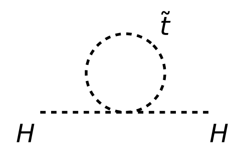
$\Rightarrow m_H^2$ insensitive to UV scales

Bose-Fermi symmetry not observed in nature $\Rightarrow$ SUSY broken

Nontrivial cancelations among diagrams:



$$\Delta m_H^2 = \frac{3y_t^2}{16\pi^2}\left[-\Lambda^2 - m_t^2 \ln\Lambda + \cdots\right]$$



$$\Delta m_H^2 = \frac{3y_t^2}{16\pi^2}\left[\Lambda^2 + m_{\tilde{t}}^2 \ln\Lambda + \cdots\right]$$

quadratic sensitivity to UV scales cancels

$\tilde{t}$ = stop

= scalar partner of the top

$$\Delta m_H^2 = -\frac{3y_t^2}{16\pi^2}(m_{\tilde{t}}^2 - m_t^2)\ln\Lambda + \cdots$$

$m_{\tilde{t}} \lesssim$ TeV $\Rightarrow$ mild logarithmic sensitivity to UV scales.

Bose-Fermi Symmetry in Quantum Mechanics

The Supersymmetric Simple Harmonic Oscillator

The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction. — Sidney Coleman

Simplest example of supersymmetry in quantum mechanics:
Define in terms of creation/annihilation operators:

$\boxed{H_b = \hbar\omega_b b^\dagger b}$ (subtract 0-point energy)

b is for "boson"

$\boxed{[b, b^\dagger] = 1}$ $[b, b] = [b^\dagger, b^\dagger] = 1$

States:

$\boxed{b|0\rangle = 0}$ $\boxed{|n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle}$ $\Rightarrow \langle n|m\rangle = \delta_{nm}$

$\boxed{H_b|n\rangle = n(\hbar\omega_b)|n\rangle}$

Fermionic simple harmonic oscillator:

$\boxed{H_f = \hbar\omega_f f^\dagger f}$ $H_f^\dagger = H_f$

f is for "fermion"

$\boxed{\{f, f^\dagger\} = 1}$ $\{f, f\} = \{f^\dagger, f^\dagger\} = 0$

$\{A, B\} = AB + BA =$ anticommutator

States:

$\boxed{f|0\rangle = 0}$ $\boxed{|1\rangle = f^\dagger|0\rangle}$

$(f^\dagger)^2|0\rangle = 0 \Rightarrow$ 2-state system
(Pauli exclusion principle)

$\boxed{H_f|n\rangle = n(\hbar\omega_f)|n\rangle}$ $n = 0, 1$

Combine bosonic and fermionic oscillators:

$\boxed{H = H_b + H_f}$

$[b, b^\dagger] = 1$ $[b, b] = [b^\dagger, b^\dagger] = 1$

$\{f, f^\dagger\} = 1$ $\{f, f\} = \{f^\dagger, f^\dagger\} = 0$

$[b, f] = [b, f^\dagger] = [b^\dagger, f] = [b^\dagger, f^\dagger] = 0$

$b|0\rangle = f|0\rangle = 0$

$\boxed{|n, 0\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle}$ $\boxed{|n, 1\rangle = f^\dagger|n, 0\rangle}$

Label states: $|n_b, n_f\rangle$ $n_b = \#$ of bosons $= 0, 1, 2, \ldots$
$n_f = \#$ of fermions $= 0, 1$

For $\omega_b = \omega_f$, this system has Bose-Fermi symmetry

Spectrum of energy levels:

$$\boxed{H|n_b, n_f\rangle = (n_b + n_f)(\hbar\omega)|n_b, n_f\rangle} \qquad \omega = \omega_b = \omega_f$$

E

$\vdots$

3ω — $|3,0\rangle,\ |2,1\rangle$ $\qquad |3,0\rangle \overset{Q}{\longleftrightarrow} |2,1\rangle$

2ω — $|2,0\rangle,\ |1,1\rangle$ $\qquad |2,0\rangle \overset{Q}{\longleftrightarrow} |1,1\rangle$

ω — $|1,0\rangle,\ |0,1\rangle$ $\qquad |1,0\rangle \overset{Q}{\longleftrightarrow} |0,1\rangle$

0 — $|0,0\rangle =$ ground state $= |0\rangle$ $\qquad Q|0,0\rangle = 0$

Note: zero point energy cancels...

Generator of symmetry:

$$\boxed{Q = b^\dagger f + f^\dagger b}$$

$$Q|n_b, n_f\rangle = |n_b - 1, n_f + 1\rangle + |n_b + 1, n_f - 1\rangle$$

$$\boxed{[Q, H] = 0} \qquad \text{(definition of symmetry in QM)}$$

Exercise: Check this using the identities

$$[AB, C] = A[B, C] + [A, C]B$$
$$= A\{B, C\} - \{A, C\}B$$

$H =$ generator of time translation symmetry

$$|\psi(t)\rangle = e^{-iHt}|\psi(t = 0)\rangle$$

$Q =$ supersymmetry generator

$$\boxed{Q^2 = b^\dagger b + f^\dagger f} \qquad \text{Exercise: Show this.}$$
$$= (\hbar\omega)^{-1} H$$

Q is "square root" of H

This example may seem trivial, but free field field theory is equivalent to an infinite number of decoupled simple harmonic oscillators, one for each $\vec{p}$.

Questions:

- How can we understand this as a symmetry?
  What are the transformations?
- Can we generalize supersymmetry to $\underbrace{\text{interesting theories}}_{\text{interacting QFT}}$?

A simple supersymmetric theory:

$$\mathcal{L} = \overline{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi + \tfrac{1}{2}\partial^\mu \phi_i \partial_\mu \phi_i - \tfrac{1}{2} m^2 \phi_i \phi_i$$

$\Psi =$ Dirac fermion

$\phi_i =$ real scalar $\qquad i = 1, \ldots, 4$

Note: same mass for fermion, boson.

Gives a spectrum with Bose-Fermi degeneracy: for each $\vec{p}$ there are 4 fermionic and 4 bosonic states with energy $\sqrt{\vec{p}^2 + m^2}$.

In fact, this theory has non-minimal ($\mathcal{N} = 2$) supersymmetry. To get theory with minimal supersymmetry need minimal fermion: Weyl spinor.

Weyl Fermions

Note on conventions:

$\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$

Spinor conventions are those of the textbook by Peskin and Schroeder.

Spinor index notation is that of Dreiner, Haber, Martin, **Phys. Rep.** 464 (2010) (`arXiv:0812.1594`.) This should be consulted for additional details and results.

These conventions are used by a majority of researchers in SUSY phenomenology.

Conventions should be conventional.

—Markus Luty

Weyl Fermions

Weyl fermions are the minimal spin $\frac{1}{2}$ field in 4D. They are the basic building blocks for all theories of fermions.

Start with Dirac representation:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

$$\Rightarrow \ \Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = SO(3,1) \text{ generators}$$

Defines Dirac spinor representation: under infinitesmal Lorentz transformations

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

$$\delta\Psi = -\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\Psi \qquad \Psi = \text{Dirac spinor}$$

Dirac representation is universal: exists for all spacetime dimensions, any metric signature.

Finite transformations:

$$\Psi \mapsto \underbrace{e^{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}}_{= S(\Lambda)}\Psi \qquad \overline{\Psi} = \Psi^\dagger\gamma^0 \mapsto \overline{\Psi}\,\underbrace{e^{+\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}}_{= [S(\Lambda)]^{-1}}$$

Index notation:

$$\Psi^a \mapsto S^a{}_b\Psi^b \qquad \overline{\Psi}_a \mapsto \overline{\Psi}_b(S^{-1})^b{}_a$$

$$a, b = 1, \ldots, 4 = \text{Dirac spinor index}$$

Dirac matrices have index structure $(\gamma^\mu)^a{}_b$

$(\gamma^\mu)^a{}_b$ is an *invariant tensor*: it is invariant when transformed according to its index structure.

Spacetime metric is the canonical example of this:

$$\eta^{\mu\nu} = \underbrace{\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\eta^{\rho\sigma}}_{\text{Lorentz transformation of } \eta^{\mu\nu}}$$

For γ^μ:

$$(\gamma^\mu)^a{}_b = \underbrace{\Lambda^\mu{}_\nu S^a{}_c(\gamma^\nu)^c{}_d(S^{-1})^d{}_b}_{\text{Lorentz transformation of } (\gamma^\mu)^a{}_b} \qquad \gamma^\mu = \Lambda^\mu{}_\nu S\gamma^\nu S^{-1}$$

$\Rightarrow$ can form Lorentz tensors by contracting spinor indices:

$$\overline{\Psi}_a\Psi^a \qquad \overline{\Psi}_a(\gamma^\mu)^a{}_b\Psi^b \qquad \cdots$$

$$(\gamma^5)^a{}_b = \frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)^a{}_b = \text{invariant tensor}$$

$\Rightarrow$ additional Lorentz tensors:

$$\overline{\Psi}_a(\gamma^5)^a{}_b\Psi^b \qquad \overline{\Psi}_a(\gamma^\mu\gamma^5)^a{}_b\Psi^b \qquad \cdots$$

Weyl basis for Dirac matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} \sigma^\mu &= (1, \vec{\sigma}) \\ \overline{\sigma}^\mu &= (1, -\vec{\sigma}) \end{aligned}} \qquad \begin{aligned} &\vec{\sigma} = \text{Pauli matrices} \\ &\text{Note: } \overline{\sigma}^\mu \neq (\sigma^\mu)^\dagger \text{ or } (\sigma^\mu)^* \end{aligned}$$

$$\Rightarrow\ \Sigma^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \overline{\sigma}^{\mu\nu} \end{pmatrix}$$

$$= \text{block diagonal}$$

$\Rightarrow$ Dirac representation is *reducible*

$$\boxed{\begin{aligned} \sigma^{\mu\nu} &= \frac{i}{4}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu) \\ \overline{\sigma}^{\mu\nu} &= \frac{i}{4}(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu) \end{aligned}}$$

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \qquad \begin{aligned} &\psi_L = \text{left-handed Weyl spinor} \\ &\psi_R = \text{right-handed Weyl spinor} \end{aligned}$$

$$\left.\begin{aligned} \delta\psi_L &= -\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\psi_L \\ \delta\psi_R &= -\frac{i}{2}\omega_{\mu\nu}\overline{\sigma}^{\mu\nu}\psi_R \end{aligned}\right\} \text{ different reps of } SO(3,1)$$

Index notation:

$$(\psi_L)_\alpha \qquad \alpha = 1, 2, = \text{Weyl spinor index}$$

$$(\psi_R)^{\dot\alpha} \qquad \dot\alpha = 1, 2, = \text{dotted Weyl spinor index}$$

$$\delta(\psi_L)_\alpha = -\frac{i}{2}\omega_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta(\psi_L)_\beta$$

$$\delta(\psi_R)^{\dot\alpha} = -\frac{i}{2}\omega_{\mu\nu}(\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}(\psi_R)^{\dot\beta}$$

Finite transformations:

$$(\psi_L)_\alpha \mapsto \underbrace{\left(e^{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}}\right)_\alpha{}^\beta}_{= S_\alpha{}^\beta(\Lambda)}(\psi_L)_\beta$$

$$(\psi_R)^{\dot\alpha} \mapsto \underbrace{\left(e^{-\frac{i}{2}\omega_{\mu\nu}\overline{\sigma}^{\mu\nu}}\right)^{\dot\alpha}{}_{\dot\beta}}_{= \overline{S}^{\dot\alpha}{}_{\dot\beta}(\Lambda)}(\psi_R)^{\dot\beta}$$

Define transformation for general tensors with upper/lower dotted/undotted Weyl spinor indices:

$$T^{\alpha\cdots\dot\beta\cdots}_{\gamma\cdots\dot\delta\cdots} \mapsto S_\gamma{}^{\gamma'}\overline{S}^{\dot\beta}{}_{\dot\beta'}(S^{-1})_{\alpha'}{}^\alpha(\overline{S}^{-1})^{\dot\delta'}{}_{\dot\delta}\cdots T^{\alpha'\cdots\dot\beta'\cdots}_{\gamma'\cdots\dot\delta'\cdots}$$

Invariant tensors:

$$\sigma^{\mu}_{\alpha\dot\beta} = \underbrace{\Lambda^{\mu}{}_{\nu} S_{\alpha}{}^{\gamma} \sigma^{\nu}_{\gamma\dot\delta} (\bar S^{-1})^{\dot\delta}{}_{\dot\beta}}_{\text{Lorentz transformation of } \sigma^{\mu}_{\alpha\dot\beta}}$$

$$\bar\sigma^{\mu\dot\alpha\beta} = \Lambda^{\mu}{}_{\nu} (\bar S^{-1})^{\dot\alpha}{}_{\dot\gamma} \bar\sigma^{\nu\dot\gamma\delta} (S^{-1})_{\delta}{}^{\beta}$$

Also:

$$\boxed{\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$$

$$\epsilon_{\alpha\beta} = \underbrace{S_{\alpha}{}^{\gamma} S_{\beta}{}^{\delta} \epsilon_{\gamma\delta}}_{\text{Lorentz transformation of } \epsilon_{\alpha\beta}}$$

$$\epsilon^{\dot\alpha\dot\beta} = \bar S^{\dot\alpha}{}_{\dot\gamma} \bar S^{\dot\beta}{}_{\dot\delta} \epsilon^{\dot\gamma\dot\delta} \qquad \textit{etc.}$$

Summarize: invariant tensors

$$\boxed{\begin{array}{cccc} \sigma^{\mu}_{\alpha\dot\beta} & \bar\sigma^{\mu\dot\alpha\beta} & & \\ \epsilon^{\alpha\beta} & \epsilon_{\alpha\beta} & \epsilon^{\dot\alpha\dot\beta} & \epsilon_{\dot\alpha\dot\beta} \end{array}}$$

can be used to form invariants by contracting indices.

Proof of invariance identities follows from identities on 2×2 matrices. For example, invariance of $\epsilon_{\alpha\beta}$:

$$0 \overset{?}{=} \delta\epsilon_{\alpha\beta} = -\frac{i}{2}\omega_{\mu\nu}(\sigma^{\mu\mu})_{\alpha}{}^{\alpha'}\epsilon_{\alpha'\beta} - \frac{i}{2}\omega_{\mu\nu}(\sigma^{\mu\mu})_{\beta}{}^{\beta'}\epsilon_{\alpha\beta'}$$

$$\Leftrightarrow 0 = \sigma^{\mu\nu}\epsilon + \sigma^{\mu\nu}\epsilon^{T} \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Leftrightarrow \epsilon\sigma^{\mu\nu}\epsilon^{T} = -\sigma^{\mu\nu}$$

Follows from $\epsilon\sigma^{\mu}\epsilon^{T} = (\bar\sigma^{\nu})^{*}$ <u>Exercise</u>: check this.

Complex conjugation relates dotted/undotted spinor indices:

$$\boxed{\left[(\sigma^{\mu\nu})_{\alpha}{}^{\beta}\right]^{*} = (\bar\sigma^{\mu\nu})^{\dot\beta}{}_{\dot\beta}}$$

$\Rightarrow$ makes sense to write

$$(\psi_{\alpha})^{\dagger} = (\psi^{\dagger})_{\dot\alpha} \qquad \textit{etc.}$$

Given L Weyl spinor ψ_{α}, we can define a R spinor by complex conjugation:

$$(\psi^{c})^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}(\psi^{\dagger})_{\dot\beta}$$

$$\psi^{c} = \text{complex conjugate spinor}$$

Any spinor Lagrangian can be written entirely in terms of L Weyl spinors.

Invariant Lagrangians (finally!)

Most general quadratic Lagrangian for a Weyl spinor ψ_{α}:

$$\boxed{\mathcal{L} = \psi^{\dagger}_{\dot\alpha} i\bar\sigma^{\mu\dot\alpha\beta}\partial_{\mu}\psi_{\beta} - \tfrac{1}{2}\left(\epsilon^{\alpha\beta}\psi_{\alpha}\psi_{\beta} + \text{h.c.}\right)}$$

The mass term is a *Majorana mass term*.

Note that it breaks any $U(1)$ symmetry acting on ψ.

Nonzero mass term requires anticommuting spinor fields:

$$\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha} \qquad \psi_{\alpha}\psi_{\beta} = -\psi_{\beta}\psi_{\alpha}$$

Canonical quantization: quantum fermion fields obey anticommutation relations, $\hbar \to 0$ limit gives anticommuting classical spinor fields.

Path integral quantization: fermion path integral is over anticommuting fields.

Check $\mathcal{L}^\dagger = \mathcal{L}$ (needed for Hermitian quantum Hamiltonian)

To agree with Hermitian conjugation of operators, complex conjugation of classical anticommuting spinors must be defined to reverse the order of spinors:

$$(\psi_\alpha \chi_\beta)^\dagger = \chi^\dagger_{\dot\beta} \psi^\dagger_{\dot\alpha} \qquad \text{(no change of sign)}$$

With this rule, we have

$$(\epsilon^{\alpha\beta}\psi_\alpha\psi_\beta)^\dagger = \epsilon^{\dot\alpha\dot\beta}\psi^\dagger_{\dot\beta}\psi^\dagger_{\dot\alpha} = -\epsilon^{\dot\alpha\dot\beta}\psi^\dagger_{\dot\alpha}\psi^\dagger_{\dot\beta}$$

$$(\psi^\dagger_{\dot\alpha} i\bar\sigma^{\mu\dot\alpha\beta}\partial_\mu\psi_\beta)^\dagger = -i\partial_\mu\psi^\dagger_{\dot\beta}\underbrace{(\bar\sigma^{\mu\dot\beta\alpha})^*}_{=\bar\sigma^{\mu\dot\beta\alpha}}\psi_\beta \qquad (\bar\sigma^\mu)^\dagger = \bar\sigma^\mu$$

$$= +\psi^\dagger_{\dot\beta} i\bar\sigma^{\mu\dot\beta\alpha}\psi_\alpha$$

$$\Rightarrow \mathcal{L}^\dagger = \mathcal{L}.$$

Expressions look cleaner when spinor indices are implicit:

$$\psi^\dagger_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta}\psi_\beta = \psi^\dagger\bar\sigma^\mu\psi$$

$$\chi^\alpha\sigma^\mu_{\alpha\dot\beta}\chi^{\dagger\dot\beta} = \chi\sigma^\mu\chi^\dagger$$

In general, omit summed indices

$${}^\alpha{}_\alpha \qquad \text{and} \qquad {}_{\dot\alpha}{}^{\dot\alpha}$$

Example:

$$\chi\psi = \chi^\alpha\psi_\alpha = \epsilon^{\alpha\beta}\chi_\beta\psi_\alpha = -\underbrace{\epsilon^{\alpha\beta}\psi_\alpha}_{= -\epsilon^{\beta\alpha}\psi_\alpha = -\psi^\beta}\chi_\beta$$

$$= \psi^\beta\chi_\beta$$

$$= +\psi\chi$$

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \qquad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \qquad etc.$$

<u>Exercise</u>:

Consider theory of a single massless Weyl fermion ψ_α with the Lagrangian given above.

(a) Show that the equation of motion is the *Weyl equation*

$$i\bar\sigma^{\mu\dot\alpha\beta}\partial_\mu\psi_\beta = 0$$

Multiply on the left by $\sigma^\nu\partial_\nu$ to show that the Weyl equation implies the massless Klein-Gordon equation:

$$\Box\psi_\alpha = 0$$

(b) Consider the most general plane wave solution

$$\psi_\alpha(x) = u_\alpha(p)e^{-ip\cdot x}$$

By going to the frame $p^\mu = (E, 0, 0, E)$, show that there is a single solution.

(c) The most general operator solution to the Weyl equation is

$$\hat\psi_\alpha(x) = \int \frac{d^3p}{(2\pi)^{3/2}(2|\vec p|)^{1/2}}\Big[\hat a(\vec p)u_\alpha(p)e^{-ip\cdot x} + \hat b^\dagger(\vec p)u_\alpha(p)e^{+ip\cdot x}\Big] \qquad p^0 = |\vec p|$$

Imposing the anticommutation relations

$$\{\hat a(\vec p), \hat a^\dagger(\vec p\,')\} = \{\hat b(\vec p), \hat b^\dagger(\vec p\,')\} = \delta^3(\vec p - \vec p\,')$$

$$\{\hat a(\vec p), \hat a(\vec p\,')\} = \{\hat b(\vec p), \hat b(\vec p\,')\} = \{\hat a(\vec p), \hat b^\dagger(\vec p\,')\} = 0$$

compute the equal-time anticommutator

$$\{\hat\psi_\alpha(t,\vec x), \hat\psi^\dagger_{\dot\beta}(t,\vec y)\}$$

You will need the identity

$$u_\alpha(p)u^\dagger_{\dot\beta}(p) = \sigma^\mu_{\alpha\dot\beta}p_\mu \qquad \text{(fixes normalization of } u_\alpha(p))$$

which you can verify in the standard frame.

(d) Show that the canonical momentum is

$$\pi^\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\alpha} = -i\psi^\dagger_{\dot\beta} i\bar{\sigma}^{0\dot\beta\alpha}$$

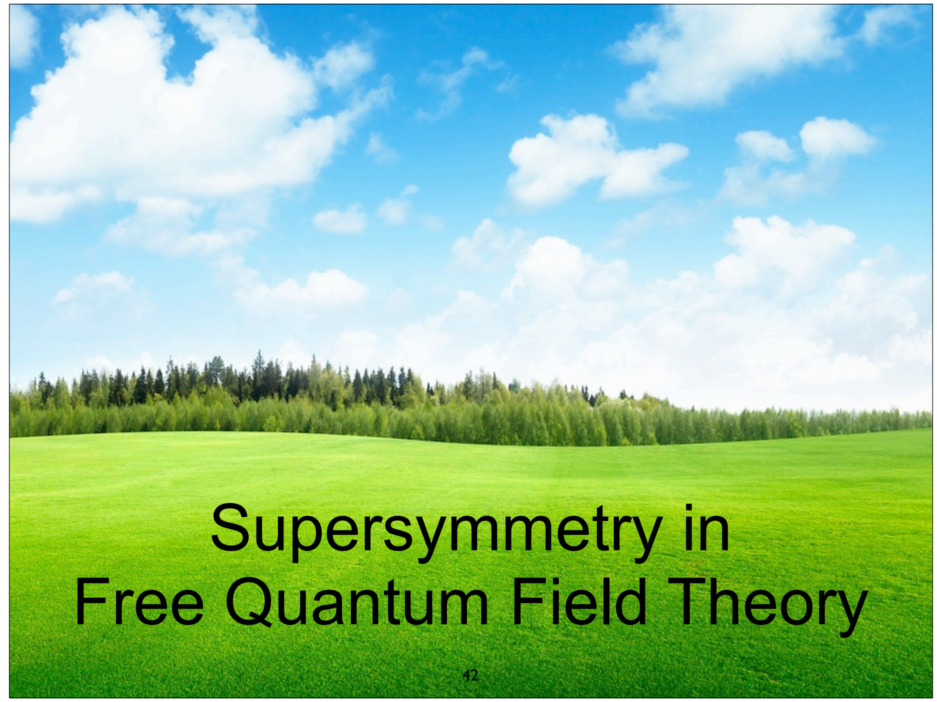
(Remember that ψ_α is a classical <u>anti</u>commuting field.)

(e) Show that the anticommutation relation you derived above is equivalent to the canonical anticommutation relation

$$\{\hat{\pi}^\alpha(t,\vec{x}), \hat{\psi}_\beta(t,\vec{y})\} = -i\delta^\alpha{}_\beta \delta^3(\vec{x}-\vec{y})$$

This exercise shows that a Weyl fermion has 2 propagating degrees of freedom.

Note: many textbook treatments of the Dirac equation change the sign of π^α and the canonical anticommutation relations to get the correct commutation relations for the creation and annihilation operators.



Simplest theory with a chance of Bose-Fermi symmetry:

ψ_α = L Weyl fermion

ϕ = complex scalar (2 degrees of freedom)

$$\mathcal{L} = \psi^\dagger i\bar{\sigma}^\mu \partial_\mu \psi + \partial^\mu \phi^\dagger \partial_\mu \phi \qquad m = 0 \text{ for now}$$

Note this preserves $U(1)$ symmetry

$$\psi_\alpha \mapsto e^{i\theta\psi}, \qquad \phi \mapsto e^{i\theta}\phi$$

Write most general SUSY transformation:

- $\delta\phi \sim \psi$, $\delta\psi \sim \phi$
- Lorentz/spinor indices match
- $U(1)$ invariant

$$\delta\phi = \underbrace{\xi\psi}_{= \xi^\alpha \psi_\alpha} \qquad \xi^\alpha = \text{spinor “parameter”}$$

$$[\phi] = 1, \quad [\psi] = \tfrac{3}{2} \quad \Rightarrow \quad [\xi] = -\tfrac{1}{2}$$

$$\delta\psi_\alpha = c\,\underbrace{(\sigma^\mu \xi^\dagger)_\alpha}_{= \sigma^\mu_{\alpha\dot\beta}\xi^{\dagger\dot\beta}} \partial_\mu\phi \qquad c = \text{constant (dimensionless)}$$

Compute $\delta\mathcal{L}$:

$$\delta(\partial^\mu\phi^\dagger\partial_\mu\phi) = \partial^\mu\phi^\dagger\partial_\mu(\delta\phi) + \text{h.c.}$$
$$= \partial^\mu\phi^\dagger\xi\partial_\mu\psi + \underbrace{h.c.}_{\text{depends on } \xi^\dagger}$$

$$\delta(\psi^\dagger i\bar\sigma^\mu\partial_\mu\psi) = \delta\psi^\dagger i\bar\sigma^\mu\partial_\mu\psi + \text{h.c.}$$
$$= ic^*\partial_\mu\phi^\dagger\xi\sigma^\mu\bar\sigma^\nu\partial_\nu\psi + \text{h.c.}$$
$$= -ic^*\partial_\mu\partial_\nu\phi^\dagger\xi\sigma^\mu\bar\sigma^\nu\psi + \text{h.c.}$$

(integrate by parts)

Use identity

$$\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = 2\eta^{\mu\nu}1_2$$

$$\delta(\psi^\dagger i\bar\sigma^\mu\partial_\mu\psi_\beta = -ic^*\Box\phi^\dagger\xi\psi + \text{h.c.}$$
$$= ic^*\partial^\mu\phi^\dagger\xi\partial_\mu\psi + \text{h.c.}$$

$\Rightarrow \delta\mathcal{L} = 0$ for $c = -i$.

Summarize: $\boxed{\delta\phi = \xi\psi \qquad \delta\psi = -i\sigma^\mu\xi^\dagger\partial_\mu\phi}$

Noether current:

$$J^\mu_\alpha = (\sigma^\nu\bar\sigma^\mu\psi)_\alpha\partial_\nu\phi^\dagger$$

Check conservation:

$$\partial_\mu J^\mu_\alpha = (\sigma^\nu \underbrace{\bar\sigma^\mu)_\alpha{}^\beta\partial_\mu\psi_\beta}_{=0}\,\partial_\nu\phi^\dagger + \underbrace{\underbrace{(\sigma^\nu\bar\sigma^\mu)_\alpha{}^\beta}_{\to\,\eta^{\mu\nu}\delta_\alpha{}^\beta}\psi_\beta\partial_\mu\partial_\nu\phi^\dagger}_{\propto\,\Box\phi = 0}$$

$\Rightarrow \partial_\mu J^\mu_\alpha = 0$ (on classical solutions)

Noether charge:

$Q_\alpha = \sqrt{2}\int d^3x\, J^0_\alpha$ (normalization is conventional)

$\dot Q_\alpha = 0$ (on classical solutions)

Quantum theory:

$$\hat\psi_\alpha(x) = \int\frac{d^3p}{(2\pi)^{3/2}(2|\vec p|)^{1/2}}\Big[\hat a(\vec p)u_\alpha(\vec p)e^{-ip\cdot x} + \hat b^\dagger(\vec p)u_\alpha(\vec p)e^{+ip\cdot x}\Big]$$

$$\hat\phi_\alpha(x) = \int\frac{d^3p}{(2\pi)^{3/2}(2|\vec p|)^{1/2}}\Big[\hat c(\vec p)e^{-ip\cdot x} + \hat d^\dagger(\vec p)e^{+ip\cdot x}\Big]$$

$$\Rightarrow \hat Q_\alpha = \sqrt{2}\int d^3p\, u_\alpha(\vec p)\Big[\hat c^\dagger(\vec p)\hat a(\vec p) + \hat b^\dagger(\vec p)\hat d(\vec p)\Big]$$

$$\hat Q^\dagger_{\dot\alpha} = \sqrt{2}\int d^3p\, u^\dagger_{\dot\alpha}(\vec p)\Big[\hat a^\dagger(\vec p)\hat c(\vec p) + \hat d^\dagger(\vec p)\hat b(\vec p)\Big]$$

fermion particle: $|\psi(\vec p)\rangle = \hat a^\dagger(\vec p)|0\rangle$

fermion antiparticle: $|\bar\psi(\vec p)\rangle = \hat b^\dagger(\vec p)|0\rangle$

scalar particle: $|\phi(\vec p)\rangle = \hat c^\dagger(\vec p)|0\rangle$

scalar antiparticle: $|\bar\phi(\vec p)\rangle = \hat d^\dagger(\vec p)|0\rangle$

$\hat Q_\alpha|\psi(\vec p)\rangle = \sqrt{2}u_\alpha(\vec p)|\phi(\vec p)\rangle \qquad \hat Q_\alpha|\bar\psi(\vec p)\rangle = 0$

$\hat Q_\alpha|\phi(\vec p)\rangle = 0 \qquad \hat Q_\alpha|\bar\phi(\vec p)\rangle = \sqrt{2}u_\alpha(\vec p)|\bar\psi(\vec p)\rangle$

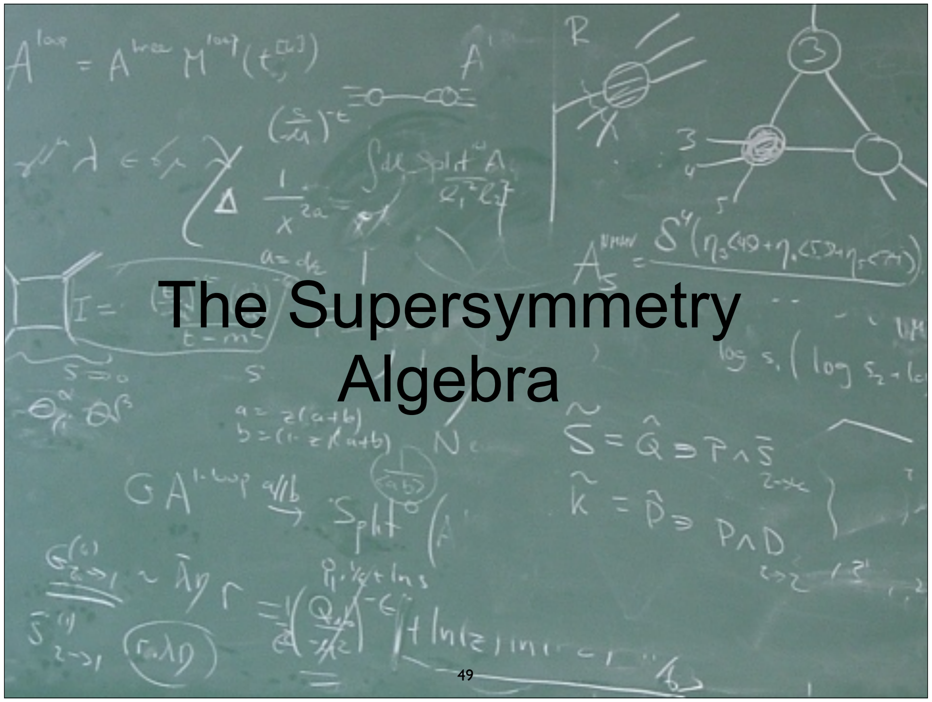
$$\psi \xrightarrow{Q} \phi \xrightarrow{Q^\dagger} \psi$$

$\hat Q^\dagger_{\dot\alpha}|\psi(\vec p)\rangle = 0 \qquad \hat Q^\dagger_{\dot\alpha}|\bar\psi(\vec p)\rangle = \sqrt{2}u^\dagger_{\dot\alpha}(\vec p)|\bar\psi(\vec p)\rangle$

$\hat Q^\dagger_{\dot\alpha}|\phi(\vec p)\rangle = \sqrt{2}u^\dagger_{\dot\alpha}(\vec p)|\psi(\vec p)\rangle \qquad \hat Q^\dagger_{\dot\alpha}|\bar\phi(\vec p)\rangle = 0$

$$\bar\psi \xrightarrow{Q^\dagger} \bar\phi \xrightarrow{Q} \bar\psi$$

The Supersymmetry Algebra



Use the free-field representation to compute

$$\{\hat{Q}_\alpha, \hat{Q}^\dagger_{\dot\beta}\} = 2\int d^3p\, d^3q\, w_\alpha(\vec{p}) w^\dagger_{\dot\beta}(\vec{q}) \times \Big[\{\hat{c}^\dagger(\vec{p})\hat{a}(\vec{p}),\, \hat{a}^\dagger(\vec{q})\hat{c}(\vec{q})\} + \{\hat{b}^\dagger(\vec{q})\hat{d}(\vec{q}),\, \hat{d}^\dagger(\vec{q})\hat{b}(\vec{q})\}\Big]$$

$$\{\hat{c}^\dagger\hat{a}, \hat{a}^\dagger\hat{c}\} = \{\hat{a}, \hat{a}^\dagger\}\hat{c}^\dagger\hat{c} + [\hat{c}, \hat{c}^\dagger]\hat{a}^\dagger\hat{a}$$

$$\{\hat{b}^\dagger\hat{d}, \hat{d}^\dagger\hat{b}\} = \{\hat{b}, \hat{b}^\dagger\}\hat{d}^\dagger\hat{d} + [\hat{d}, \hat{d}^\dagger]\hat{b}^\dagger\hat{b}$$

$a,\ b =$ fermion
$c, d =$ boson

$$\{\hat{Q}_\alpha, \hat{Q}^\dagger_{\dot\beta}\} = 2\int d^3p\, \underbrace{w_\alpha(\vec{p}) w^\dagger_{\dot\beta}(\vec{p})}_{= \sigma^\mu_{\alpha\dot\beta} p_\mu} \Big[\hat{a}^\dagger(\vec{p})\hat{a}(\vec{p}) + \cdots + \hat{d}^\dagger(\vec{p})\hat{d}(\vec{p})\Big]$$

$$= 2\sigma^\mu_{\alpha\dot\beta}\hat{P}_\mu \qquad \hat{P}_\mu = \text{4-momentum operator}$$

$$\boxed{\{\hat{Q}_\alpha, \hat{Q}^\dagger_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}\hat{P}_\mu}$$

Similarly,

$$\boxed{\{\hat{Q}_\alpha, \hat{Q}_\beta\} = 0 \qquad \{\hat{Q}^\dagger_{\dot\alpha}, \hat{Q}^\dagger_{\dot\beta}\} = 0}$$

$$\boxed{[\hat{P}_\mu, \hat{Q}_\alpha] = 0 \qquad [\hat{P}_\mu, \hat{Q}^\dagger_{\dot\alpha}] = 0}$$

This is the famous ($\mathcal{N} = 1$) SUSY algebra.

Drop the hats from now on...

Consequences of SUSY Algebra:

$$\hat{P}^\mu = \bar\sigma^{\mu\dot\alpha\beta}\{\hat{Q}^\dagger_{\dot\alpha}, \hat{Q}_\beta\}$$

$$\hat{P}^0 = \hat{Q}^\dagger_1\hat{Q}_1 + \hat{Q}_1\hat{Q}^\dagger_1 + \hat{Q}^\dagger_2\hat{Q}_2 + \hat{Q}_2\hat{Q}^\dagger_2$$

For any state $|\psi\rangle$

$$\langle\psi|\hat{P}^0|\psi\rangle = \|\hat{Q}^\dagger_1|\psi\rangle\|^2 + \|\hat{Q}_1|\psi\rangle\|^2 + \|\hat{Q}^\dagger_2|\psi\rangle\|^2 + \|\hat{Q}_2|\psi\rangle\|^2 \geq 0$$

If SUSY is unbroken, the vacuum state is SUSY invariant:

$$\hat{Q}_\alpha|0\rangle = 0, \qquad \hat{Q}^\dagger_{\dot\alpha}|0\rangle = 0$$

$$\Rightarrow\ \hat{P}^\mu|0\rangle = 0$$

In particular, the vacuum energy vanishes.

A clue to the cosmological constant problem?

Requires supergravity...

Massless 1-particle states:

$|\vec{p}, \lambda\rangle \qquad \lambda = \hat{p}\cdot\vec{S} = \text{helicity}$

Choose frame $p^\mu = (E, 0, 0, E) \qquad E > 0$

$$\{Q_1, Q_1^\dagger\} = 0$$

$$\{Q_2, Q_2^\dagger\} = 4E$$

This is the algebra of one fermionic creation and annihilation operator $Q_2^\dagger$, Q_2 [$Q_1^\dagger$, Q_1 act trivially].

$$[Q_\alpha, \hat{p}\cdot\vec{S}] = [Q_\alpha, M^{12}] = -(\sigma^{12})_\alpha{}^\beta Q_\beta$$

$$[Q_2, \hat{p}\cdot\vec{S}] = +\tfrac{1}{2}Q_2 \qquad [Q_2^\dagger, \hat{p}\cdot\vec{S}] = -\tfrac{1}{2}Q_2^\dagger$$

$\Rightarrow Q_2$ ($Q_2^\dagger$) acts as raising (lowering) operator for helicity.

Irreducible 1-particle representations:

$$\begin{matrix} |\vec{p},\lambda\rangle \\ |\vec{p},\lambda+\frac{1}{2}\rangle \end{matrix} \quad \overset{CPT}{\longleftrightarrow} \quad \begin{matrix} |\vec{p},-\lambda\rangle \\ |\vec{p},-\lambda-\frac{1}{2}\rangle \end{matrix}$$

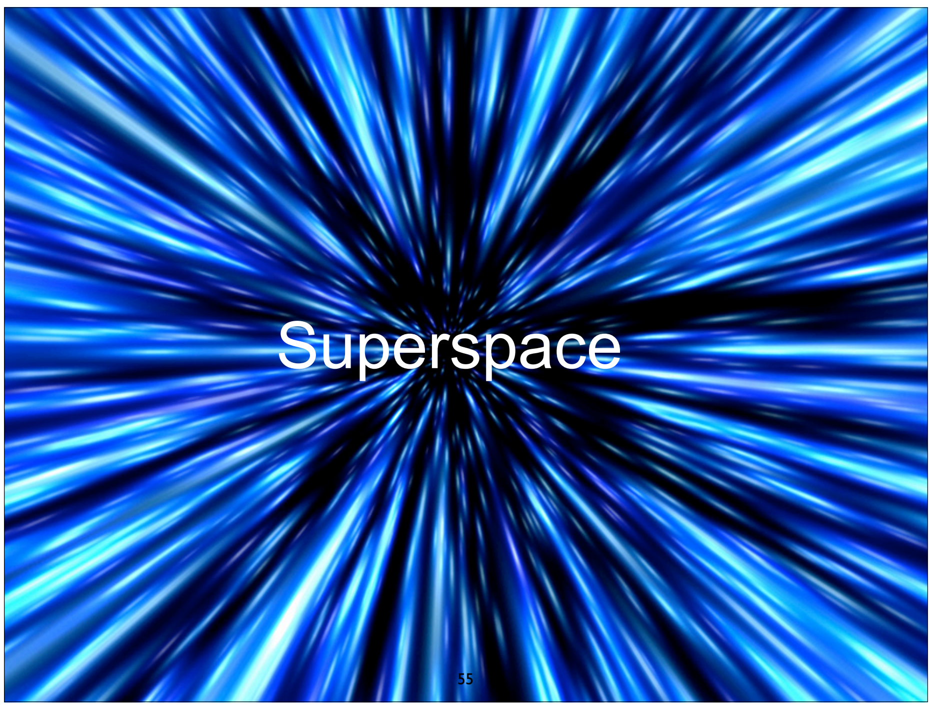
$$\lambda = 0: \quad \begin{matrix} |\vec{p},0\rangle & CPT|\vec{p},0\rangle & \leftrightarrow \text{ complex scalar} \\ |\vec{p},\frac{1}{2}\rangle & |\vec{p},-\frac{1}{2}\rangle & \leftrightarrow \text{ Weyl fermion} \end{matrix}$$

This is the *chiral multiplet*.

$$\lambda = \tfrac{1}{2}: \quad \begin{matrix} |\vec{p},\frac{1}{2}\rangle & |\vec{p},-\frac{1}{2}\rangle & \leftrightarrow \text{ Weyl fermion} \\ |\vec{p},1\rangle & |\vec{p},-1\rangle & \leftrightarrow \text{ massless gauge field} \end{matrix}$$

This is the *massless vector multiplet*.

These are the multiplets that describe massless particles of spin ≤ 1.



A good pedagogical discussion for this subject:

D. Bertonini, J. Thaler, Z. Thomas, “Super-tricks for Superspace” (TASI 2012 lectures), `arXiv:1302.6229`.

Warning: although this uses the same spinor conventions as we do, but uses a non-conventional definition of the SUSY generators. It is easy to translate between them:

$$Q_\alpha^{(\text{them})} = -Q_\alpha^{(\text{us})}$$

$$P_\mu^{(\text{them})} = -P_\mu^{(\text{us})}$$

Lorentz transformations act naturally on spacetime:

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$$

SUSY acts naturally on *superspace*.

superspace = $\{(x^\mu, \theta^\alpha, \bar\theta^{\dot\alpha})\}$

θ^α, $\bar\theta^{\dot\alpha} =$ anticommuting "coordinates"

$\bar\theta^{\dot\alpha} = \theta^{\dagger\dot\alpha} = (\theta^\alpha)^\dagger$

$$\boxed{\{\theta^\alpha, \theta^\beta\} = 0} \qquad \boxed{\{\bar\theta^{\dot\alpha}, \bar\theta^{\dot\beta}\} = 0}$$

$\Rightarrow \quad \theta^1\theta^1 = \theta^2\theta^2 = 0,$ *etc.*

The natural variables for SUSY quantum field theory are therefore *superfields* = functions of superspace.

Superfields are defined by Taylor expanding in θ, $\bar\theta$.

θ, $\bar\theta$ anticommute $\Rightarrow$ expansion contains finitely many terms.

Function of one real anticommuting variable θ:

$$f(\theta) = a + b\theta \qquad \theta^2 = 0$$

Superfield = function of θ^α, $\bar\theta^{\dot\alpha}$:

Highest component $= \theta^1\theta^2\bar\theta^1\bar\theta^2$

Simplify expansion using identities

$\theta_\alpha\theta_\beta = 2 \times 2$ antisymmetric matrix $\propto \epsilon_{\alpha\beta}$

$$\boxed{\theta_\alpha\theta_\beta = \tfrac{1}{2}\epsilon_{\alpha\beta}\theta\theta} \quad \boxed{\theta^\alpha\theta^\beta = -\tfrac{1}{2}\epsilon^{\alpha\beta}\theta\theta} \quad \theta\theta = \theta^\alpha\theta_\alpha$$

$$\boxed{\bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = \tfrac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta} \quad \boxed{\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = -\tfrac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta} \quad \bar\theta\bar\theta = \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}$$

General (scalar) superfield:

$$\boxed{\begin{aligned} S(x,\theta,\bar\theta) = A(x) &+ \theta^\alpha\psi_\alpha(x) + \bar\theta_{\dot\alpha}\bar\chi^{\dot\alpha}(x) \\ &+ \theta\theta B(x) + \bar\theta\bar\theta C(x) \\ &+ \theta\sigma^\mu\bar\theta V_\mu(x) \\ &+ (\bar\theta\bar\theta)\theta^\alpha\lambda_\alpha(x) + (\theta\theta)\bar\theta_{\dot\alpha}\bar\eta^{\dot\alpha}(x) \\ &+ (\theta\theta)(\bar\theta\bar\theta)D(x) \end{aligned}}$$

$S \longleftrightarrow (A, \psi_\alpha, \bar\chi_{\dot\alpha}, B, C, V_\mu, \lambda_\alpha, \bar\eta^{\dot\alpha}, D)$

θ^α, $\bar\theta^{\dot\alpha} =$ algebraic placeholders

Analogy: complex numbers:

$z = x + iy \qquad i^2 = -1$

$z \longleftrightarrow (x, y) \qquad i =$ placeholder

$z_1 + z_2 \longleftrightarrow (x_1 + x_2,\ y_1 + y_2)$

$z_1 z_2 \longleftrightarrow (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$

Superfields naturally add and multiply together:

$$S_1(x,\theta,\bar\theta) + S_2(x,\theta,\bar\theta) = A_1 + A_2 + \theta^\alpha(\psi_{1\alpha} + \psi_{2\alpha}) + \cdots$$

$$S_1 + S_2 \longleftrightarrow (A_1 + A_2,\ \psi_{1\alpha} + \psi_{2\alpha},\ \ldots)$$

$$S_1(x,\theta,\bar\theta)\,S_2(x,\theta,\bar\theta) = A_1A_2 + \theta^\alpha(A_1\psi_{2\alpha} + A_2\psi_{1\alpha}) + \cdots$$

$$S_1S_2 \longleftrightarrow (A_1A_2,\ A_1\psi_{2\alpha} + A_2\psi_{1\alpha},\ \ldots)$$

Spacetime translations generated by derivative operator:

$$\phi(x) \mapsto \phi(x-a) = e^{ia^\mu P_\mu}\phi(x) \qquad \boxed{P_\mu = i\partial_\mu}$$

$$\delta\phi(x) = ia^\mu P_\mu \phi(x)$$

Define SUSY transformation of superfields:

$$\boxed{\delta S(x,\theta,\bar\theta) = i(a^\mu P_\mu + \xi^\alpha Q_\alpha + \bar\xi_{\dot\alpha}\bar Q^{\dot\alpha}) S(x,\theta,\bar\theta)}$$

$Q_\alpha, \bar Q^{\dot\alpha} =$ derivative operators

$$(\xi^\alpha Q_\alpha)^\dagger = \bar\xi_{\dot\alpha}\bar Q^{\dot\alpha}$$

Derivative operators: in addition to $\partial_\mu = \dfrac{\partial}{\partial x^\mu}$,

define $\dfrac{\partial}{\partial\theta^\alpha}, \dfrac{\partial}{\partial\bar\theta_{\dot\alpha}}$:

$$\boxed{\frac{\partial}{\partial\theta^\alpha}\theta^\beta = \delta_\alpha{}^\beta} \qquad \boxed{\frac{\partial}{\partial\theta^\alpha}\bar\theta_{\dot\beta} = 0}$$

$$\boxed{\frac{\partial}{\partial\bar\theta_{\dot\alpha}}\bar\theta_{\dot\beta} = \delta^{\dot\alpha}{}_{\dot\beta}} \qquad \boxed{\frac{\partial}{\partial\bar\theta_{\dot\alpha}}\theta^\beta = 0}$$

Careful about signs:

$$\frac{\partial}{\partial\theta^\alpha}(\psi^\beta\theta^\gamma) = \frac{\partial}{\partial\theta^\alpha}(-\theta^\gamma\psi^\beta) = -\delta^\alpha{}_\gamma\psi^\beta$$

Anticommute spinors to the left before acting with $\dfrac{\partial}{\partial\theta^\alpha}$

<u>Exercise</u>: Show that $\dfrac{\partial}{\partial\theta_\alpha} = -\epsilon^{\alpha\beta}\dfrac{\partial}{\partial\theta^\beta}$

Define SUSY generators acting on superfields:

$$\boxed{\begin{aligned} iQ_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\bar\theta)_\alpha\partial_\mu \\ i\bar Q^{\dot\alpha} &= \frac{\partial}{\partial\bar\theta_{\dot\alpha}} + i(\bar\sigma^\mu\theta)^{\dot\alpha}\partial_\mu \end{aligned}}$$

Check that they satisfy SUSY algebra:

$$\boxed{\begin{aligned} &\{Q_\alpha, \bar Q_{\dot\beta}\} = 2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu \\ &\{Q_\alpha, Q_\beta\} = \{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0 \end{aligned}}$$

Remember $P_\mu = i\partial_\mu$

SUSY covariant derivatives:

$S =$ superfield $\Rightarrow$ $\partial_\mu S =$ superfield?

Does $\partial_\mu S$ transform like a superfield?

$$\begin{aligned} \delta(\partial_\mu S) = \partial_\mu(\delta S) &= \partial_\mu\left[i(\xi Q + \bar\xi\bar Q)S\right] \\ &= i(\xi Q + \bar\xi\bar Q)\partial_\mu S \end{aligned}$$

Works because $[\partial_\mu, Q_\alpha] = [\partial_\mu, \bar Q_{\dot\alpha}] = 0$

$$\Leftrightarrow\ [P_\mu, Q_\alpha] = [P_\mu, \bar Q_{\dot\alpha}] = 0$$

On the other hand,

$$\frac{\partial}{\partial\theta^\alpha}S \neq \text{superfield}$$

$$\left\{\frac{\partial}{\partial\theta^\alpha}, \bar Q_{\dot\beta}\right\} \neq 0$$

Define SUSY covariant derivatives that anticommute with Q_α, $\overline{Q}_{\dot\alpha}$:

$$\boxed{\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu\overline{\theta})_\alpha\partial_\mu \\ \overline{D}^{\dot\alpha} &= \frac{\partial}{\partial\overline{\theta}_{\dot\alpha}} - i(\overline{\sigma}^\mu\theta)^{\dot\alpha}\partial_\mu \end{aligned}} \qquad \begin{aligned} iQ_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\overline{\theta})_\alpha\partial_\mu \\ i\overline{Q}^{\dot\alpha} &= \frac{\partial}{\partial\overline{\theta}_{\dot\alpha}} + i(\overline{\sigma}^\mu\theta)^{\dot\alpha}\partial_\mu \end{aligned}$$

$$\boxed{0 = \{D_\alpha, Q_\beta\} = \{D_\alpha, \overline{Q}_{\dot\beta}\} = \{\overline{D}_{\dot\alpha}, Q_\beta\} = \{\overline{D}_{\dot\alpha}, \overline{Q}_{\dot\beta}\}}$$

$$\boxed{\{D_\alpha, \overline{D}_{\dot\beta}\} = 2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu}$$

$$\boxed{0 = \{D_\alpha, D_\beta\} = \{\overline{D}_{\dot\alpha}, \overline{D}_{\dot\beta}\}}$$

$D_\alpha S =$ superfield:

$$\begin{aligned} \delta(D_\alpha S) = D_\alpha(\delta S) &= D_\alpha\left[i(\xi Q + \overline{\xi}\overline{Q})S\right] \\ &= +i(\xi Q + \overline{\xi}\overline{Q})D_\alpha S \end{aligned}$$

$+$ sign because $\{D_\alpha, \xi^\beta\} = \{D_\alpha, \overline{\xi}_{\dot\beta}\} = 0$

$$e.g.\ \frac{\partial}{\partial\theta^\alpha}(\xi^\beta\theta^\gamma) = -\frac{\partial}{\partial\theta^\alpha}(\theta^\gamma\xi^\beta) = -\delta^\gamma{}_\alpha\xi^\beta$$

Chiral superfields:

SUSY covariant derivative allows us to construct simpler superfields with fewer component fields.

Define *chiral superfield* Φ by condition

$$\boxed{\overline{D}_{\dot\alpha}\Phi(x, \theta, \overline{\theta}) = 0}$$

$\Rightarrow$ Φ is independent of something?

Change variables in superspace: $(x^\mu, \theta, \overline{\theta}) \to (y^\mu, \theta, \overline{\theta})$

$$y^\mu = x^\mu + i\underbrace{\overline{\theta}\overline{\sigma}^\mu\theta}_{= -\theta\sigma^\mu\overline{\theta}} \qquad \text{note } [\theta] = [\overline{\theta}] = -\tfrac{1}{2}$$

Work out D_α, $\overline{D}_{\dot\alpha}$ in terms of new variables:

$$\begin{aligned} \overline{D}^{\dot\alpha} &= \frac{\partial}{\partial\overline{\theta}_{\dot\alpha}} - i(\overline{\sigma}^\mu\theta)^{\dot\alpha}\frac{\partial}{\partial x^\mu} \\ &= \frac{\partial}{\partial\overline{\theta}_{\dot\alpha}} + \underbrace{\frac{\partial y^\mu}{\partial\overline{\theta}_{\dot\alpha}}}_{= i(\overline{\sigma}^\mu\theta)^{\dot\alpha}}\frac{\partial}{\partial y^\mu} - i(\overline{\sigma}^\mu\theta)^{\dot\alpha}\underbrace{\frac{\partial y^\nu}{\partial x^\mu}}_{= \delta^\nu{}_\mu}\frac{\partial}{\partial y^\nu} \\ &= \frac{\partial}{\partial\overline{\theta}_{\dot\alpha}} \end{aligned}$$

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu\overline{\theta})_\alpha\frac{\partial}{\partial x^\mu} \\ &= \frac{\partial}{\partial\theta^\alpha} + \underbrace{\frac{\partial y^\mu}{\partial\theta^\alpha}}_{= -i(\sigma^\mu\theta)_\alpha}\frac{\partial}{\partial y^\mu} - i(\sigma^\mu\overline{\theta})_\alpha\underbrace{\frac{\partial y^\nu}{\partial x^\mu}}_{= \delta^\nu{}_\mu}\frac{\partial}{\partial y^\nu} \\ &= \frac{\partial}{\partial\theta^\alpha} - 2i(\sigma^\mu\overline{\theta})_\alpha\frac{\partial}{\partial y^\mu} \end{aligned}$$

Summarize:

$$\boxed{\bar{D}^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}} \qquad D_\alpha = \frac{\partial}{\partial\theta^\alpha} - 2i(\sigma^\mu\bar\theta)_\alpha \frac{\partial}{\partial y^\mu}}$$

$\bar{D}_{\dot\alpha}\Phi = 0 \;\Rightarrow\; \Phi =$ function of (y, θ) (independent of $\bar\theta$)

Component fields: "chiral representation"

$$\boxed{\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta^\alpha\psi_\alpha(y) + \theta\theta F(y)}$$

$\Phi \longleftrightarrow (\phi, \psi_\alpha, F)$

Can expand to write as function of $(x, \theta, \bar\theta)$:

$$\phi(y) = \phi(x) + i\partial_\mu\phi(x)\bar\theta\bar\sigma^\mu\theta + \cdots$$

$\bar{D}_{\dot\alpha}\Phi^n = 0 \quad\Rightarrow\quad \Phi^n =$ chiral superfield

In fact, for any function $f(\Phi)$

$\bar{D}_{\dot\alpha} f(\Phi) = 0 \quad\Rightarrow\quad f(\Phi) =$ chiral superfield

To get a chiral superfield, $f(\Phi)$ cannot depend on $\Phi^\dagger$.
That is, $f(\Phi)$ must be a *holomorphic* function of Φ.
This has far-reaching implications, as we will see below.

Note that

$D_\alpha\Phi^\dagger = 0$

Superfields satisfying this constraint are called *anti-chiral*.

The properties of anti-chiral superfields can be worked out by complex conjugating the results for chiral superfields.

Can use D_α, $\bar{D}_{\dot\alpha}$ to define component fields by projection:

$$\phi = \Phi\big|_{\theta,\bar\theta=0} \equiv \Phi|$$

$$\psi_\alpha = \frac{1}{\sqrt{2}}\frac{\partial}{\partial\theta^\alpha}\Phi| = \frac{1}{\sqrt{2}}D_\alpha\Phi|$$

extra terms vanish for $\theta, \bar\theta = 0$

$$F = -\tfrac{1}{4}\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta}\Phi| = -\tfrac{1}{4}DD\Phi|$$

Use algebra of derivative operators to compute SUSY transformation of component fields:

$$\begin{aligned}\delta\phi &= i(\xi Q + \bar\xi\bar Q)\Phi| \\ &= (\xi D + \bar\xi D)\Phi| = \underbrace{\frac{1}{\sqrt{2}}}_{\text{because we defined } Q_\alpha = \sqrt{2}\int d^3x J^0_\alpha}\xi\psi\end{aligned}$$

$$\begin{aligned}\delta\psi_\alpha &= \frac{i}{\sqrt{2}}(\xi Q + \bar\xi\bar Q)D_\alpha\Phi| \\ &= \frac{i}{\sqrt{2}}D_\alpha(\xi Q + \bar\xi\bar Q)\Phi| \\ &= \frac{1}{\sqrt{2}}D_\alpha(\xi D + \bar\xi\bar D)\Phi| \\ &= \frac{1}{\sqrt{2}}\left[-\tfrac{1}{2}\xi_\alpha DD\Phi| - 2i(\sigma\bar\xi)_\alpha\partial_\mu\Phi|\right] \\ &= \sqrt{2}\xi_\alpha F - i\sqrt{2}(\sigma\bar\xi)_\alpha\partial_\mu\phi\end{aligned}$$

Similarly,

$$\delta F = -i\sqrt{2}\bar\xi\bar\sigma^\mu\partial_\mu\psi \qquad \text{(exercise)}$$

Summarize:

$$\boxed{\begin{aligned} \phi &= \Phi| \\ \psi_\alpha &= \frac{1}{\sqrt{2}} D_\alpha \Phi| \\ F &= -\tfrac{1}{4} DD\Phi| \end{aligned}}$$

$$\boxed{\begin{aligned} \delta\phi &= \frac{1}{\sqrt{2}} \xi\psi \\ \delta\psi_\alpha &= \sqrt{2}\xi_\alpha F - i\sqrt{2}(\sigma\bar{\xi})_\alpha \partial_\mu \phi \\ \delta F &= -i\sqrt{2}\bar{\xi}\bar{\sigma}^\mu \partial_\mu \psi \end{aligned}}$$

Write SUSY invariant Lagrangians using D_α, $\bar{D}_{\dot{\alpha}}$:

$$\boxed{\mathcal{L}_D = \frac{1}{16} D^2 \bar{D}^2 K|} \qquad K = K^\dagger = \text{real superfield} \;\Rightarrow\; \mathcal{L}_D^\dagger = \mathcal{L}_D$$

Here we use shorthand

$$D^2 = DD = D^\alpha D_\alpha \qquad \bar{D}^2 = \bar{D}\bar{D} = \bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \qquad \textit{etc.}$$

$\mathcal{L}_D$ is called a "D-term." Reason:

$$K = \cdots + \theta^2 \bar{\theta}^2 D_K \qquad D_K = \text{highest component field of } K$$

50% of the symbols in SUSY are some form of the letter D...

$$[\mathcal{L}_D] = 4 \;\Rightarrow\; [K] = 2$$

$$\begin{aligned} \delta\mathcal{L}_D &= \frac{i}{16} D^2 \bar{D}^2 (\xi Q + \bar{\xi}\bar{Q}) K| \\ &= \frac{i}{16} (\xi Q + \bar{\xi}\bar{Q}) D^2 \bar{D}^2 K| \\ &= \frac{1}{16} (\xi D + \bar{\xi}\bar{D}) D^2 \bar{D}^2 K| \quad = \text{total derivative} \end{aligned}$$

Reason: $\quad D_\alpha D^2 = 0$

$$\bar{D}_{\dot{\alpha}} D^2 \bar{D}^2 K = \underbrace{[\bar{D}_{\dot{\alpha}}, D^2]}_{\propto\, \partial_\mu} D^2 K$$

$$\boxed{[\bar{D}_{\dot{\alpha}}, D^2] = -4i\sigma^\mu_{\alpha\dot{\alpha}} D^\alpha \partial_\mu \qquad [D_\alpha, \bar{D}^2] = 4i\sigma^\mu_{\alpha\dot{\alpha}} \bar{D}^{\dot{\alpha}} \partial_\mu}$$

<u>Exercise</u>: Show that if K is a chiral superfield, then $\mathcal{L}_D$ is a total derivative.

Can use similar ideas to construct SUSY invariant from a chiral superfield:

$$\boxed{\mathcal{L}_F = -\frac{1}{4} D^2 W| + \text{h.c.}} \qquad \begin{aligned} W &= \text{chiral superfield} \\ &= \cdots \theta^2 \bar{\theta}^2 F_W \qquad \text{"F term"} \end{aligned}$$

$$\Rightarrow\; \mathcal{L}_F^\dagger = \mathcal{L}_F$$

$$\begin{aligned} \delta\mathcal{L}_F &= -\frac{i}{4} D^2 (\xi Q + \bar{\xi}\bar{Q}) W| \\ &= -\frac{i}{4} (\xi Q + \bar{\xi}\bar{Q}) D^2 W| \\ &= -\frac{1}{4} (\xi D + \bar{\xi}\bar{D}) D^2 W| = \text{total derivative} \end{aligned}$$

$$D_\alpha D^2 = 0 \qquad \bar{D}_{\dot{\alpha}} D^2 W = [\bar{D}_{\dot{\alpha}}, D^2] W \propto \partial_\mu$$

$$[\mathcal{L}_F] = 4 \;\Rightarrow\; [W] = 3$$

In the literature, it is common to use the notation

$$\mathcal{L}_D = \int d^4\theta\, K \qquad \mathcal{L}_F = \int d^2\theta\, W + \text{h.c.}$$

This arises because integration and differentiation are identical for anticommuting variables:

$$\int d\theta\,(a + b\theta) = b = \frac{\partial}{\partial\theta}(a + b\theta)$$

We will use this notation with the understanding that it is defined by

$$\boxed{\int d^4\theta\, K = \frac{1}{16} D^2 \bar{D}^2 K|} \qquad \boxed{\int d^2\theta\, W = -\frac{1}{4} D^2 W|}$$

We now have all the tools we need to write interacting SUSY invariant Lagrangians!

$$\Phi = \text{chiral superfield}$$
$$= \phi(y) + \theta\psi(y) + \theta^2 F(y)$$

$$[\phi] = 1 \;\Rightarrow\; [\psi] = \tfrac{3}{2},\; [F] = 2$$
$$\Rightarrow\; [\Phi] = 1$$

Write the most general SUSY invariant Lagrangian with dimensionless couplings:

$$\mathcal{L} = \int d^4\theta\, \Phi^\dagger \Phi + \left(\int d^2\theta\, \frac{\lambda}{3} \Phi^3 + \text{h.c.} \right)$$

$$[d^4\theta] = 2, \quad [d^2\theta] = 1$$

$$\int d^4\theta\, \Phi^\dagger\Phi = \frac{1}{16} D^2\bar{D}^2(\Phi^\dagger\Phi)| = \frac{1}{16} D^2 \left[(\bar{D}^2\Phi^\dagger)\Phi \right]\Big|$$
$$= \frac{1}{16}\Big[D^2\bar{D}^2\Phi^\dagger|\,\Phi|$$
$$+ 2D^\alpha\bar{D}^2\Phi^\dagger| D_\alpha\Phi|$$
$$+ \bar{D}^2\Phi| D^2\Phi| \Big]$$

$$D^2\bar{D}^2\Phi^\dagger| = [D^2, \bar{D}^2]\Phi^\dagger = -16\Box\phi^\dagger \qquad \boxed{[D^2, \bar{D}^2] = -16\Box}$$

$$D_\alpha\bar{D}^2\Phi^\dagger| = [D_\alpha, \bar{D}^2]\Phi^\dagger| = 4\sqrt{2}(\sigma^\mu\partial_\mu\psi^\dagger)_\alpha$$

$$\int d^4\theta\, \Phi^\dagger\Phi = -(\Box\phi^\dagger)\phi + \psi i\sigma^\mu\partial_\mu\psi^\dagger + F^\dagger F$$

$$\boxed{\int d^4\theta\, \Phi^\dagger\Phi = \partial^\mu\phi^\dagger\partial_\mu\phi + \psi^\dagger i\bar{\sigma}^\mu\partial_\mu\psi + F^\dagger F}$$

Evaluate F term for general function of Φ:

$$\int d^2\theta\, W(\Phi) = -\frac{1}{4} D^2 W(\Phi)|$$
$$= -\frac{1}{4}\left[W'(\Phi) D^2\Phi + W''(\Phi) D^\alpha\Phi D_\alpha\Phi \right]\Big|$$
$$= W'(\phi)F - \tfrac{1}{2} W''(\phi)\psi\psi$$

$$\boxed{\int d^2\theta\, W(\Phi) = W'(\phi)F - \tfrac{1}{2} W''(\phi)\psi\psi}$$

$$W(\Phi) = \text{superpotential}$$

For our theory,

$$\mathcal{L} = \int d^4\theta\, \Phi^\dagger \Phi + \left(\int d^2\theta\, \frac{\lambda}{3} \Phi^3 + \text{h.c.} \right)$$

$$= \partial^\mu \phi^\dagger \partial_\mu \phi + \psi^\dagger i \bar{\sigma}^\mu \partial_\mu \psi + F^\dagger F$$

$$+ \lambda \phi^2 F - \tfrac{1}{2} \lambda \phi \psi \psi + \text{h.c.}$$

We recognize kinetic terms for ϕ, ψ and Yukawa coupling.

Note no derivatives act on F, and $\mathcal{L}$ is quadratic on F. We say that F is an *auxiliary field*.

The significance of this is that we can integrate out F exactly:

$$|F^\dagger + \lambda\phi^2|^2 = F^\dagger F + (\lambda \phi^2 F + \text{h.c.}) + |\lambda \phi^2|^2$$

$$\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi + \psi^\dagger i \bar{\sigma}^\mu \partial_\mu \psi$$

$$- \tfrac{1}{2}(\lambda \phi\psi\psi + \text{h.c.}) + |F^\dagger + \lambda\phi^2|^2 - |\lambda\phi^2|^2$$

Write functional integral:

$$Z = \int [d\phi]\,[d\psi]\,[dF]\, e^{i\int \mathcal{L}}$$

Integral over F is trivial:

$$\int [dF]\, e^{i\int |F^\dagger + \lambda\phi^2|^2} = \int [dX]\, e^{i\int |X|^2} \qquad X = F + (\lambda\phi^2)^\dagger$$

This change of variables is a simple shift, and therefore has trivial Jacobian.

The functional integral over F is therefore independent of other fields, and does not affect correlation functions.

$\Rightarrow$ integrating out F gives

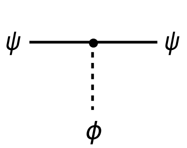
scalar potential!

$$\mathcal{L} \to \partial^\mu \phi^\dagger \partial_\mu \phi + \psi^\dagger i \bar{\sigma}^\mu \partial_\mu \psi$$

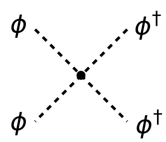
$$- \tfrac{1}{2}(\lambda\phi\psi\psi + \text{h.c.}) - |\lambda\phi^2|^2$$

SUSY relates the Yukawa coupling and the quartic scalar coupling!

Schematically,



$= \lambda$



$= |\lambda|^2$

This is exactly the structure we described in the first lecture for the Higgs-top/stop couplings.

Consider a general theory with N chiral superfields

$$\Phi^a \qquad a = 1, \ldots, N$$

$$\boxed{\mathcal{L} = \int d^4\theta\, \Phi^\dagger_a \Phi^a + \left(\int d^2\theta\, W(\Phi) + \text{h.c.} \right)}$$

After integrating out auxiliary fields F^a

$$\boxed{\mathcal{L}_{\text{int}} = \frac{1}{2} \frac{\partial^2 W(\phi)}{\partial\phi^a \partial\phi^b} \psi^a \psi^b - \left(\frac{\partial W}{\partial \phi^a} \right)^\dagger \frac{\partial W}{\partial \phi^a}}$$

The most general renormalizable superpotential is

$$\boxed{W(\Phi) = \kappa_a \Phi^a + \tfrac{1}{2} m_{ab} \Phi^a \Phi^b + \tfrac{1}{3} \lambda_{abc} \Phi^a \Phi^b \Phi^c}$$

$$[W] = 3 \qquad [\Phi] = 1$$

Note that integrating out F^a is equivalent to imposing its equation of motion

$$F^\dagger_a = \frac{\partial W(\phi)}{\partial \phi^a}$$

We can therefore write the F-term potential as

$$V_F = F^\dagger_a F_a$$

Note that $V_F \geq 0$, and unbroken SUSY requires

$$\langle F^a \rangle = 0 \qquad \text{for all } a$$

Non-renormalization theorem:

The UV divergences of SUSY theories are very constrained. We will show that the coupling constants in the superpotential are not renormalized.

The UV divervences of a QFT can be parameterized by local terms in the 1PI effective action that are relevant or marginal:

$$\Gamma[\Phi] = \text{1PI effective action} = \int d^4x\, \mathcal{L}_{\text{1PI}}[\Phi]$$

$$\begin{aligned}\mathcal{L}_{\text{1PI}} = &\int d^4\theta\, (\delta Z)^a{}_b \Phi^\dagger_a \Phi^b \\ &+ \int d^2\theta \left(\delta\kappa_a \Phi^a + \tfrac{1}{2}\delta m_{ab}\Phi^a\Phi^b + \tfrac{1}{3}\delta\lambda_{abc}\Phi^a\Phi^b\Phi^c\right) \\ &\qquad + \text{h.c.} \\ &+ \text{finite (and non-local)}\end{aligned}$$

$\Lambda = $ UV cutoff

$$\delta Z \sim \ln\Lambda$$

$$\delta\kappa \sim \Lambda^2 + \Lambda m + \kappa \ln\Lambda$$

$$\delta m \sim \Lambda + m \ln\Lambda$$

$$\delta\lambda \sim \ln\Lambda$$

The UV divergent terms must respect the symmetries of the original theory.

This is true as long as the UV regulator that preserves the symmetries in question.

In the present class of theories we can use *e.g.* Pauli-Villars or higher derivative regulator to regulate the theory while preserving SUSY.

A very powerful technique is to promote the couplings κ, m, λ to background chiral superfields that transform under SUSY.

This generalized Lagrangian is SUSY invariant as long as we keep κ, m, λ inside the superspace integrals.

$$\mathcal{L}_{\text{int}} = \int d^2\theta \left(\kappa_a \Phi^a + \tfrac{1}{2} m_{ab}\Phi^a\Phi^b + \tfrac{1}{3}\lambda_{abc}\Phi^a\Phi^b\Phi^c\right) + \text{h.c.}$$

The Lagrangian is also invariant under a $U(N)$ symmetry

$$\Phi^a \mapsto U^a{}_b \Phi^b$$

$$\kappa_a \mapsto (U^{-1})^b{}_a \kappa_b$$

$$m_{ab} \mapsto (U^{-1})^c{}_a (U^{-1})^d{}_b m_{cd}$$

$$\lambda_{abc} \mapsto (U^{-1})^d{}_a (U^{-1})^e{}_b (U^{-1})^f{}_c \lambda_{def}$$

That is, all quantities transform according to their index structure.

The coefficients $\delta Z, \delta\kappa, \delta m, \delta\lambda$ must be functions of κ, m, λ that respect this $U(N)$ symmetry.

SUSY also requires the coefficients $\delta\kappa, \delta m, \delta\lambda$ to be holomorphic functions of κ, m, λ (*i.e.* independent of $\kappa^\dagger, m^\dagger, \lambda^\dagger$).

Claim: most general allowed form is

$$\delta\lambda_{abc} = c_\lambda \, \lambda_{abc} \ln\Lambda$$

$$\delta m_{ab} = c_m \, m_{ab} \ln\Lambda$$

$$\delta\kappa_a = c_\kappa \, \kappa_a \ln\Lambda$$

$$c_\lambda, c_m, c_\kappa = \text{independent of couplings}$$

Note there are no couplings with upper $U(N)$ indices we can use to contract indices. If not for holomorphy, we could use

$$\kappa^{\dagger a}, \ m^{\dagger ab}, \ \lambda^{\dagger abc}$$

[For $N = 1$, this follows from $U(1)$ symmetry, since κ, m, λ have different charges.]

Also, we cannot have divergent term such as $m_{ab} \propto \kappa_a \kappa_b$ because of dimensional analysis.

$$[\kappa_a] = 2, \quad [m_{ab}] = 1, \quad [\lambda_{abc}] = 0$$

Because $\delta\kappa, \delta m, \delta\kappa$ are linear in the couplings, they can be computed in perturbation theory. But all loop diagrams have at least 2 powers of the couplings.

$$\mathcal{L}_{\text{int}} = \lambda_{abc} \phi^a \psi^b \psi^c + \text{h.c.}$$
$$- \sum_a \left| \kappa_a + m_{ab} \phi^b + \lambda_{abc} \phi^b \phi^c \right|^2$$

$$\Rightarrow c_\kappa, c_m, c_\lambda = 0$$

$$\Rightarrow \delta\kappa, \delta m, \delta\lambda = 0 \qquad \text{QED}$$

This is a symmetry argument
$\Rightarrow$ valid beyond perturbation theory.

The coefficient δZ is nonzero:

$$\delta Z^a{}_b = c_Z \lambda^{\dagger acd} \lambda_{bcd} \ln\Lambda + \underbrace{O(\lambda^4)}_{\sim (\lambda^\dagger \lambda)^2 \text{ by } U(N) \text{ invariance}}$$

$$c_Z = -\frac{1}{4\pi^2}$$

δZ cannot depend on κ or m by dimensional analysis.

Treat Λ dependence using standard renormalization theory.

For simplicity, focus on the case of one chiral superfield Φ:

$$\mathcal{L} = \int d^4\theta \, \Phi^\dagger \Phi + \int d^2\theta \left(\kappa\Phi + \tfrac{1}{2} m \Phi^2 + \tfrac{1}{3} \lambda \Phi^3 \right) + \text{h.c.}$$

Cancel Λ dependence in the 1PI effective action by adding a counterterm to the Lagrangian:

$$\Delta\mathcal{L}_{\text{ct}} = \int d^4\theta \left[-c_Z \lambda^\dagger \lambda \ln\frac{\Lambda}{\mu} + O(\lambda^4) \right] \Phi^\dagger \Phi$$

This eliminates the dependence on Λ, at the price of introducing dependence on the renormalization scale μ.

Renormalized Lagrangian:

$$\mathcal{L} = \mathcal{L}_{\text{R}} + \Delta\mathcal{L}_{\text{ct}}$$

$$\mathcal{L}_{\text{R}} = \int d^4\theta \, Z_{\text{R}}(\mu) \Phi^\dagger \Phi + \int d^2\theta \left(\kappa\Phi + \tfrac{1}{2} m \Phi^2 + \tfrac{1}{3} \lambda \Phi^3 \right) + \text{h.c.}$$

$$\mu \frac{d}{d\mu} \ln Z_{\text{R}} = -c_Z \lambda^\dagger \lambda + O(\lambda^4) \qquad \text{wavefunction RG equation}$$

Define canonically normalized fields

$$\Phi = [Z_R(\mu)]^{-1/2}\,\hat{\Phi}$$

Note $Z_R \sim 1 + \lambda^\dagger\lambda \;\neq$ chiral superfield

$\Rightarrow \hat{\Phi}$ is a chiral superfield only in the case where λ is a constant (independent of $x, \theta, \bar{\theta}$).

For this case, we can write

$$\mathcal{L}_R = \int d^4\theta\, \hat{\Phi}^\dagger\hat{\Phi} + \int d^2\theta \left[\kappa_R(\mu)\hat{\Phi} + \tfrac{1}{2}m_R(\mu)\hat{\Phi}^2 + \tfrac{1}{3}\lambda_R(\mu)\hat{\Phi}^3\right] + \text{h.c.}$$

$$\kappa_R(\mu) = [Z_R(\mu)]^{-1/2}\kappa$$

$$m_R(\mu) = [Z_R(\mu)]^{-1}m$$

$$\lambda_R(\mu) = [Z_R(\mu)]^{-3/2}\lambda$$

We see that the physical couplings are multiplicatively renormalized:

$$\mu\frac{d}{d\mu}\kappa_R = -\tfrac{1}{2}c_Z\lambda_R^2\kappa_R + O(\lambda_R^4)$$

$$\mu\frac{d}{d\mu}m_R = -c_Z\lambda_R^2 m_R + O(\lambda_R^4) \qquad (\lambda_R = \text{real})$$

$$\mu\frac{d}{d\mu}\lambda_R = -\tfrac{3}{2}c_Z\lambda_R^3 + O(\lambda_R^4)$$

One implication of this is that if a superpotential coupling is set to zero at some scale, it remains zero at all scales, whether or not there is an enhanced symmetry.

SUSY Gauge Theory

Guiding princple: gauge invariance

Suppose ϕ, ψ are components of a chiral superfield Φ with charge q under a $U(1)$ gauge group:

$$\psi_\alpha(x) \mapsto e^{iq\alpha(x)}\psi_\alpha(x) \qquad \psi_\alpha(x) \mapsto e^{iq\alpha(x)}\psi_\alpha(x)$$

$\alpha(x) =$ gauge transformation parameter

$q =$ charge (*e.g.* ± 1)

Generalize to superspace:

$$\boxed{\Phi \mapsto e^{qg\Omega}\Phi} \qquad \alpha = \mathrm{Im}\,\Omega|$$

Transformed superfield = chiral $\Rightarrow \Omega =$ chiral superfield

Kinetic term is not invariant:

$$\int d^4\theta\,\Phi^\dagger\Phi \mapsto \int d^4\theta\, e^{qg(\Omega+\Omega^\dagger)}\,\Phi^\dagger\Phi$$

($\Omega^\dagger = -\Omega \Rightarrow \Omega =$ independent of x.)

Make kinetic term gauge invariant by introducing a real superfield V transforming as

$$\boxed{V \mapsto V - \tfrac{1}{2}(\Omega + \Omega^\dagger)} \qquad V^\dagger = V$$

$$\Rightarrow \int d^4\theta\,\Phi^\dagger e^{2qV}\Phi = \text{gauge invariant}$$

Define components by projection:

$$
\boxed{
\begin{aligned}
&C = V| \\
&\chi_\alpha = D_\alpha V| \\
&B = D^2 V| \\
&A^\mu = \frac{i}{4}\bar{\sigma}^{\mu\dot{\alpha}\beta}[\bar{D}_{\dot{\alpha}}, D_\beta]V| \\
&\lambda_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V| \\
&D = \frac{1}{32}\{D^2, \bar{D}^2\}V|
\end{aligned}
}
\qquad
\begin{aligned}
&A_\mu^\dagger = A_\mu \\
\\
&D^\dagger = D
\end{aligned}
$$

Compute gauge transformation of these components.

Define components of Ω:

$$\omega + i\alpha = \Omega| \qquad \omega, \theta = \text{real}$$

$$\eta_\alpha = D_\alpha \Omega|$$

$$E = D^2 \Omega|$$

Gauge transformation:

$$\delta C = -\tfrac{1}{2}(\Omega + \Omega^\dagger)| = -\omega$$

$$\delta\chi_\alpha = D_\alpha\left[-\tfrac{1}{2}(\Omega + \Omega^\dagger)\right]\Big| = -\tfrac{1}{2}\eta_\alpha$$

$$\delta A^\mu = \frac{i}{4}\bar{\sigma}^{\mu\dot{\alpha}\beta}\bar{D}_{\dot{\alpha}} D_\beta\left[-\tfrac{1}{2}(\Omega + \Omega^\dagger)\right]\Big| + \text{h.c.}$$

$$= \partial^\mu \alpha \longleftarrow \text{conventional gauge transformation!}$$

$$\delta\lambda_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha\left[-\tfrac{1}{2}(\Omega + \Omega^\dagger)\right]\Big| = 0$$

To understand δA_μ and $\delta\lambda_\alpha$, note that

$$
\begin{aligned}
\bar{D}_{\dot{\alpha}} D_\beta(\Omega + \Omega^\dagger) &= \bar{D}_{\dot{\alpha}} D_\beta \Omega \\
&= \{\bar{D}_{\dot{\alpha}}, D_\beta\}\Omega \\
&= 2i\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \Omega
\end{aligned}
$$

$$\bar{D}^{\dot{\alpha}}\bar{D}_{\dot{\alpha}} D_\beta(\Omega + \Omega^\dagger) = 2i\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu \bar{D}^{\dot{\alpha}}\Omega = 0$$

Summarize:

$$
\boxed{
\begin{array}{lll}
\delta C = -\omega & \delta\chi_\alpha = -\tfrac{1}{2}\eta_\alpha & \delta B = E \\
\delta A_\mu = \partial_\mu \alpha & \delta\lambda_\alpha = 0 & \delta D = 0
\end{array}
}
$$

We can use gauge freedom in ω, η_α, E to fix

$$C = 0 \qquad \chi_\alpha = 0 \qquad B = 0$$

(Wess-Zumino gauge)

In Wess-Zumino gauge, the gauge-invariant kinetic term is

$$
\begin{aligned}
\int d^4\theta\, \Phi^\dagger e^{2qV}\Phi = D^\mu\phi^\dagger D_\mu\phi + \psi^\dagger i\bar{\sigma}^\mu D_\mu\psi + F^\dagger F \\
- \sqrt{2}q\left(\phi^\dagger\lambda\psi + \text{h.c.}\right) + q\phi^\dagger\phi D
\end{aligned}
$$

where

$$D_\mu\phi = (\partial_\mu - iqA_\mu)\phi \qquad D_\mu\psi = (\partial_\mu - iqA_\mu)\psi$$

The form of the kinetic terms is dictated by the residual gauge invariance

$$\phi \mapsto e^{iq\alpha}\phi \qquad \psi \mapsto e^{iq\alpha}\psi \qquad F \mapsto e^{iq\theta}F$$

We see that A_μ is a conventional gauge field.

$$[A_\mu] = 1 \qquad [\lambda] = \tfrac{3}{2} \qquad [D] = 2$$

$A_\mu =$ gauge field

$\lambda =$ propagating spin $\frac{1}{2}$ field

$=$ superpartner of gauge particle

$D =$ auxiliary field

We can get conventional form of the Lagrangian by rescaling $V \to gV$, where g is the gauge coupling:

$$\boxed{\begin{aligned}\int d^4\theta\, \Phi^\dagger e^{2qgV} \Phi = D^\mu \phi^\dagger D_\mu \phi + \psi^\dagger i\bar{\sigma}^\mu D_\mu \psi + F^\dagger F \\ - \sqrt{2} qg \left(\phi^\dagger \lambda_\alpha \psi^\alpha + \text{h.c.} \right) \\ + qg\phi^\dagger \phi D\end{aligned}}$$

$$D_\mu \phi = (\partial_\mu - iqgA_\mu)\phi \qquad D_\mu \psi = (\partial_\mu - iqgA_\mu)\psi$$

To write gauge kinetic term, use superfield whose lowest component is λ_α:

$$\boxed{W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V} \quad = \text{gauge invariant}$$

$$\bar{D}_{\dot\beta} W_\alpha = 0 \qquad \textit{i.e. } W_\alpha = \text{chiral}$$

W_α has nothing to do with superpotential W (sorry!)

$$[W_\alpha] = \tfrac{3}{2}$$

$\Rightarrow$ we can write dimension-4 gauge- and SUSY-invariant term

$$\int d^2\theta\, W^\alpha W_\alpha = -\frac{1}{2} F^{\mu\nu} F_{\mu\nu} + \frac{i}{4} \epsilon^{\mu\nu\rho\tau} F_{\mu\nu} F_{\rho\tau} - 2\lambda^\dagger i\bar{\sigma}^\mu \partial_\mu \lambda + D^2$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\begin{aligned}\mathcal{L}_{\text{gauge}} &= \int d^2\theta \left(\frac{1}{4g^2} - \frac{i\Theta}{32\pi^2} \right) W^\alpha W_\alpha + \text{h.c.} \\ &= -\frac{1}{4g^2} F^{\mu\nu} F_{\mu\nu} + \frac{\Theta}{64\pi^2} \epsilon^{\mu\nu\rho\tau} F_{\mu\nu} F_{\rho\tau} \\ &\quad + \frac{1}{g^2} \lambda^\dagger i\bar{\sigma}^\mu \partial_\mu \lambda + \frac{1}{2g^2} D^2\end{aligned}$$

$g =$ gauge coupling

$\epsilon^{\mu\nu\rho\tau} F_{\mu\nu} F_{\rho\tau} =$ total derivative

$\Theta =$ vacuum angle

Θ term is a total derivative, and does not give any observable effects for a $U(1)$ gauge theory.

It plays an important role in non-abelian gauge theories.

To get canonically normalized kinetic terms, write (neglecting Θ term)

$$\boxed{\begin{aligned}\mathcal{L}_{\text{gauge}} &= \int d^2\theta\, \tfrac{1}{4} W^\alpha W_\alpha + \text{h.c.} \\ &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \lambda^\dagger i\bar{\sigma}^\mu \partial_\mu \lambda + \tfrac{1}{2} D^2\end{aligned}}$$

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V$$

D-term Potential

The component field D appears quadratically in $\mathcal{L}$ and without derivatives. It is therefore an auxiliary field and can be integrated out exactly.

Consider a general theory with N chiral superfields and one $U(1)$ gauge group:

$$\Phi^a \mapsto e^{gq_a\Omega}\Phi^a \qquad\qquad a = 1,\ldots,N$$

We have rescaled $V \to gV$ so that gauge fields are canonically normalized.

$$\begin{aligned}\mathcal{L} &= \int d^4\theta \sum_a \Phi_a^\dagger e^{2gq_aV}\Phi^a \\ &\quad + \int d^2\theta\, \tfrac{1}{4}W^\alpha W_\alpha + \text{h.c.} \\ &\quad + \int d^2\theta\, W(\Phi) + \text{h.c.} \\ &= gD\sum_a q_a\phi_a^\dagger\phi^a + \tfrac{1}{2}D^2 + \text{independent of } D \\ &= \tfrac{1}{2}\left(D + gq_a\phi_a^\dagger\phi^a\right)^2 - \tfrac{1}{2}g^2\left(\sum_a q_a\phi_a^\dagger\phi^a\right)^2 + \cdots\end{aligned}$$

Integrating out D generates potential

$$\boxed{V_D = \tfrac{1}{2}g^2\left(\sum_a q_a\phi_a^\dagger\phi^a\right)^2}$$

Integrating out D is equivalent to imposing its equation of motion

$$D = g\sum_a q_a\phi_a^\dagger\phi_a$$

We can therefore write the D-term potential as

$$V_F = \tfrac{1}{2}D^2$$

Note that $V_D \geq 0$, and unbroken SUSY requires

$$\langle D\rangle = 0$$

Exercise: Consider SUSY QED, a $U(1)$ SUSY gauge theory with 2 chiral superfields $\Phi_\pm$ with gauge charge ± 1.

The superpotential is

$$W = m\Phi_+\Phi_-$$

Work out the scalar potential for this model.

Show that the only minimum of the potential is at $\langle\phi_+\rangle = \langle\phi_-\rangle = 0$.

Exercise: Consider the same theory with the addition of a chiral superfield S that is neutral under the gauge group.

The superpotential is

$$W = \lambda S\Phi_+\Phi_- + \frac{\kappa}{3}S^3$$

Show that this theory has a minimum of the potential for any value of $\langle S\rangle$.

Non-Abelian Gauge Theory

The formulation of SUSY non-Abelian gauge theory follows the same steps as Abelian gauge theory, with some technical complications.

$\Phi =$ chiral superfield in fundamental representation of $SU(N)$ gauge group

$$\Phi^a \mapsto (e^{\Omega_A T_A})^a{}_b \Phi^b \qquad \begin{array}{l} a, b = 1, \ldots, N \\ (T_A)^a{}_b = SU(N) \text{ generator} \\ \qquad A = 1, \ldots, N^2 - 1 \\ \mathrm{tr}(T_A T_B) = \tfrac{1}{2}\delta_{AB} \end{array}$$

Kinetic term is not gauge invariant:

$$\int d^4\theta\, \Phi^\dagger \Phi \mapsto \int d^4\theta\, \Phi^\dagger e^{\Omega_A^\dagger T_A} e^{\Omega_A T_A} \Phi$$

Introduce one gauge superfield for each gauge generator:

$$V = V_A T_A \qquad V_A^\dagger = V_A$$

Invariant kinetic term:

$$\int d^4\theta\, \Phi^\dagger e^{2V} \Phi = \text{gauge invariant}$$

$$e^{2V} \mapsto e^{-\Omega^\dagger} e^{2V} e^{-\Omega} \qquad \Omega = \Omega_A T_A$$

$$\Rightarrow\; V_A \mapsto V_A - \tfrac{1}{2}(\Omega_A + \Omega_A^\dagger) + \cdots$$

To define components with simple gauge transformation properties, note that

$$e^{-2V} D_\alpha e^{2V} \mapsto e^{\Omega} e^{-2V} e^{\Omega^\dagger} D_\alpha (e^{-\Omega^\dagger} e^{2V} e^{-\Omega})$$
$$= e^{\Omega}(e^{-2V} D_\alpha e^{2V}) e^{-\Omega} + e^{\Omega} D_\alpha e^{-\Omega}$$

Looks like spinor version of

$$A_\mu \mapsto e^{i\theta_A T_A} A_\mu e^{-i\theta_A T_A} + e^{i\theta_A T_A} \partial_\mu e^{-i\theta_A T_A}$$

Define component fields

$$A^\mu = \frac{i}{8} \bar{\sigma}^{\mu\dot{\alpha}\beta} \bar{D}_{\dot{\alpha}} (e^{-2V} D_\alpha e^{2V})| + \text{h.c.}$$

$$\lambda_\alpha = -\frac{1}{4} \bar{D}^2 (e^{-2V} D_\alpha e^{2V})|$$

$$D = \frac{1}{64} D^\alpha \bar{D}^2 (e^{-2V} D_\alpha e^{2V})| + \text{h.c.}$$

Other components vanish in Wess-Zumino gauge.

Work out gauge invariant kinetic term in terms of component fields:

$$\boxed{\begin{aligned} \int d^4\theta\, \Phi^\dagger e^{2V} \Phi &= D^\mu \phi^\dagger D_\mu \phi + \psi^\dagger i \bar{\sigma}^\mu D_\mu \psi + F^\dagger F \\ &\quad + \sqrt{2}(\phi^\dagger T_A \lambda_A \psi + \text{h.c.}) + D_A \phi^\dagger T_A \phi \end{aligned}}$$

$$D_\mu \phi = \partial_\mu \phi - i A_{\mu A} T_A \phi$$

To write gauge kinetic term, note that

$$\bar{D}^2(e^{-2V}D_\alpha e^{2V}) \mapsto e^\Omega \bar{D}^2(e^{-2V}D_\alpha e^{2V})e^{-\Omega} + e^\Omega \underbrace{\bar{D}^2 D_\alpha e^{-\Omega}}$$

$$= [\bar{D}^2, D_\alpha]e^{-\Omega}$$

$$\propto \partial_\mu \bar{D}_{\dot\alpha} e^{-\Omega} = 0$$

$$\boxed{W_\alpha = -\frac{1}{8}\bar{D}^2(e^{-2V}D_\alpha e^{2V})} \quad = \text{chiral superfield}$$

$$W_\alpha \mapsto e^\Omega W_\alpha e^{-\Omega}$$

$$W_\alpha = W_{\alpha A} T_A$$

Write invariant Lagrangian:

$$\mathcal{L} = \int d^2\theta \left(\frac{1}{4g^2} - \frac{i\Theta}{32\pi^2}\right) W^\alpha_A W_{\alpha A} + \text{h.c.}$$

$$+ \int d^4\theta \Phi^\dagger e^{2V_A T_A}\Phi$$

$$= -\frac{1}{4g^2}F^{\mu\nu}_A F_{\mu\nu A} - \frac{i\Theta}{64\pi^2}\epsilon^{\mu\nu\rho\tau}F_{\mu\nu A}F_{\rho\tau A}$$

$$+ \frac{1}{g^2}\lambda^\dagger_A i\bar\sigma^\mu (D_\mu\lambda)_A + \frac{1}{2g^2}D_A D_A$$

$$+ D^\mu\phi^\dagger D_\mu\phi + \psi^\dagger i\bar\sigma^\mu D_\mu\psi + F^\dagger F + D_A\phi^\dagger T_A\phi$$

$$(D_\mu\lambda)_A = \partial_\mu\lambda_A + f_{ABC}A_{\mu B}\lambda_C \qquad [T_A, T_B] = if_{ABC}T_C$$

Integrate out auxiliary fields D_A:

$$\Rightarrow \quad \boxed{V_D = \frac{g^2}{2}\left(\sum_A \phi^\dagger T_A \phi\right)^2}$$

If there are chiral superfields Φ^i, $i = 1, \ldots, n$, we have

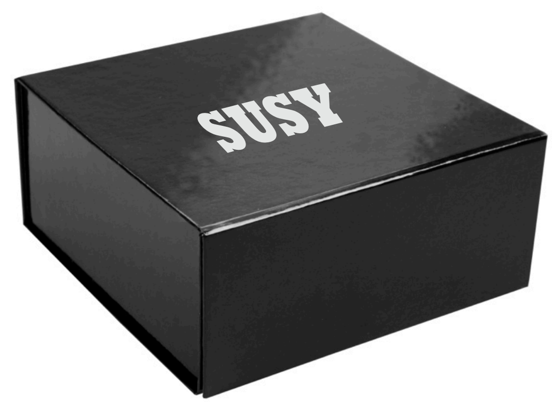
Exercise: Consider $SU(N)$ SUSY QCD with one flavor. This is the theory of two chiral superfields Q and $\tilde{Q}$ transforming in the fundamental and antifundamental representation.

$T_A =$ generators of fundamental representation

$-T_A^T =$ generators of antifundamental representation

Write the gauge invariant kinetic term for Q and $\tilde{Q}$ and work out the D-term potential.

SUSY in a Black Box



Can understand the rest of the lectures treating the superfield formalism as a black box.

Chiral superfields: $\Phi = (\phi, \psi_\alpha, F)$

$U(1)$ gauge superfields: $V = (\lambda_\alpha, A_\mu, D)$ (WZ gauge)

$$\int d^2\theta\, W(\Phi) = \frac{\partial W(\phi)}{\partial \phi^a} F_a + \frac{\partial^2 W(\phi)}{\partial \phi^a \phi^b} \psi^a \psi^b$$

$$\int d^4\theta\, \Phi^\dagger e^{2gV} \Phi = D^\mu \phi^\dagger D_\mu \phi + \psi^\dagger i\bar{\sigma}^\mu D_\mu \psi + F^\dagger F + \sqrt{2} g(\phi^\dagger T_A \lambda_A \psi + \text{h.c.}) + D_A \phi^\dagger T_A \phi$$

$$W_\alpha = -\tfrac{1}{4} \bar{D}^2 D_\alpha V$$

$$\int d^2\theta\, \tfrac{1}{4} W^\alpha W_\alpha = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + \lambda^\dagger i\bar{\sigma}^\mu D_\mu \lambda + \tfrac{1}{2} D^2$$

Supersymmetry Breaking

If SUSY exists in nature, it must be broken.

SUSY is a spacetime symmetry, so it is gauged when we include gravity (supergravity).

Because of this, SUSY must be broken spontaneously.

This means we must look for a SUSY invariant Lagrangian whose ground state breaks SUSY.

From the SUSY algebra, we know that SUSY is broken if and only if the vacuum energy is nonzero.

$$\cancel{\text{SUSY}} \quad \Leftrightarrow \quad \langle 0|H|0\rangle > 0$$

Classical potential:

$$V = F_a^\dagger F_a + \tfrac{1}{2} D_A D_A$$

$$F_a^\dagger = \frac{\partial W(\phi)}{\partial \phi^a} \qquad D_A = \frac{g_A^2}{2} \phi_a^\dagger (T_A)^a{}_b \phi^a$$

$T_A =$ gauge generators/charges

g_A depends on A for product gauge group

We see that SUSY breaking requires nonzero VEVs for some of the axiliary fields F^a and D_A.

Polonyi Model:

Simplest model of SUSY breaking.

$$\mathcal{L} = \int d^4\theta\, \Phi^\dagger \Phi + \left(\int d^2\theta\, \kappa \Phi + \text{h.c.} \right)$$

$$F^\dagger = \frac{\partial W}{\partial \phi} = \kappa \neq 0 \qquad \text{“}F\text{-type breaking”}$$

Nonzero vacuum energy, but is SUSY really broken?

$$\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi + \psi^\dagger i \bar{\sigma}^\mu \partial_\mu \psi$$

$= \text{free field theory with SUSY spectrum}$

$\langle \phi \rangle = \text{undetermined}\ldots$

Can turn this into a theory with real SUSY breaking by generalizing Kahler potential:

$$\mathcal{L} = \int d^4\theta \left[\Phi^\dagger \Phi - \frac{1}{4M^2} (\Phi^\dagger \Phi)^2 \right] + \left(\int d^2\theta\, \kappa \Phi + \text{h.c.} \right)$$

Potential terms (no spacetime derivatives):

$$\mathcal{L} = F^\dagger F - \frac{1}{M^2} \phi^\dagger \phi F^\dagger F + (\kappa F + \text{h.c.})$$

Integrate out $F \Rightarrow$

$$V = \frac{|\kappa|^2}{1 - \phi^\dagger \phi / M^2}$$

Minimizing potential fixes $\langle \phi \rangle = 0$

Can show:

$$m_\phi^2 = \frac{|\kappa|^2}{M^2} \qquad m_\psi = 0$$

Massless fermion is the Goldstino = Goldstone fermion

The Goldstino

When a global symmetry is spontaneously broken, the theory contains a massless Nambu-Goldstone excitation with the quantum numbers of the broken symmetry generator. So it is not surprising that spontaneously broken SUSY leads to a massless neutral Weyl fermion, the Goldstino.

$$Q_\alpha \text{ broken } \Rightarrow \text{ massless Goldstino } \chi_\alpha$$

This can be proven in great generality.

We will content ourselves with showing it in the simplest possible case of a theory with only chiral superfields.

$$V = \left(\frac{\partial W}{\partial \phi^a}\right)^\dagger \frac{\partial W}{\partial \phi^a}$$

$$0 = \frac{\partial V}{\partial \phi^b} = \left(\frac{\partial W}{\partial \phi^a}\right)^\dagger \frac{\partial^2 W}{\partial \phi^a \partial \phi^b}$$

$$\cancel{\text{SUSY}} \Rightarrow \left\langle \frac{\partial W}{\partial \phi^a} \right\rangle \neq 0$$

$$\Rightarrow \underbrace{\left\langle \frac{\partial^2 W}{\partial \phi^a \partial \phi^b} \right\rangle}_{\text{fermion mass matrix}} \text{ has a zero eigenvalue}$$

When a gauge theory is spontaneously broken, the Nambu-Goldstone excitation becomes the longitudinal polarization of a massive gauge particle. So it is not surprising that in supergravity the Goldstino becomes the longitudinal mode of the gravitino, the spin $\frac{3}{2}$ superpartner of the graviton.

$$\begin{aligned} \text{massless gravity multiplet} &= \text{massless spin 2} \\ &\quad + \text{massless spin } \tfrac{3}{2} \end{aligned}$$

$$m_{3/2} = \text{gravitino mass} \sim \frac{\langle F \rangle}{M_{\text{Pl}}} \qquad \langle F \rangle = \cancel{\text{SUSY}} \text{ VEV}$$

Note that for $\langle F \rangle \ll M_{\text{Pl}}^2$ the gravitino is a light particle with possible consequences for phenomenology and cosmology.

If SUSY solves the naturalness problem of the Standard Model, then the superpartners of the observed particles must have masses at the TeV scale.*

It would be very exciting if SUSY were spontaneously broken at the TeV scale by a "super Higgs" sector whose particles have masses at the TeV scale. Unfortunately, this idea does not work.

The reason is that renormalizable SUSY theories do not contain a Yukawa coupling of the form

$$\phi\lambda\lambda \qquad \begin{aligned} \phi &= \text{scalar from chiral multiplet} \\ \lambda &= \text{gaugino} \end{aligned}$$

This means there is no way to generate a mass for the gauginos at tree level.

* We'll discuss some fine print later.

Some possibilities:

- Strong SUSY breaking at the TeV scale
- Superpartner masses from loop effects
  e.g. gauge mediation or anomaly mediation
- Hidden sector SUSY breaking

Hidden Sector SUSY Breaking

Assume SUSY is broken in a hidden sector that is coupled via higher-dimension operators to the visible sector, which contains the Standard Model fields. These higher-dimension operators can arise from new physics at high scales, for example new heavy particles or string theory.

Assume that the hidden sector breaks SUSY through the F term of a chiral superfield X:

$$\langle F_X \rangle \neq 0 \qquad X = \text{hidden sector chiral superfield}$$

Assume $\langle X \rangle = 0$ without loss of generality (shift field if necessary).

$$\langle X \rangle = F_X \theta^2 \qquad F_X \neq 0$$

Now consider higher-dimension couplings between X and the visible sector fields.

Explore this with a toy hidden sector consisting of a single chiral superfield Φ and a gauge superfield V, with

$$W(\Phi) = \frac{\lambda}{3}\Phi^3$$

Possible couplings:

$$\int d^2\theta \, \frac{X}{M} W^\alpha W_\alpha + \text{h.c.} = \frac{F_X}{M}\lambda\lambda + \text{h.c.} \qquad \Rightarrow m_\lambda \sim \frac{F_X}{M}$$

$$\int d^4\theta \, \frac{1}{M^2} X^\dagger X \Phi^\dagger \Phi = \left|\frac{F_X}{M}\right|^2 \phi^\dagger \phi \qquad \Rightarrow \; m_\phi^2 \sim \left(\frac{F_X}{M}\right)^2$$

$\Rightarrow$ gaugino and scalar masses $\sim F_X/M$

But wait, there's more!

$$\int d^2\theta \, \frac{X}{M}\Phi^3 + \text{h.c.} = \frac{F_X}{M}\phi^3 + \text{h.c.} \qquad \Rightarrow \text{cubic scalar couplings}$$

$$\int d^4\theta \left(\frac{X}{M} + \text{h.c.}\right)\Phi^\dagger\Phi = \frac{F_X}{M} F_\phi^\dagger \phi + \text{h.c.}$$

To see what this does, integrate out F_ϕ:

$$\begin{aligned} \mathcal{L} &= F_\phi^\dagger F_\phi + \left[F_\phi \lambda \phi^2 + \text{h.c.}\right] \\ &= \left|F_\phi^\dagger + \lambda\phi^2 + \frac{F_X^\dagger}{M}\right|^2 - \left|\lambda\phi^2 + \frac{F_X^\dagger}{M}\phi^\dagger\right|^2 \end{aligned}$$

$$\Rightarrow \; \Delta V_F = \left|\frac{F_X}{M}\right|^2 \phi^\dagger\phi + \left(\frac{F_X}{M}\lambda\phi^3 + \text{h.c.}\right)$$

Note: all SUSY breaking mass parameters $\sim F_X/M$!

Higher order terms give smaller effects...

Summary: Hidden sector SUSY breaking models can naturally give gaugino masses, scalar masses, and cubic scalar couplings all of the same order of magnitude.

$$\frac{F_X}{M} \sim \text{TeV}$$

Note that if the particles in the hidden sector have masses large compared to the TeV scale, then the only observable effect of the hidden sector is the VEV F_X.

This looks just like explicit breaking...

Soft SUSY Breaking

A more phenomenological approach: break SUSY explicitly, with all mass parameters chosen to be $\sim$ TeV.

We must ensure that this explicit breaking does not give rise to quadratic sensitivity to UV physics. That is, the breaking must be *soft*.

To explore this, consider again our toy visible sector, but now allow the most general renormalizable superpotential:

$$\begin{aligned}\mathcal{L} = &\int d^2\theta\, S W^\alpha W_\alpha + \text{h.c.}\\ &+ \int d^4\theta\, Z \Phi^\dagger \Phi\\ &+ \int d^2\theta \left[\kappa\Phi + \tfrac{1}{2}\mu\Phi^2 + \tfrac{1}{3}\lambda\Phi^3\right] + \text{h.c.}\end{aligned}$$

Note we have written the couplings as superfields and included an arbitrary normalization for the chiral superfield kinetic term:

$$S = \frac{1}{2g^2} + \cdots = \text{chiral superfield}$$

$$Z = 1 + \cdots = \text{real superfield}$$

$$\kappa, \mu, \lambda = \text{chiral superfield}$$

The SUSY non-renormalization theorem guarantees that this theory has only logarithmic renormalization of Z and S.

Logarithmic UV divergences $\Leftrightarrow$ logarithmic sensitivity to UV physics. We want to preserve this feature wen we include SUSY breaking.

A simple way to do this is to break SUSY by turning on θ^2 and $\theta^2\bar{\theta}^2$ components of the coupling superfields.

$$Z \to 1 + (\theta^2 B + \text{h.c.}) + \theta^2\bar{\theta}^2(-m_0^2 + |B|^2)$$

$$S \to \frac{1}{2g^2} - \theta^2 \frac{m_{1/2}}{g^2}$$

$$\kappa \to \hat{\kappa}\left(1 + \theta^2 B_\kappa\right)$$

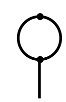
$$\mu \to \hat{\mu}\left(1 + \theta^2 B_\mu\right)$$

$$\lambda \to \hat{\lambda}\left(1 + \theta^2 B_\mu\right)$$

The UV divergent terms in the effective action can be written in terms of superfields, and the fact that the higher components of the superfield couplings are nonvanishing does not change the log divergent terms. For example,

$$\Delta\mathcal{L}_{\text{1PI}} \sim \int d^4\theta\, (Z\lambda^\dagger\lambda \times \ln\Lambda)\Phi^\dagger\Phi$$

There is one subtle point in this argument: there is an additional allowed UV divergence:



$$\Rightarrow \quad \Delta\mathcal{L}_{\text{1PI}} \sim \int d^4\theta\, (\lambda\mu^\dagger \times \ln\Lambda)\Phi + \text{h.c.}$$

This term is a total derivative in the SUSY limit, but a contribution to the linear term in the scalar potential when SUSY breaking is turned on.

Note: allowed only if Φ is a gauge singlet...

We can write an arbitrary breaking term using superfield spurions, but all other terms are higher-dimension operators in superspace, and therefore give rise to power divergences.

For example, a non-holomorphic cubic coupling:

$$\Delta\mathcal{L} = \int d^4\theta\, X \Phi^\dagger \Phi^2 + \text{h.c.} \qquad X = \theta^2\bar{\theta}^2 h \qquad [h] = 1$$

$$= h\phi^\dagger\phi^2 + \text{h.c.}$$

Even though the coupling h has positive mass dimension, the breaking is not soft: there is a counterterm

$$\Delta\mathcal{L}_{1\text{PI}} \sim \int d^4\theta\, \Lambda^2 X \Phi + \text{h.c.} \sim \Lambda^2 h\phi + \text{h.c.}$$

Summarize: softly broken SUSY is equivalent to turning on higher components of superfield couplings.

These are also the effects that we expect if SUSY is broken in a hidden sector.

This is a good starting point for a phenomenological treatment of models with broken SUSY.

The Minimal Supersymmetric Standard Model

An excellent reference with many more details:

S. Martin, "A Supersymmetry Primer," `arXiv:9709356 (v6)`.

But note: uses $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$...

Start by writing the Standard Model in terms of L Weyl fermions:

	$SU(3)_C$	$SU(2)_W$	$U(1)_Y$	
q	3	2	$\frac{1}{6}$	$\times 3$
u^c	$\bar{3}$	1	$-\frac{2}{3}$	$\times 3$
d^c	$\bar{3}$	1	$\frac{1}{3}$	$\times 3$
ℓ	1	2	$-\frac{1}{2}$	$\times 3$
e^c	1	1	1	$\times 3$
h	1	2	$\frac{1}{2}$	

Notation:

$$q_\alpha = q_{L\alpha} \qquad (u^c)_\alpha = \epsilon_{\alpha\beta} u_R^{\dagger\alpha} = \epsilon_{\alpha\beta}(u_R^{\dot{\alpha}})^\dagger$$

To embed this in a SUSY model, each fermion multiplet must be part of a chiral superfield. This means that we are introducing a complex scalar superpartner for each Weyl fermion.

Denote the superfield by a capital letter and the scalar partners with a tilde, *e.g.*

$$Q(x, \theta, \bar{\theta}) = \tilde{q}(x) + \sqrt{2}\theta^\alpha q_\alpha(x) + \cdots$$

It is traditional to name the scalar superpartners of fermions by adding an "s" in front of the name, *e.g.*

quark $\longleftrightarrow$ squark

lepton $\longleftrightarrow$ slepton

top $\longleftrightarrow$ stop

The $SU(3) \times SU(2) \times U(1)$ gauge fields of the Standard Model must embedded in gauge superfields. This means that we are introducing a Weyl fermion superpartner for each gauge boson.

It is traditional to name the fermion superpartners of gauge by adding an "ino" to the end of the name, *e.g.*

gauge boson $\longleftrightarrow$ gaugino

gluon $\longleftrightarrow$ gluino

W boson $\longleftrightarrow$ wino

photon $\longleftrightarrow$ photino

What about the Higgs doublet?

It has the same quantum numbers as a left-handed slepton* if we complex conjugate it. We could therefore think about identifying it with a linear combination of left-handed sleptons.

But this is a really bad idea, since the Higgs VEV would then break lepton number.

We therefore embed the Higgs in its own chiral superfield.

$$H = h + \sqrt{2}\theta^\alpha \tilde{h}_\alpha + \cdots$$

* the scalar partner of ℓ

Now let us write the most general allowed couplings of these fields in the SUSY limit.

The gauge boson (and gaugino) couplings are dictated by gauge invariance and SUSY.

The most general superpotential has the form

$$W = QHU^c + U^cD^cD^c + QLD^c + LLE^c + LH$$

• The last 4 terms violate baryon number, lepton number, or both.

• There is no Yukawa coupling of the Higgs to d^c or e^c.

Note that holomorphy of the superpotential does not allow us to write a term $QH^\dagger D^c$ or $LH^\dagger E^c$.

We could choose to embed the Higgs scalar doublet in a chiral superfield $\tilde{H}$ with hypercharge $-\frac{1}{2}$. Then we could write

$$W = Q\tilde{H}D^c + L\tilde{H}E^c + U^cD^cD^c + QLD^c + LLE^c$$

Now we don't have Yukawa couplings of the Higgs to u^c.

To have Yukawa couplings to all standard model fermions, we need two Higgs superfields, traditionally called H_u and H_d instead of H and $\tilde{H}$.

This is the (super)field content of the minimal supersymmetric standard model, the MSSM.

The allowed superpotential terms are now

$$\begin{aligned} W = {} & QH_uU^c + QH_dD^c + LH_dE^c + H_uH_d \\ & + U^cD^cD^c + QLD^c + LLE^c + LH_d \end{aligned}$$

The MSSM wallet card:

	$SU(3)_C$	$SU(2)_W$	$U(1)_Y$	
Q	3	2	$\frac{1}{6}$	$\times 3$
U^c	$\bar{3}$	1	$-\frac{2}{3}$	$\times 3$
D^c	$\bar{3}$	1	$\frac{1}{3}$	$\times 3$
L	1	2	$-\frac{1}{2}$	$\times 3$
E^c	1	1	1	$\times 3$
H_u	1	2	$\frac{1}{2}$	
H_d	1	2	$-\frac{1}{2}$	

We will assume for now that the terms that violate baryon/lepton number are absent.

(For reasons we will explain later, these are commonly referred to as "R-parity violating terms.")

As we will see, this has far-reaching implications, and we will revisit it later.

With this assumption, the superpotential is given by

$$\boxed{\begin{aligned} W = {} & (y_u)_{ij}Q^iH_uU^{cj} + (y_d)_{ij}Q^iH_dD^{cj} + (y_e)_{ij}L^iH_dE^{cj} \\ & + \mu H_uH_d \end{aligned}}$$

$i, j = 1, 2, 3 =$ generation index

How "minimal" is the MSSM?

Although the number of degrees of freedom is (slightly more than) double that of the Standard Model, the extra degrees of freedom are related by a symmetry.

We need an additional Higgs superfield, and we need to suppress the interactions that violate baryon/lepton number.

Soft SUSY Breaking

We take a phenomenological approach to breaking SUSY in the MSSM, and break SUSY softly.

- Gaugino mass terms

$$\boxed{\Delta\mathcal{L} = -M_1 \tilde{B}\tilde{B} - M_2 \tilde{W}_a \tilde{W}_a - M_3 \tilde{G}_A \tilde{G}_A + \text{h.c.}}$$

$\tilde{B} = \text{bino}$ $\quad a = 1, 2, 3 = SU(2)_W$ adjoint index

$\tilde{W}_a = \text{wino}$

$\tilde{G}_A = \text{gluino}$ $\quad A = 1, \ldots, 8 = SU(3)_C$ adjoint index

- Non-holomorphic scalar mass terms

$$\begin{aligned} \Delta V = {} & (m^2_{\tilde{q}})^i{}_j \tilde{q}^\dagger_i \tilde{q}^j + (m^2_{\tilde{u}^c})^i{}_j \tilde{u}^{c\dagger}_i \tilde{u}^{cj} + (m^2_{\tilde{d}^c})^i{}_j \tilde{d}^{c\dagger}_i \tilde{d}^{cj} \\ & + (m^2_{\tilde{\ell}})^i{}_j \tilde{\ell}^\dagger_i \tilde{\ell}^j + (m^2_{\tilde{e}^c})^i{}_j \tilde{e}^{c\dagger}_i \tilde{e}^{cj} \\ & + m^2_{H_u} H^\dagger_u H_u + m^2_{H_d} H^\dagger_d H_d \end{aligned}$$

Note I am using $H_{u,d}$ for scalar fields instead of $h_{u,d}$...

Important: scalar masses depend on flavor in general.

- Holomorphic cubic scalar couplings ("A terms")

$$\boxed{\Delta V = (A_u)_{ij} \tilde{q}^i H_u \tilde{u}^{cj} + (A_d)_{ij} \tilde{q}^i H_d \tilde{d}^{cj} (A_e)_{ij} \tilde{\ell}^i H_d \tilde{e}^{cj} + \text{h.c.}}$$

More flavor dependence...

- Holomorphic mass term ("$B\mu$ term")

$$\boxed{\Delta V = B H_u H_d + \text{h.c.}}$$

$$H_u H_d = \epsilon_{ab} H^a_u H^b_d$$

$a, b = 1, 2 = SU(2)_W$ fundamental index

$$\epsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

There are other more exotic soft terms that do not arise from higher components of superfield couplings. These are unlikely to arise from complete models of SUSY breaking, so we will neglect them.

In any case, we have plenty to deal with already...

The Higgs Potential

The main motivation for SUSY is to explain the naturalness of the light Higgs, so begin exploration of the MSSM phenomenology with the Higgs potential.

We will assume that the squarks and sleptons have positive mass-square terms and vanishing VEVs. This means we can consider the scalar potential with these scalar fields set to zero. The only scalar fields we need to consider are then H_u and H_d.

In the absence of SUSY breaking, the only contributions to the Higgs potential come from the superpotential term

$$W = \mu H_u H_d$$

and the D-term potential from the $SU(2)_W \times U(1)_Y$ gauge interactions.

$$\boxed{V_F = |\mu|^2 (H_u^\dagger H_u + H_d^\dagger H_d)}$$

$$V_D = \frac{g^2}{2} \sum_{a=1}^{3} \left(H_u^\dagger T_a H_u + H_d^\dagger T_a H_d \right)^2 + \frac{g'^2}{2} \left(\tfrac{1}{2} H_u^\dagger H_u - \tfrac{1}{2} H_d^\dagger H_d \right)^2$$

Simplify V_D using

$$(T_a)^i{}_j (T_a)^k{}_\ell = \tfrac{1}{2} \left(\delta^i{}_\ell \delta^k{}_j - \tfrac{1}{2} \delta^i{}_j \delta^k{}_\ell \right) \qquad T_a = \tfrac{1}{2} \tau_a$$

$$\boxed{V_D = \frac{g^2 + g'^2}{8} (H_u^\dagger H_u - H_d^\dagger H_d)^2 + \frac{g^2}{2} \left| H_u^\dagger H_d \right|^2}$$

This is a 2 Higgs doublet model with positive quadratic and quartic terms. We see that electroweak symmetry is unbroken in the SUSY limit.

Now add soft SUSY breaking terms involving H_u and H_d:

$$\boxed{V_{\text{soft}} = m_{H_u}^2 H_u^\dagger H_u + m_{H_d}^2 H_d^\dagger H_d + (B H_u H_d + \text{h.c.})}$$

The mass terms $m_{H_u}^2$ and $m_{H_d}^2$ can be negative, so electroweak symmetry can be broken.

Note we can rephase h_u, h_d to make B real and positive.

The full Higgs potential is now given by

$$\begin{aligned} V_{\text{Higgs}} = {} & m_{H_u,\text{eff}}^2 H_u^\dagger H_u + m_{H_d,\text{eff}}^2 H_d^\dagger H_d + (B H_u H_d + \text{h.c.}) \\ & + \frac{g^2 + g'^2}{8} (H_u^\dagger H_u - H_d^\dagger H_d)^2 + \frac{g^2}{2} \left| H_u^\dagger H_d \right|^2 \end{aligned}$$

$$m_{H_u,\text{eff}}^2 = m_{H_u}^2 + |\mu|^2 \qquad m_{H_d,\text{eff}}^2 = m_{H_d}^2 + |\mu|^2$$

Make $SU(2)_W \times U(1)_Y$ gauge transformation to choose VEV

$$\langle H_u \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_u \end{pmatrix}$$

Most general form of H_d VEV is then

$$\langle H_d \rangle = \frac{e^{i\theta}}{\sqrt{2}} \begin{pmatrix} v_d \cos\delta \\ v_d e^{i\eta} \sin\delta \end{pmatrix}$$

Minimizing the potential with respect to the angles θ, δ, and η can be shown to require that they vanish.

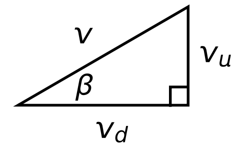
$$\langle H_d \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v_d \\ 0 \end{pmatrix}$$

$\Rightarrow U(1)_{\mathrm{EM}}$ unbroken.

$$v_u^2 + v_d^2 = v^2 \qquad\qquad v = 246\ \mathrm{GeV}$$

It is conventional to define

$$\boxed{\tan\beta = \frac{v_u}{v_d}}$$



The minimization conditions can be written

$$B = \frac{m^2_{H_u,\mathrm{eff}} + m^2_{H_d,\mathrm{eff}}}{2\sin(2\beta)}$$

$$\left| \frac{m^2_{H_u,\mathrm{eff}} - m^2_{H_d,\mathrm{eff}}}{\cos(2\beta)} \right| = m^2_{H_u,\mathrm{eff}} + m^2_{H_d,\mathrm{eff}} + m_Z^2$$

We have written the quartic coupling in terms of m_Z:

$$g^2 + g'^2 = \frac{2m_Z^2}{v^2}$$

The important point is that the mass terms must be of order m_Z if we want to avoid large cancelations.

This arises because SUSY fixes the quartic to be proportional to $g^2 + g'^2$.

$$v^2 \sim -\frac{m_H^2}{g^2 + g'^2}$$

$$\Rightarrow \quad m_H^2 \sim (g^2 + g'^2) v^2 \sim m_Z^2$$

$$H_u = \begin{pmatrix} H_u^+ \\ \frac{1}{\sqrt{2}}(v_u + H_u^0 + iA_u^0) \end{pmatrix} \qquad H_d = \begin{pmatrix} \frac{1}{\sqrt{2}}(v_d + H_d^0 + iA_d^0) \\ H_d^- \end{pmatrix}$$

$H_u^0, H_d^0 =$ CP even neutral scalars

$A_u^0, A_d^0 =$ CP odd neutral scalars

$H_u^\pm, H_d^\pm =$ charged scalars

Mass eigenstates:

3 linear combinations of

$$A_u^0, A_d^0 \qquad\qquad H_u^\pm, H_d^\pm$$

are Nambu-Goldstone bosons that become the lonigutudinal polarizations of the Z, $W^\pm$.

Physical scalar Higgs mass eigenstates:

$h^0, H^0 = \text{CP even} \qquad m_{h^0} < m_{H^0}$

$A^0 = \text{CP even}$

$H^\pm = \text{charged}$

Mass eigenvalues:

$$\boxed{\begin{aligned} m_{A^0}^2 &= m_{H_u,\text{eff}}^2 + m_{H_d,\text{eff}}^2 \\ m_{H^\pm}^2 &= m_{A^0}^2 + m_W^2 \\ m_{h^0,H^0}^2 &= \frac{1}{2}\Big[m_{A^0}^2 + m_Z^2 \\ &\qquad \mp \sqrt{(m_{A^0}^2 - m_Z^2)^2 + 4 m_Z^2 m_{A^0}^2 \sin^2(2\beta)}\Big] \end{aligned}}$$

Identify h^0 with the Higgs boson observed at the LHC
$\Rightarrow m_{h^0} = 125$ GeV.

m_{h^0} is a monotonically increasing function of m_{A^0}. Taking the $m_{A^0} \to \infty$ limit, we obtain

$$m_{h^0} < m_Z \cos(2\beta) < 91 \text{ GeV}$$

This was already ruled out in the 1990s by LEP.

The root of the problem is the fact that SUSY fixes the Higgs quartic couplings to be proportional to $g^2 + g'^2$:

$$m_h^2 \sim (g^2 + g'^2) v^2 \sim m_Z^2$$

We need some additional contribution to the Higgs quartic.

In fact, such contributions exist within the MSSM itself: loops!

Consider the 1-loop contributions from the particles with the largest coupling to the Higgs: the top and stops.

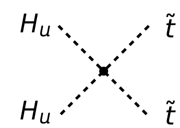
$$\text{top:} \quad q^3 = \begin{pmatrix} t_L \\ b_L \end{pmatrix} \qquad u_\alpha^{c3} = \epsilon_{\alpha\beta} (t_R^{\dot\alpha})^\dagger$$

$$\text{stops:} \quad \tilde{q}^3 = \begin{pmatrix} \tilde{t}_L \\ \tilde{b}_L \end{pmatrix} \qquad \tilde{u}^{c3} = \tilde{t}_R^\dagger$$

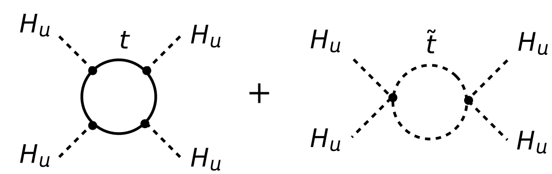
After electroweak symmetry breaking, $\tilde{t}_L$ and $\tilde{t}_R$ have the same quantum numbers, and therefore mix.

Simplify the discussion by neglecting stop mixing and assuming that all stops have the same mass.

$$\Delta W = y_t Q^3 H_u U^{c3} \quad \Rightarrow \quad \Delta V_F = y_t^2 H_u^\dagger H_u (\tilde{q}^{3\dagger}\tilde{q}^3 + \tilde{u}^{c3\dagger}\tilde{u}^{c3})$$

$\Rightarrow$ 

$\sim y_t^2$



$$\Rightarrow \Delta\lambda_H \sim \frac{3 y_t^4}{16\pi^2} \ln \frac{m_{\tilde{t}}^2}{m_t^2}$$

Note: must go to zero when $m_{\tilde{t}} = m_t$.

$$\Delta m_{h^0}^2 \sim \Delta\lambda_H v^2$$

Formula including all factors:

$$\Delta m_{h^0}^2 = \frac{3y_t^2}{8\pi^2} \ln \frac{m_{\tilde{t}}^2}{m_t^2} \times v^2 \qquad \text{Assumes } h^0 \simeq h_u^0 \text{ (optimistic)}$$

To get $m_{h^0} = 125$ GeV, need $m_{\tilde{t}} \simeq 880$ GeV.

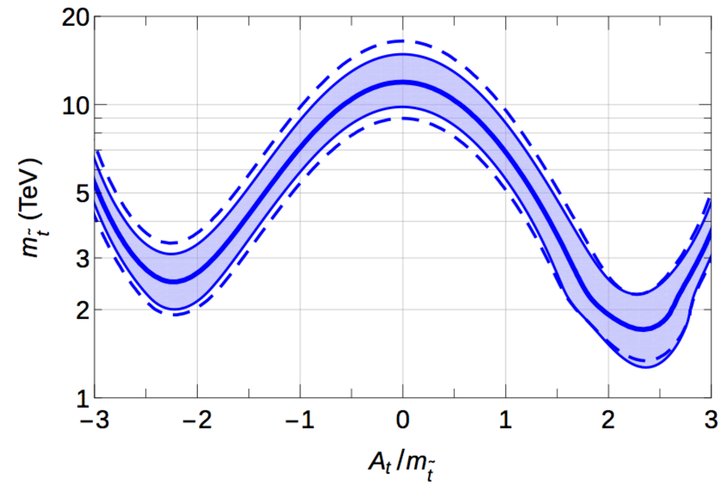
But higher loop corrections are very important for this estimate. For example, the QCD correction to the top mass gives

$$m_t = \frac{y_t v}{\sqrt{2}} \left[1 + \frac{4}{3} \frac{g_s^2}{4\pi^2} + \cdots \right]$$

$$\Rightarrow \frac{y_t v}{\sqrt{2}} = 163 \text{ GeV} \qquad \text{instead of } 173 \text{ GeV}$$

Including this correction, we find that in order to reproduce the Higgs mass we need $m_{\tilde{t}} \simeq 1.3$ TeV.

Full result, including stop mixing:



$m_{\tilde{t}} \sim 10$ TeV (no stop mixing)

Because the stop mass is heavy, we have to worry about contributions to the Higgs mass parameter. Putting in all the factors, this is

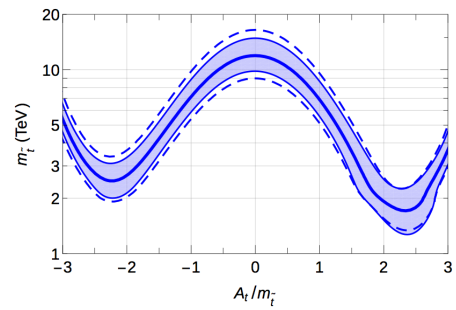
$$\Delta m_{H_u}^2 = -\frac{3y_t^2}{4\pi^2}(m_{\tilde{t}}^2 - m_t^2) \ln \frac{\Lambda}{m_{\tilde{t}}}$$

Assuming $\ln \frac{\Lambda^2}{m_t^2} \sim 1$ (opmistic!)

$$\frac{\Delta m_H^2}{m_H^2} \sim 10^3$$

$\Rightarrow$ obtaining the observed Higgs mass requires a fine-tuning of on part in 10^{-3}.

Large stop mixing helps a bit:



$m_{\tilde{t}} \simeq 2$ TeV for $A_t \simeq \pm 5$ TeV

$$\Delta m_H^2 = -\frac{3y_t^2}{8\pi^2}(2m_{\tilde{t}}^2 + A_t^2) \ln \frac{\Lambda}{m_{\tilde{t}}} \sim 10^2 \times m_H^2$$

Problem is worse for high-scale SUSY breaking ($\Lambda \gg$ TeV)

Note that eliminating the quadratic sensitivity of the Higgs mass to heavy masses was the motivation for SUSY!

This motivates extensions of the MSSM that can generate a larger Higgs quartic.

Example: next-to-minimal supersymmetric standard model

Add to MSSM gauge singlet chiral superfield S

$$\Delta W = \lambda S H_u H_d + \frac{\kappa}{3} S^3$$

$$\frac{\partial W}{\partial S} = \lambda H_u H_d + \kappa S^2$$

$$\Delta V_F = \left| \lambda H_u H_d + \kappa S^2 \right|^2 = \lambda^2 |H_u H_d|^2 + \cdots$$

The SUSY Spectrum

What is the expected spectrum of superpartners?

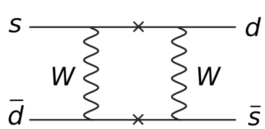
A strong hint: flavor dependence of general soft breaking terms:

$$V_{\text{soft}} = (m^2_{\tilde{q}})^i{}_j \, \tilde{q}^\dagger_i \tilde{q}^j + (m^2_{\tilde{u}^c})^i{}_j \, \tilde{u}^{c\dagger}_i \tilde{u}^{cj} + (m^2_{\tilde{d}^c})^i{}_j \, \tilde{d}^{c\dagger}_i \tilde{d}^{cj} + \cdots$$

There is no *a priori* reason that these matrices should be diagonal in the same basis as quark mass matrix $\Rightarrow$ new sources of flavor mixing.

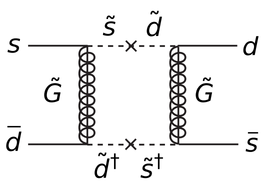
Example: K^0–$\bar{K}^0$ mixing

SM:



$$\Delta\mathcal{L}_{\text{eff}} \sim \frac{g^4}{16\pi^2} \frac{m_c^2}{m_W^4} (\bar{s}d)(\bar{d}s)$$

MSSM:



$$\Delta\mathcal{L}_{\text{eff}} \sim \frac{g_s^4}{16\pi^2} \frac{\left(m^2_{\tilde{d}\tilde{s}}\right)^2}{M^6_{\text{SUSY}}} (\bar{s}d)(\bar{d}s)$$

$$\Rightarrow \frac{m^2_{\tilde{d}\tilde{s}}}{M^2_{\text{SUSY}}} \lesssim 10^{-3} \left(\frac{M}{\text{TeV}} \right)$$

Many other entries have constraints at the level of 10^{-2} from B^0–$\bar{B}^0$ and D^0–$\bar{D}^0$ mixing.

Suggests a simple *Ansatz*:

$$(m^2_{\tilde{q}})^i{}_j \simeq m^2_{\tilde{q}} \delta^i{}_j \qquad (m^2_{\tilde{u}^c})^i{}_j \simeq m^2_{\tilde{u}^c} \delta^i{}_j \quad \cdots$$

This form is invariant under the field rotations that diagonalize the quark and lepton mass matrices $\Rightarrow$ no new flavor violation.

A terms:

$$(A_u)_{ij} = A_u (y_u)_{ij} \qquad (A_d)_{ij} = A_d (y_d)_{ij} \quad \cdots$$

$\Rightarrow$ A terms also diagonal in flavor basis.

Reduced soft SUSY breaking parameter space:

scalar masses:	$m^2_{\tilde{q}},\ m^2_{\tilde{u}^c},\ m^2_{\tilde{d}^c},\ m^2_{\tilde{\ell}},\ m^2_{\tilde{e}^c},$ $m^2_{H_u},\ m^2_{H_d}$
gaugino masses:	$M_1,\ M_2,\ M_3$
A terms:	$A_u,\ A_d,\ A_e$
and don't forget:	$\mu,\ B$

This is still a lot of parameter space.

Look at some simple theory ideas...

High Scale SUSY Breaking

Assume SUSY broken at high scales in a hidden sector by $\langle F_X \rangle \neq 0$:

$$\Delta\mathcal{L} \sim \int d^4\theta \left(\frac{X^\dagger X}{M^2} + \frac{X}{M} + \text{h.c.} \right) Q^\dagger Q \qquad \text{scalar masses, } A \text{ terms}$$

$$+ \int d^2\theta \frac{X}{M} W^\alpha W_\alpha + \text{h.c.} \qquad \text{gaugino masses}$$

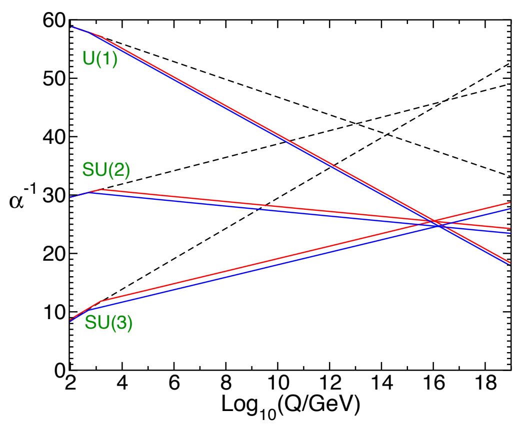
$$+ \int d^4\theta \left(\frac{X^\dagger X}{M^2} + \frac{X^\dagger}{M} \right) H_u H_d + \text{h.c.} \qquad \mu,\ B \text{ terms}$$

All required SUSY breaking generated with size $\sim F_X/M$, including μ and B terms!

Very compelling...

Grand Unification

Further support for extrapolating the MSSM to high scales comes from grand unification:



$M_{\text{GUT}} \simeq 2 \times 10^{16}$ GeV

Unification suggests that at the GUT scale $M_1 = M_2 = M_3$.

RG equations: $\mu \dfrac{d}{d\mu} M_a = \dfrac{b_a}{8\pi^2} g_a^2 M_a \qquad b_a = (\tfrac{33}{5},\ 1,\ -3)$

$$\Rightarrow \frac{M_1}{g_1} = \frac{M_2}{g_2} = \frac{M_3}{g_3}$$

At TeV scale: $M_1 : M_3 : M_3 \simeq 5 : 2 : 1$

Scalar masses:

$$\mu \frac{d}{d\mu} m^2_{\tilde{q}} = \frac{1}{8\pi^2} \left[-\tfrac{16}{3} g_3^2 M_3^2 + \cdots \right]$$

$$\Rightarrow m_{\tilde{q}} \sim M_3 \text{ at TeV scale}$$

$$\mu \frac{d}{d\mu} m^2_{H_u} = \frac{1}{8\pi^2} \left[y_t^2 (m^2_{\tilde{q}_3} + m^2_{\tilde{t}^c}) + \cdots \right]$$

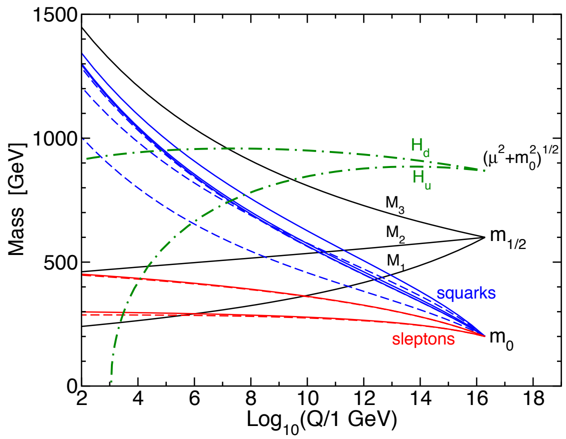
$$\Rightarrow m^2_{H_u} \sim m^2_{\tilde{t}} \text{ at TeV scale}$$

Uh oh...

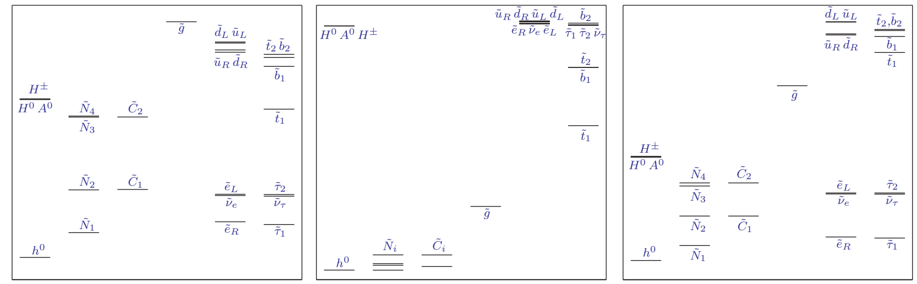
"MSUGRA" boundary conditions at $M_{\rm GUT}$:

$$m^2_{\tilde q} = m^2_{\tilde u^c} = \cdots = m^2_{H_u} = m^2_{H_d} = m^2_0$$

$$M_1 = M_2 = M_3 = m_{1/2}$$



(Too) many other possibilities...



R Parity

Recall that we had to suppress the baryon and lepton number violating superpotential terms

$$\Delta W = U^c D^c D^c + QLD^c + LLE^c + LH_d$$

There is a striking consequence of forbidding these terms: there is a Z_2 symmetry under which

$$\underbrace{\begin{array}{r} q,\ u^c,\ d^c,\ \ell,\ e^c = \text{even} \\ B_\mu,\ W_\mu,\ G_\mu = \text{even} \\ H_u,\ H_d = \text{even} \end{array}}_{\text{SM particles + Higgs}} \qquad \underbrace{\begin{array}{r} \tilde q,\ \tilde u^c,\ \tilde d^c,\ \tilde \ell,\ \tilde e^c = \text{odd} \\ \tilde B_\mu,\ \tilde W_\mu,\ \tilde G_\mu = \text{odd} \\ \tilde H_u,\ \tilde H_d = \text{odd} \end{array}}_{\text{"supersymmetric particles"}}$$

This "R-parity" symmetry has the important consequence that the lightest supersymmetric particle (LSP) is absolutely stable.

If this particle is electrically neutral, it is a candidate for the dark matter particle.

Well-motivated possibilities:

- neutralino = mixture of $\tilde B$, $\tilde W_3$, $\tilde H^0_u$, $\tilde H^0_d$.
- gravitino

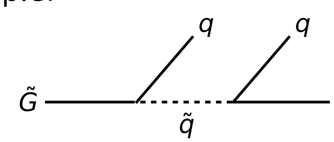
$$m_{3/2} \sim \frac{F}{M_{\rm Pl}} \ll \frac{F}{M} \sim \text{TeV} \qquad \text{for } M \ll M_{\rm Pl}$$

Generic SUSY signal (R parity conserved):

- Supersymmetric particles produced in pairs
- Each supersymmetric particle decays to ordinary particles plus LSP
- LSP leads to missing transverse energy

$$\vec{p}_{T,\text{miss}} = -\sum_i \vec{p}_{T,i} \qquad E_{T,\text{miss}} = \left|\vec{p}_{T,\text{miss}}\right|$$

Example:



$\Rightarrow$ 4 jets + missing E_T

Many possibilities $\Rightarrow$ look everywhere.

Search reach determined by rate: need 10–100 events to see signal, depending on backgrounds.

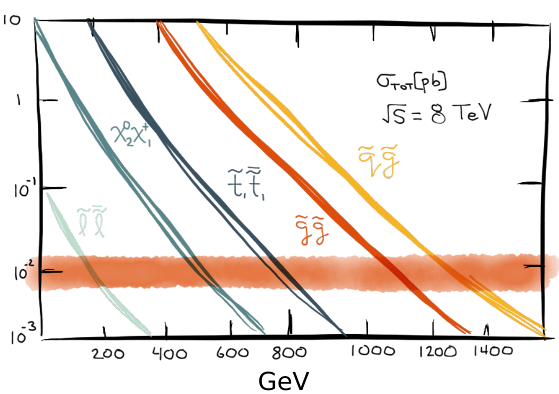
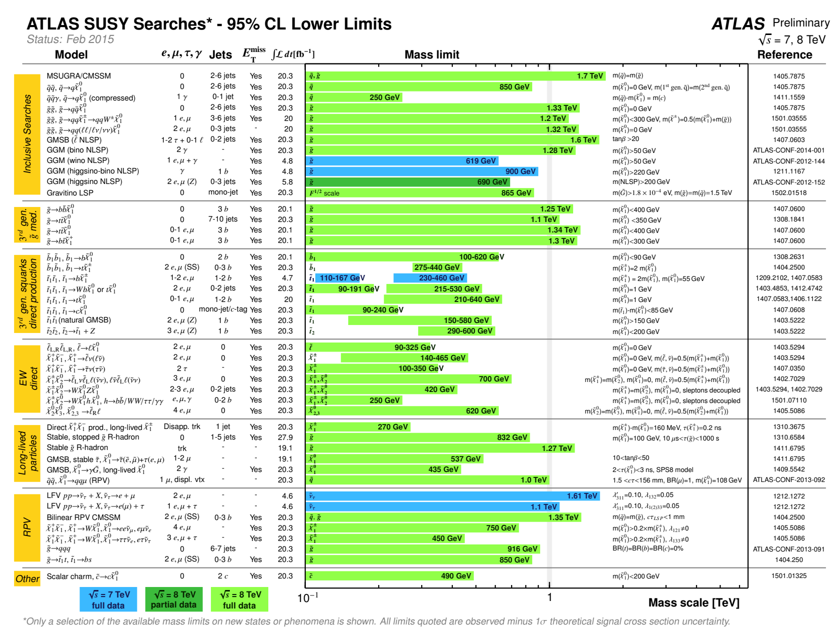


Figure from N. Craig, `arXiv:1309.0528`
(excellent overview of LHC8 searches)



In the absence of any signal, how natural is SUSY?

Supersymmetric particles most directly connected to naturalness:

Higgsino mass: $m^2_{H_{u,d},\mathrm{eff}} = m^2_{H_{u,d}} + |\mu|^2$

μ = Higgsino mass

$$\text{tuning} \sim 20\% \left(\frac{\mu}{200\ \text{GeV}} \right)^{-2}$$

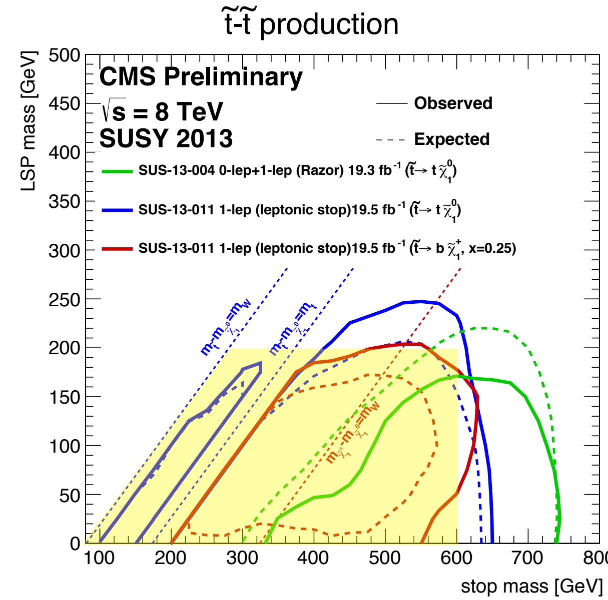
Top mass:

$$\text{tuning} \sim 20\% \left(\frac{m_{\tilde{t}}}{600\ \text{GeV}} \right)^{-2}$$

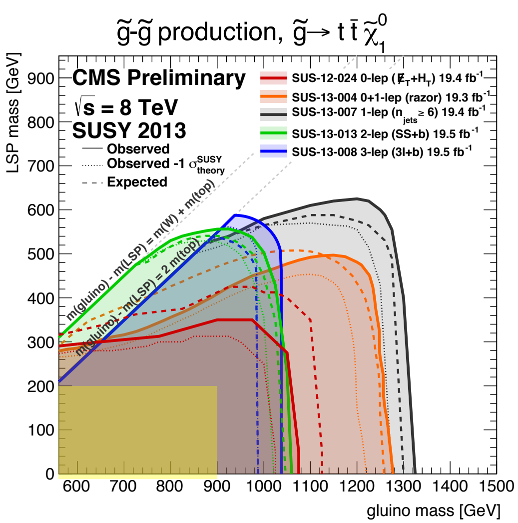
Gluino mass: $\mu \frac{d}{d\mu} m_{\tilde{t}} = -\frac{2}{3\pi^2} g_3^2 M_3^2 + \cdots$

$$\text{tuning} \sim 20\% \left(\frac{m_{\tilde{G}}}{900\ \text{GeV}} \right)^{-2}$$

Stop Searches

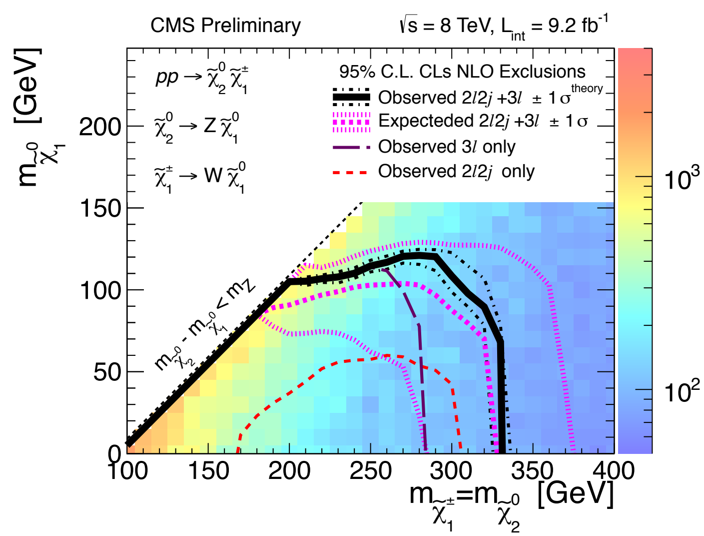


Gluino Searches

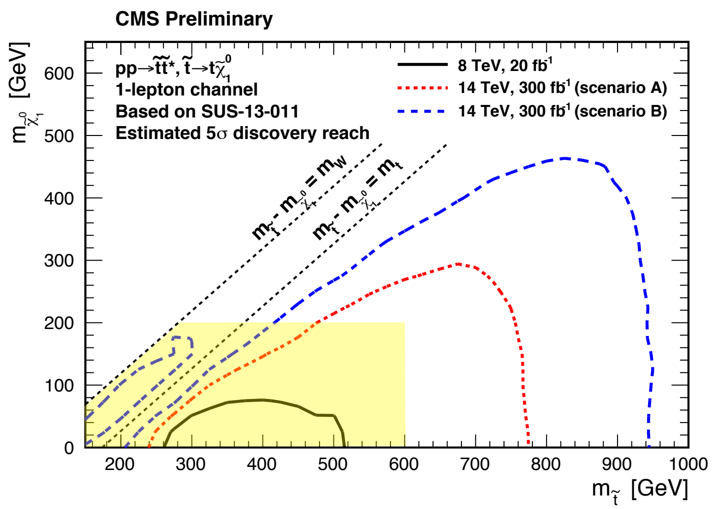


$M_3 > 1300$ GeV $\Rightarrow$ tuning $\lesssim 10\%$

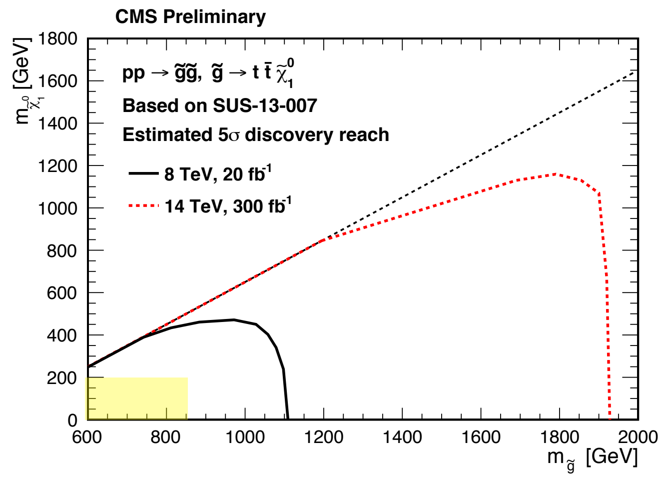
Electroweakino Searches



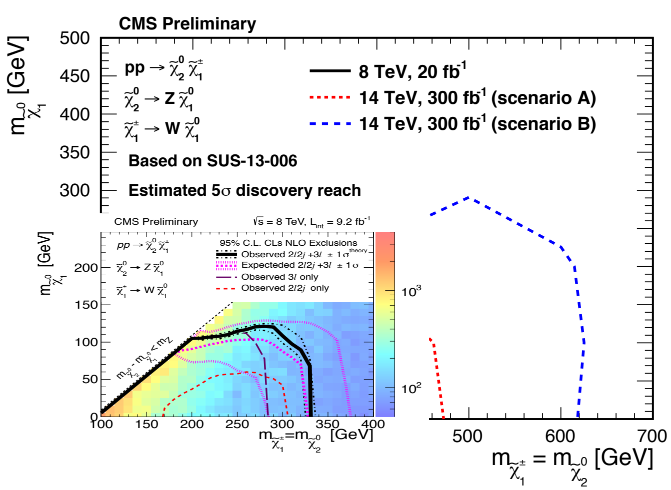
Stop Searches at 14 TeV



Gluino Searches at 14 TeV



Electroweakino Searches at 14 TeV



Final Thoughts

G. Altarelli: "The train of SUSY is late." (1990's)

SUSY needs either tuning or non-minimal structure.

But: only *one* tuning.

$m_{H_u}^2 \ll m_{\tilde{t}}^2 \Rightarrow$ recover SM

Occam's Razor: "Entities must not be multiplied beyond necessity."

MSSM with one percent-level tuning is arguably the most compelling framework for particle physics.

Implications:

- The most likely place to find SUSY is just around the corner.
- Keep looking for "standard" SUSY signals.

Thanks!

Backup

Classical Fermion Fields

Classical mechanics is a limiting case ($\hbar \to 0$) of quantum mechanics. So why do we define a quantum theory by "quantizing" a classical action?

QFT: Heisenberg fields obey Lorentz covariant equations of motion:

$$(\Box + m^2)\underbrace{\hat{\phi}(x)}_{\text{hats on operators for emphasis}} = \frac{\lambda}{3!}\hat{\phi}^3(x) \qquad (\phi^4 \text{ theory})$$

$\Rightarrow$ time dependence is Lorentz covariant.

Unitarity of time evolution guaranteed by Heisenberg equations of motion:

$$\frac{\partial}{\partial t}\hat{\phi}(t,\vec{x}) = i[\hat{H}, \hat{\phi}(t,\vec{x})] \quad \Rightarrow \quad \hat{\phi}(t,\vec{x}) = \underbrace{e^{i\hat{H}t}\hat{\phi}(t=0,\vec{x})e^{-i\hat{H}t}}_{\text{unitary transformation}}$$

Quantization: construct $\hat{H}$ so that Heisenberg equation of motion is equivalent to classical equation of motion:

$$\frac{\partial}{\partial t}\hat{\phi}(t,\vec{x}) = i[\hat{H}, \hat{\phi}(t,\vec{x})] \quad \Leftrightarrow \quad \left.\frac{\delta S}{\delta\phi(x)}\right|_{\phi\to\hat{\phi}} = 0$$

From this point of view, classical action is the “data” that defines the quantum theory.

For bosonic fields, the classical action does more: it describes the classical limit of the theory. But for fermions we have to take the more formal viewpoint described above.

Fermion fields are quantized by imposing equal-time *anti*commutation relations:

Dirac spinor field:

$$\{\hat{\Psi}^a(t,\vec{x}), \hat{\Psi}^\dagger_b(t,\vec{y}\} = \hbar\delta^a{}_b\delta^3(\vec{x}-\vec{y})$$

$$\{\hat{\Psi}^a(t,\vec{x}), \hat{\Psi}^b(t,\vec{y}\} = \{\hat{\Psi}^\dagger_a(t,\vec{x}), \hat{\Psi}^\dagger_b(t,\vec{y}\} = 0$$

$$a = 1,\ldots,4$$
$$= \text{Dirac spinor index}$$

The classical fields can be thought of as the $\hbar \to 0$ limit of the quantum fields:

$$\hbar \to 0: \ \hat{\Psi}^a(x) \to \Psi^a(x)$$

$$\Rightarrow \left.\begin{aligned} &\{\Psi^a(x), \Psi^\dagger_b(y)\} = 0 \\ &\{\Psi^a(x), \Psi^b(y)\} = \{\Psi^\dagger_a(x), \Psi^\dagger_b(y)\} = 0 \end{aligned}\right\} \text{ for all } x, y$$

Alternately: in path integral quantization, fermion path integral is over anticommuting fields.

Gauge Coupling Renormalization

We can learn a lot about the renormalization of the gauge coupling by promoting it to a superfield.

Discuss using SUSY QED, the theory of two chiral superfields $\Phi_\pm$ with charges $q = \pm 1$.

Use normalization with $1/g^2$ in front of gauge kinetic term:

$$\mathcal{L} = \int d^2\theta \, \tfrac{1}{2} S W^\alpha W_\alpha + \text{h.c.}$$
$$+ \int d^4\theta \, Z \left[\Phi_+^\dagger e^{2V}\Phi_+ + \Phi_-^\dagger e^{-2V}\Phi_-\right]$$

$$S = \frac{1}{2g^2} - \frac{i\Theta}{16\pi^2} + \cdots$$
$$= \text{chiral multiplet}$$

RG equation for gauge coupling:

$$\mu\frac{dg^2}{d\mu}g^2 = b_1 g^4 + b_2 g^6 + O(g^8)$$

$$\Rightarrow \mu\frac{d}{d\mu}\left(\frac{1}{g^2}\right) = -b_1 - b_2 g^2 + O(g^4)$$

$$\mu\frac{d}{d\mu}S = -\tfrac{1}{2}b_1 - \tfrac{1}{2}b_2 \underbrace{S^{-1}}_{\text{holomorphic!}} + O(S^{-2})$$

$$\text{Re}\, S^{-1} = \frac{g^2}{1 + (g^2\Theta/16\pi^2)^2}$$

This depends on Θ, which is impossible (at least in perturbation theory).

We conclude that

$$\mu \frac{d}{d\mu} S = -\tfrac{1}{2} b_1 \qquad \text{exactly!}$$