PROBLEM LIST 4. ROBUST TRANSITIVITY.

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(1) Consider an homeomorphism f on a compact metric space X. Prove that:

(a) f is transitive (i.e. has a dense forward orbit) iff for any non-empty open sets U, V, there is $n \geq 1$ such that $f^n(U) \cap V \neq \emptyset$.

(b) Assuming that X has no isolated point, f is transitive if and only if for any non-empty open sets U, V, there exists $n \in \mathbb{Z}$ such that $f^n(U) \cap V \neq \emptyset$.

(2) For an Anosov C^3-diffeomorphism of $\mathbb{T}^2$ which preserves the volume, the stable foliation is $C^{1+\alpha}$ for any $\alpha \in (0,1)$.

(3) Consider $f \in \mathrm{Diff}^1(M)$ and a partially hyperbolic set K with a splitting $E^c \oplus E^u$. Assume that K is contained in a submanifold Σ tangent to E^c which is locally invariant (there exists a neighborhood U of K such that $f(U \cap \Sigma) \subset \Sigma$).

Prove that for any $x \in K$ the *global* strong unstable manifold $W^{uu}(x)$ intersects K only at x.

(4) Consider $f \in \mathrm{Diff}^1(M)$ with a global partially hyperbolic splitting $TM = E^c \oplus E^u$. The strong stable foliation is said *dynamically minimal* if and only if the only invariant compact set which is a union of strong unstable leaves (and non-empty) is M.

(a) Show that the strong stable foliation is dynamically minimal if and only if every disk D in a strong stable leaf verifies that $\overline{\bigcup_{n \leq 0} f^n(D)} = M$.

(b) Show that if $f \in \mathsf{PH}$ has a dynamically minimal strong stable foliation, then it is transitive.

(c) Give an example of $f \in \mathsf{PH}$ whose strong stable foliation is dynamically minimal but not minimal. (*Hint*: Consider a product example.)