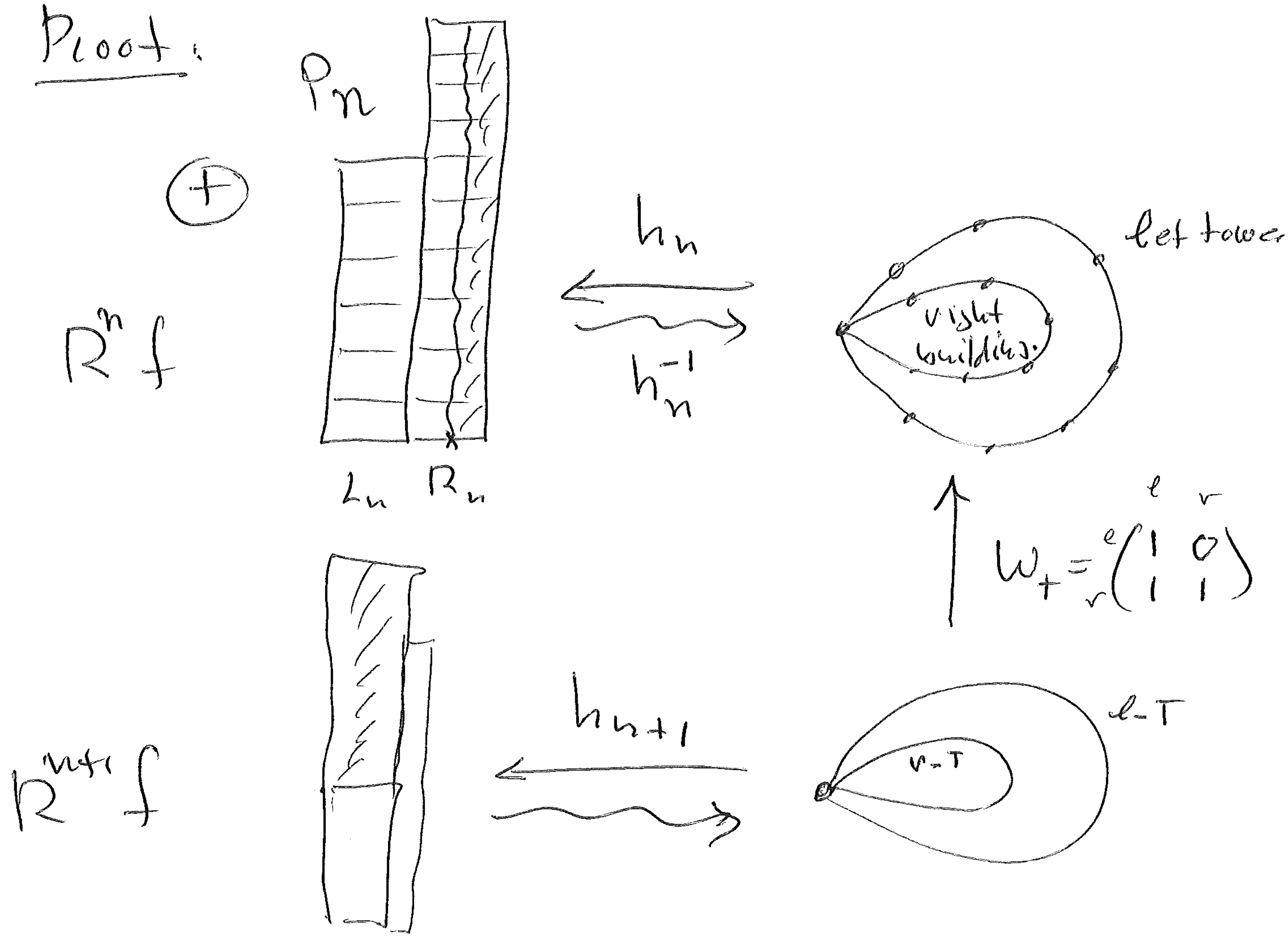
Thm: $S^1 \circlearrowleft f$ ∞-renormalizable circle map.

f has only 1 invariant probability measure ($\dim M = 1$).

Proof:



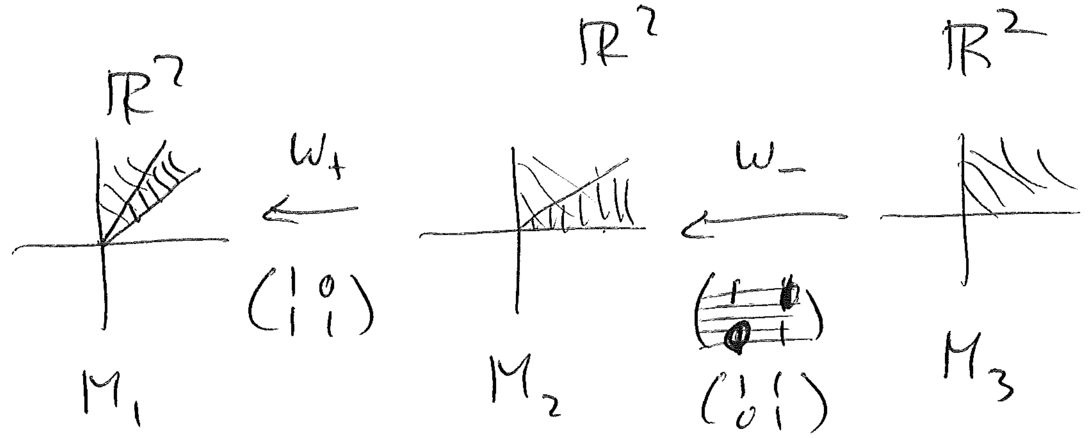
$$w_- = \begin{pmatrix}1&1\\0&1\end{pmatrix}$$

<u>Lemma:</u> $h_n \to h$ Continuous map

$$\begin{tikzcd} S^1 \arrow[loop below, "f"] & X = \varprojlim_{w_n} X_n \arrow[l, "h"'] \end{tikzcd}$$

h is 1-1 except on ~~countab~~ the orbit of c. (innocent for measure theory)

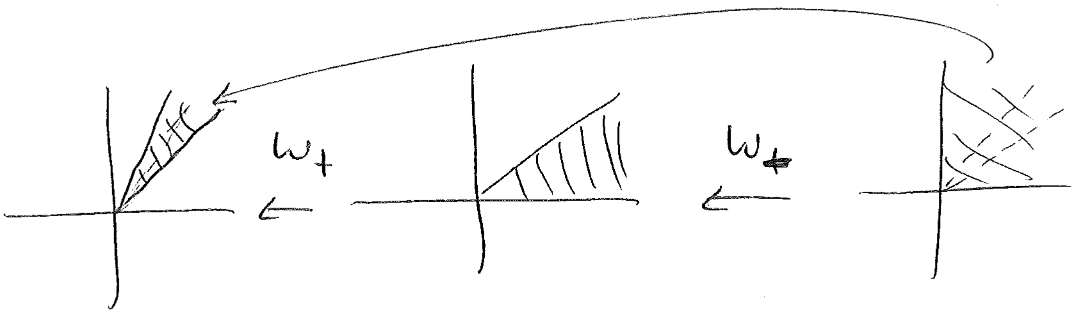
<u>Proof:</u> exercise $\square$.

M: 

$$w_+ w_- = \begin{pmatrix}1&0\\1&2\end{pmatrix}$$

$$w_- w_+ = \begin{pmatrix}2&1\\1&1\end{pmatrix}$$

$$M = \bigcap_{n \geq 1} w_1 w_2 \cdots w_n (\mathbb{R}^2_+)$$



$w_+ w_-$ (and $w_- w_+$) contracts the cone. In the limit there will be only a line: $\dim M = 1$.*

X is uniquely ergodic

Hence $f : S^1 \circlearrowleft$ has a unique probability measure.

* Use PF-Theorem.

Example $f = R_\rho$: $\nu = \lambda$. (Lebesgue)

$$h_*(\nu) = \lambda.$$

There are cases when

$$\nu \ll \lambda$$

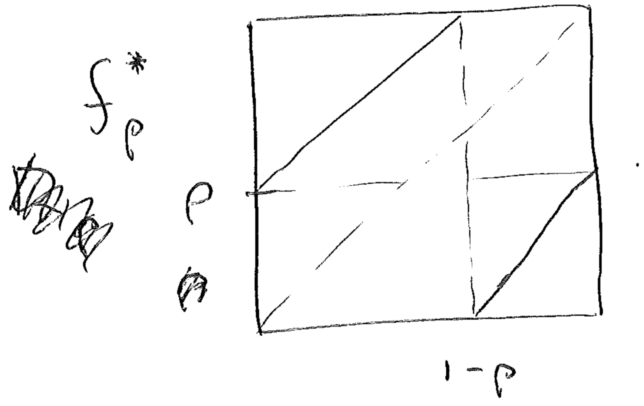
and

$$\nu \perp \lambda \quad \text{(Arnold Example)}$$

This depends on the "quality" of h.

Renormalization and Hyperbolicity

Let $A = \{ f^*_\rho \}_{\rho \in S^1}$ rotations.

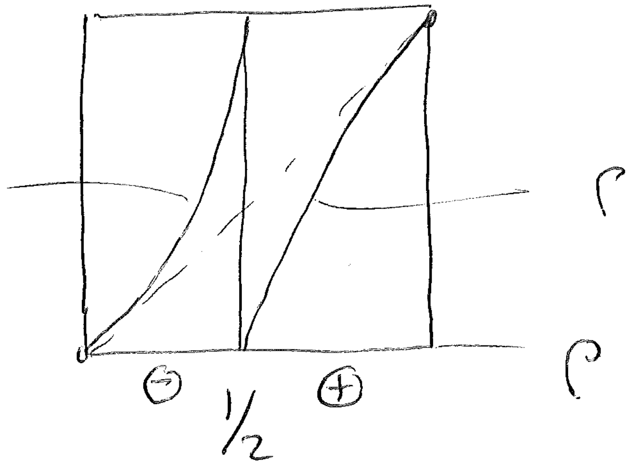


$v = \rho$

$c = 1-\rho$

Then $R : A \to A$

$$\rho' = \frac{\rho}{1-\rho}$$



$$\rho' = \frac{2\rho - 1}{\rho}$$

Thm $f \in C^3$ ∞-renormalizable circle diffeo.

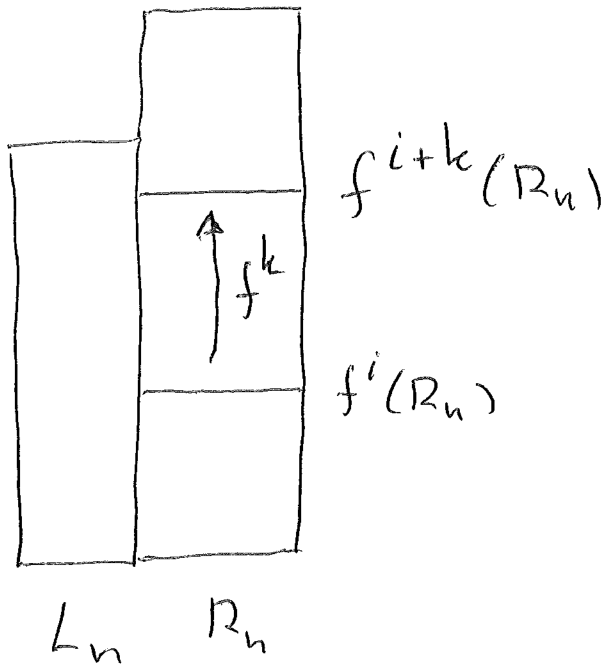
then $R^n f \xrightarrow{C^1} A$

In particular, if $a(f) = a(g)$

then $|R^n f - R^n g| \longrightarrow 0$

Moreover,

$R^n f$



$$f^k : f^i(R_n) \longrightarrow f^{i+k}(R_n)$$

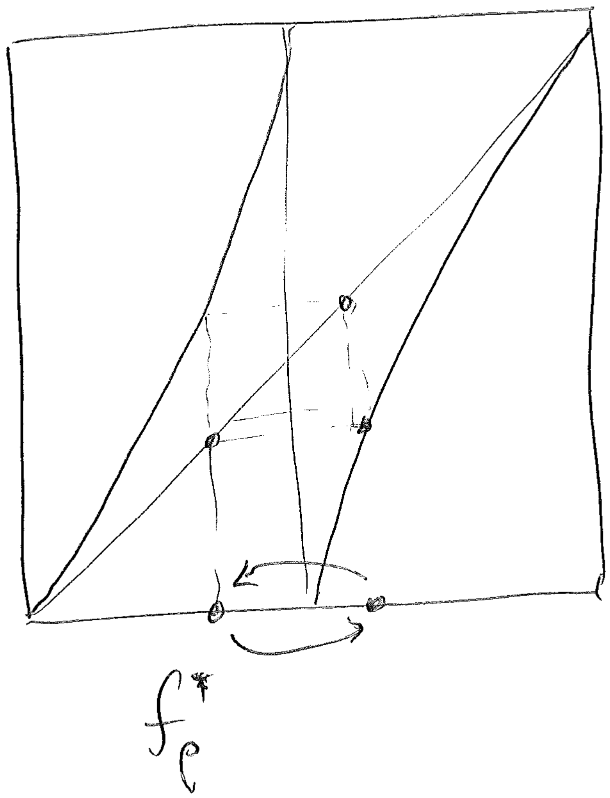
converges uniformly (in n and i,j) to affine maps.

<u>Example</u> $\underline{a} = (a_n) \quad a_n = 1$

$$\rho = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 \ddots}}} = \frac{\sqrt{5}-1}{2}$$

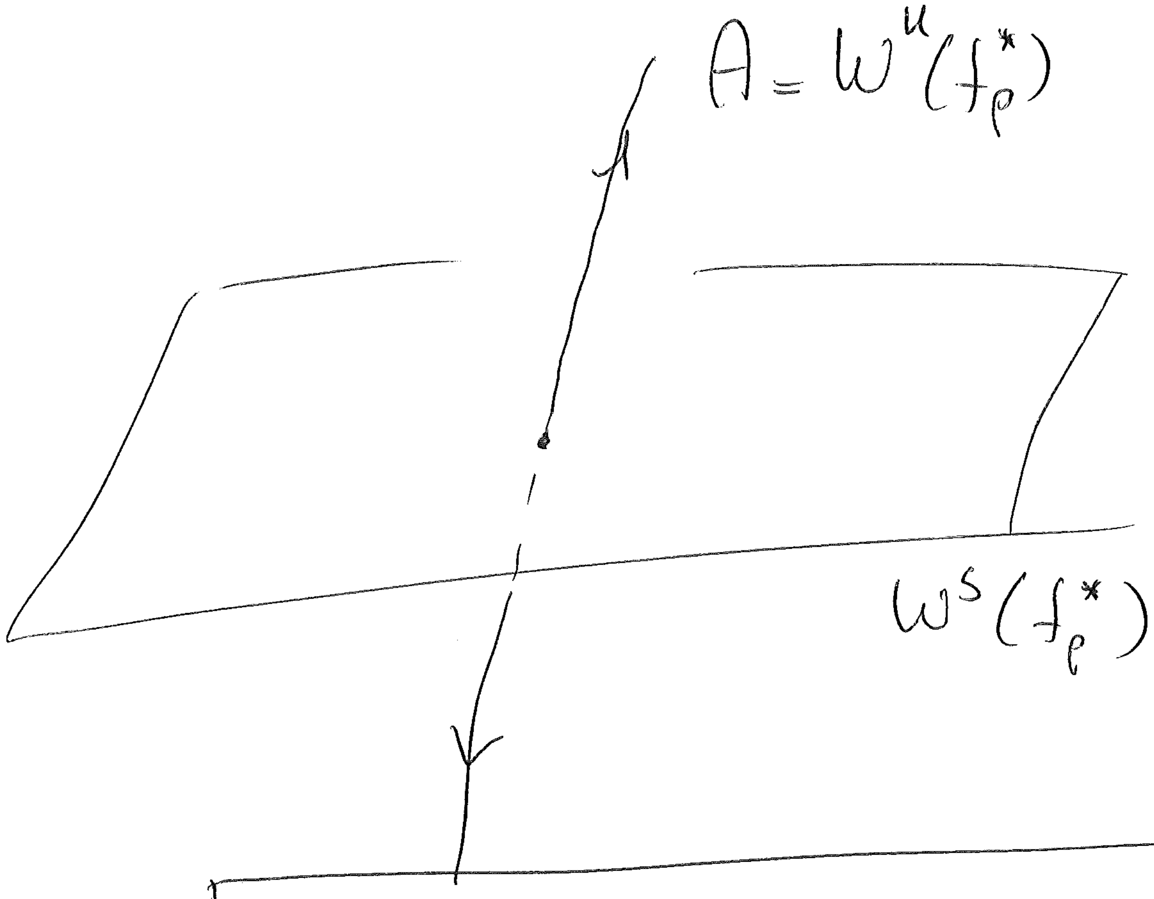
Let f^*_ρ be the corresponding rotation of $\rho = \frac{\sqrt{5}-1}{2}$

Then $R^2 f^*_\rho = f^*_\rho$: f^*_ρ is a periodic point of period 2 of renormalization



RIA

In the space of C^3 circle diffeos we see a hyperbolic picture around f_ρ^*



and
$$\boxed{\begin{aligned} W^s(f_\rho^*) &= \left\{ f \;\middle|\; a_n(f) = 1 \right\}. \\ &= \left\{ \text{top. class} \right\} \end{aligned}}$$

In particular, the topological class of maps with $\rho_f = \rho$ is a manifold of codim $= 1$.

Hence, its easy to deform maps and turn the recurrent orbits into period orbits (Closing Lemma).

Closing-Lemma.

This shows the relation between R and Bifurcation

Geometry

Thm (Herman)

$f \in C^3$ circle diffeo.

$\rho_f = [a_n]$

$h: S^1 \longrightarrow S^1$ conjugation

betwee f and $f^*_{\rho_f}$ (rot)

__If__ a_n does not go to infinity too fast

__then__ h is $C^{1+\alpha}$

(h is cont. diff. ~~and~~ (C^1)

and

Dh is α-Hölder

$(|Dh(x) - Dh(y)| \leq K|x-y|^\alpha)$

__Remark__: If a conjugation is differentiable the on small scale it looks affine. This implies that on small scale

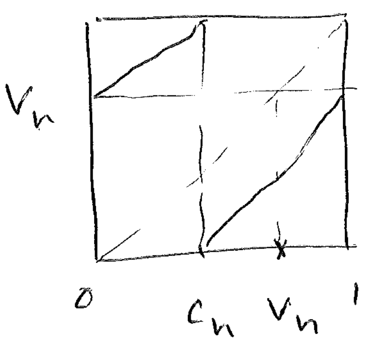
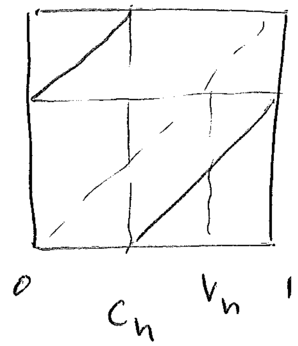
The geometry of f- orbits is the same as the geometry of f_p^* orbits.

~~This is called: the~~

The Dynamics has Rigid geometry.

Proof Herman Theorem (for $a_n \equiv 1$)

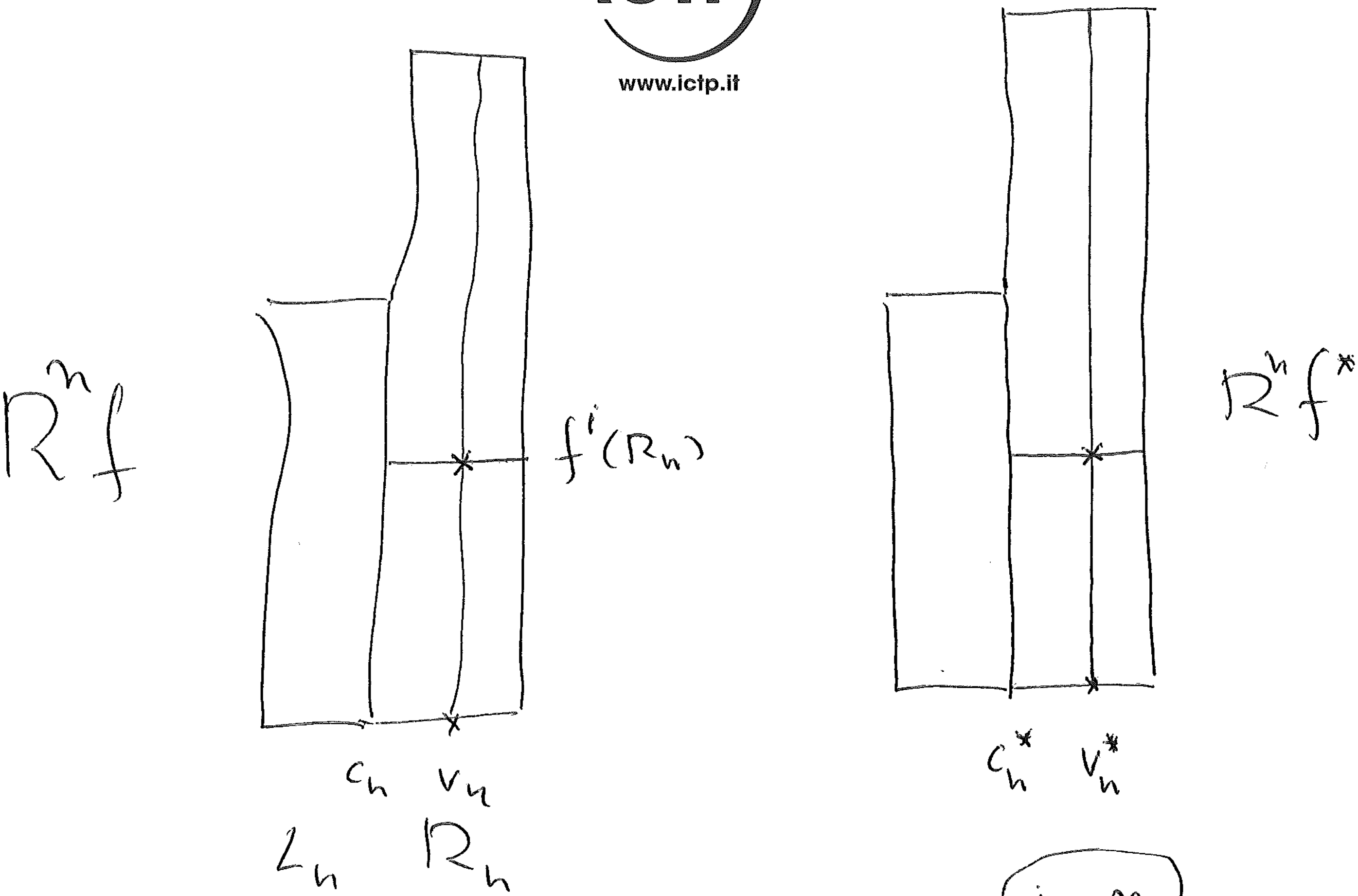
Let $f^* = f^*_{pf}$.

$R^n f$   $R^n f^*$

$$|R^n f - R^n f^*| \leq \lambda^n \quad , \lambda < 1$$

so

$$|c_n - c_n^*| \lesssim \lambda^n$$
$$|v_n - v_n^*| \lesssim \lambda^n$$



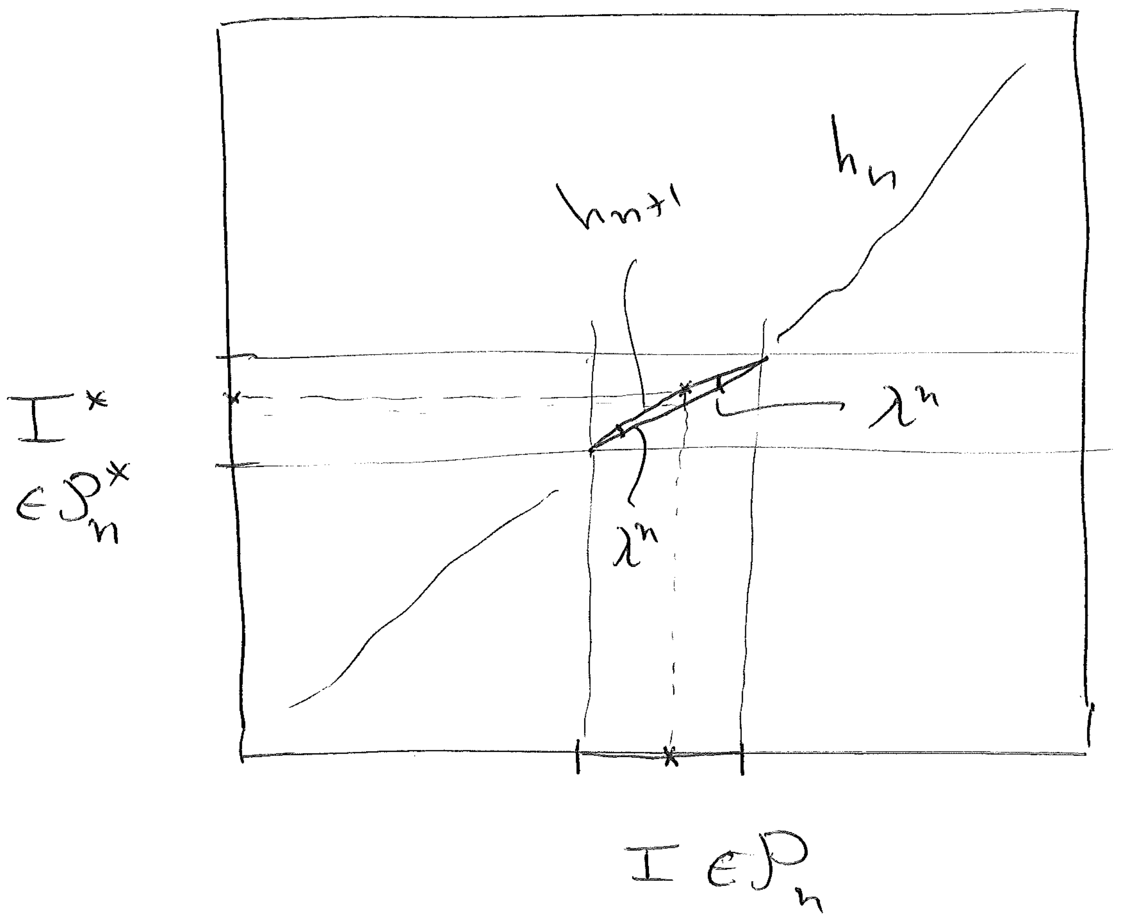
$f^i: R_n \longrightarrow f^i(R_n)$ is exp. close to (in n) affine. $\Longrightarrow$

The cutting of $f^i(R_n)$ happen at the same place as R_n is cut (exp. close to v_n)

But $|v_n - v_n^*| = O(\lambda^n)$

So $f^i(R_n)$ and $(f^*)^i(R^*_n)$ are cut at the same place up to exponential small error.

Draw h_n:



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So $\left| Dh_n - Dh_{n+1} \right| = O(\lambda^n)$

$\Longrightarrow$ Exercise $\Longrightarrow$

$Dh_n \longrightarrow Dh \in C^1$

$\square$