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On the Finiteness of Attractors for Piecewise C^2 Maps of the Interval

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This is a joint work with
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Let $f : [0,1] \to [0,1]$ be a map of the interval.

Definition (Attractor)

A compact transitive set $A \subset [0,1]$ is an attractor if its *basin of attraction*

$$\beta_f(A) = \{x; \omega_f(x) \subset A\}$$

has positive Lebesgue measure.

Problem:

How many attractors a map of the interval with some regularity can have? Can it be zero? Infinitely many?

Problem:

Has the union of the basins of attraction of all attractors of f full Lebesgue measure?

That is,

$$\mathrm{Leb}\left(\bigcup_{A\in\text{attractors}} \beta_f(A)\right) = 1?$$

Problem:

It is possible to classify the attractors?

Theorem (Singer, 1978)

If f is a C^3 map whit negative Schwarzian derivative, $Sf < 0$ then f has at most $\#\mathcal{C}_f + 2$ periodic attractors.

Schwarzian derivative

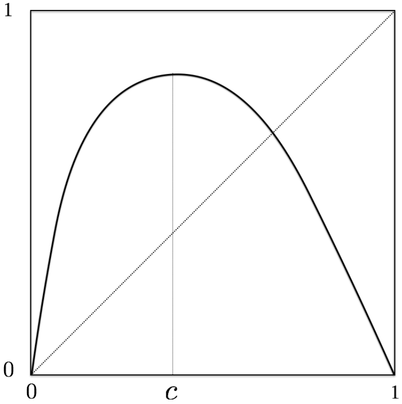
$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

Critical set

$$\mathcal{C}_f = \{c \in [0,1]\,;\, f'(c) = 0\}$$

Definition (S-unimodal map)

A C^3 map $f : [0,1] \to [0,1]$, with $f(0) = f(1) = 0$, is called a *S-unimodal* if $Sf < 0$, it has at most two fixed points and a single critical point $c \in (0,1)$.



Example (Logistic family)

$f_t(x) := 4\,t\,x\,(1-x)$, with $t \in (0,1)$

Theorem (Blokh and Lyubich, $\sim$ 1989)

If f is a non-flat S-unimodal map with $Sf < 0$ then f has a single attractor A and $\omega_f(x) = A$ for almost every $x \in [0,1]$. In particular,

$$Leb(\beta_f(A)) = 1.$$

Furthermore,

1. *either A is a periodic attractor (an attracting periodic orbit)*
2. *or A is a cycle of intervals*
3. *or A is a cantor set with $A = \omega_f(c)$, with c being the critical point of f.*

Non-flat (or non-degenerated)

A map is non-flat if

$$f(x) \approx f(c) + a|x - c|^{\alpha},$$

$a \neq 0$, $\alpha > 1$, for every $c \in \mathcal{C}_f$.

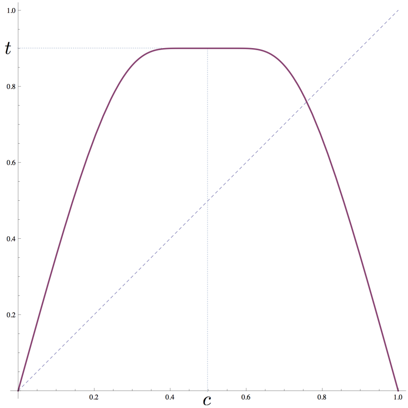
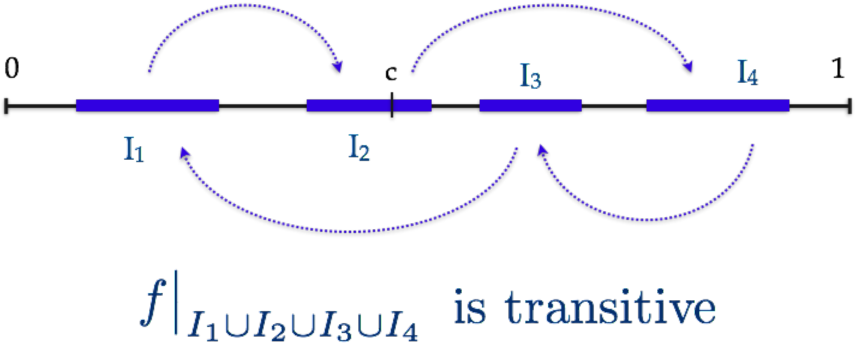


Figura: Let $0 < t \leq 1$ and $f_t : [0,1] \to [0,1]$ be given by $f_t(1/2) = t$ and $f_t(x) = t(1 - e^{2-1/|x-1/2|})$ if $x \neq 1/2$. The only critical point of f_t is $c = 1/2$. In the picture, we draw the graphic of f_t with $t = 0.9$. Notice that f_t is C^∞, $f_t^{(n)}(1/2) = 0 \; \forall \, n \geq 1$ and $Sf(x) = -(8/(1-2x)^4)$. Thus, f_t is a family of flat S-unimodal maps.

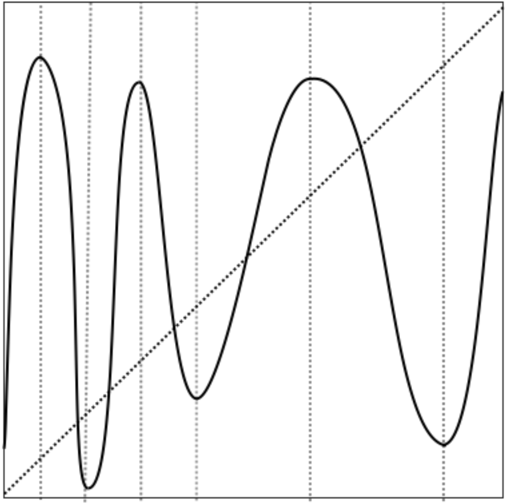
Cycle of intervals

Finite union of compact $I_1 \cup \cdots \cup I_s$ such that $f|_{I_1 \cup \cdots \cup I_s}$ is transitive.



Definition (S-multimodal map)

A C^3 map $f : [0,1] \to [0,1]$ is called a S-*multimodal* if $Sf < 0$, $\mathcal{C}_f \subset (0,1)$, $\#\mathcal{C}_f < +\infty$ and every $c \in \mathcal{C}_f$ is a local maximum or minimum .



Theorem (Blokh and Lyubich, $\sim$ 1989)

If f is a non-flat S-multimodal map then f has at most $\#\mathcal{C}_f + 2$ attractors,

$$Leb\Big(\bigcup_{A \in attractors} \beta_f(A)\Big) = 1$$

and $\omega_f(x) = A$ for almost every $x \in \beta_f(A)$ and $A \in$ attractors.

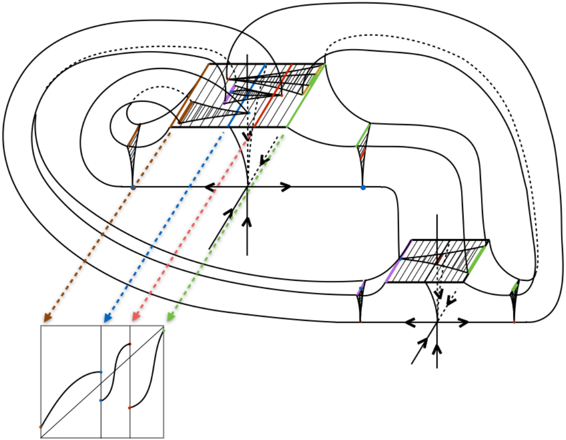
Furthermore, if A is one of the attractors then

1. either A is a periodic attractor (an attracting periodic orbit)
2. or A is a cycle of intervals
3. or A is a cantor set with $A = \omega_f(c)$,
   for some $c \in \mathcal{C}_f$.

Maps with discontinuities

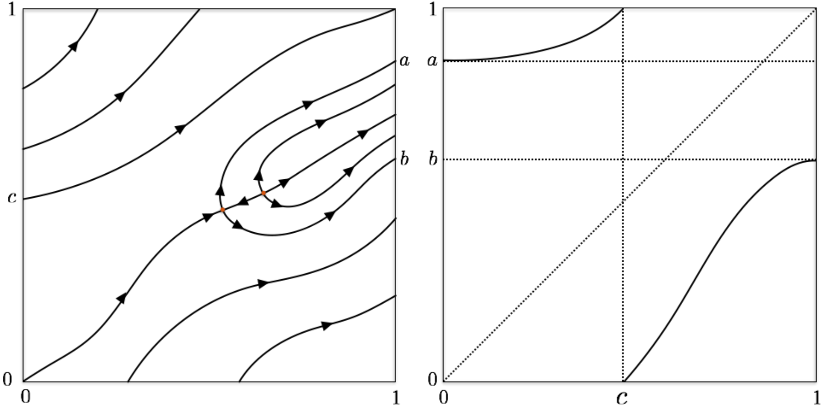
A particular interesting type of maps with discontinuities are those one with discontinuous critical points.

Piecewise C^r maps with non-flat discontinuous critical points can be obtained as the quotient by stable manifolds of a Poincaré map of some dissipative flow.

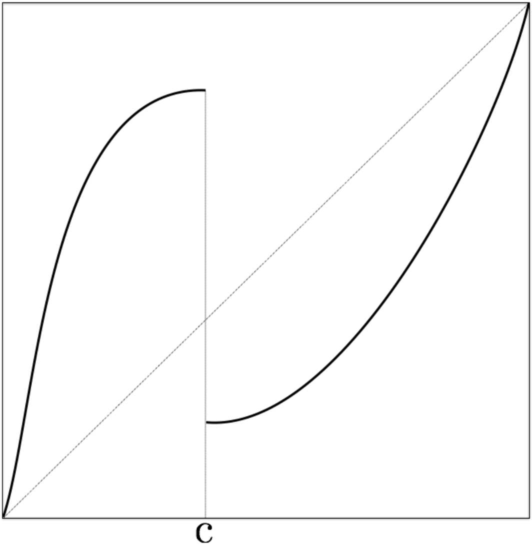


A particular interesting type of maps with discontinuities are those one with discontinuous critical points.

An one dimensional piecewise C^r map induced by a Cherry flow.



The contracting Lorenz maps are the simplest case of maps the interval with $Sf < 0$ and a discontinuous critical point.



Wandering intervals

An interval $I = (a, b) \subset [0, 1]$ is a called a *wandering interval* if

1. $f^j(I) \cap f^k(I) = \emptyset \; \forall \, 0 \leq j < k$.
2. $f^j(I) \cap \mathcal{C}_f = \emptyset \; \forall \, j \geq 0$;
3. I does not intersect the basin of attraction of a periodic attractor.

Remark

One of the main ingredients of most of the proofs of finiteness of attractors was the non-existence of wandering intervals.

Theorem (Denjoy, 1932)

A C^1 diffeomorphism f of the circle such that $\log |f'|$ has bounded variation does not admit wandering intervals.

Theorem (Guckenheimer, 1979)

A non-flat S-unimodal map does not admit wandering intervals.

Theorem (Yoccoz, 1984)

A C^∞ homeomorphisms of the circle having only non-flat critical points does not admit wandering intervals.

Theorem (Lyubich, 1987)

A non-flat S-multimodal map does not admit wandering intervals.

Theorem (de Melo and van Strien, 1989)

A non-flat C^2 unimodal map does not admit wandering intervals.

Theorem (Blokh and Lyubich, 1989)

A non-flat C^2 multimodal map does not admit wandering intervals.

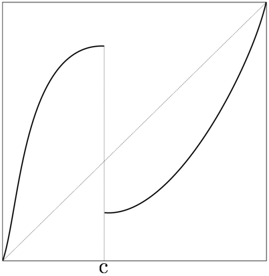
Theorem (Martens, de Melo and van Strien, 1992)

A non-flat C^2 map does not admit wandering intervals.

Warning!

We know almost nothing about wandering intervals even for the contracting Lorenz maps with negative Schwarzian derivative.

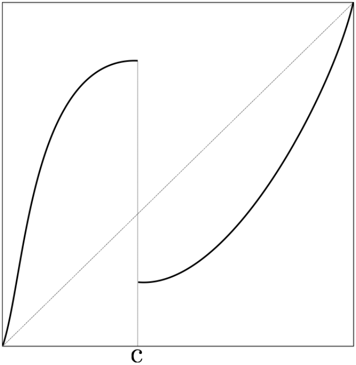
1. There are simple examples of contracting Lorenz maps with wandering intervals associated to Cherry attractors.
2. It was conjectured that all wandering intervals are associated to Cherry attractors, but except for very few cases this conjecture is open.
3. Even for ∞-renormalizable maps it is not known if there exist wandering intervals.



Using an induced map in a Hoffbauer-Keller tower:

Theorem (Keller and St. Pierre, 2000)

If f is a C^3 non-flat contracting Lorenz map with $Sf < 0$ then f has almost a single non periodic attractor. Furthermore, if A is a non periodic attractor for f then $\omega_f(x) = A$ for almost every $x \in [0,1]$. In particular, $Leb(\beta_f(A)) = 1$.

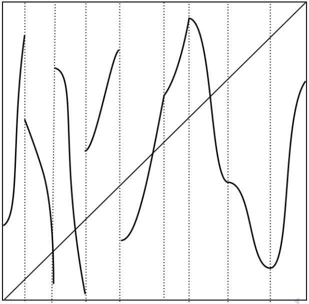


Theorem (Brandão, Palis, P, 2013)

Let $f : [0,1] \to [0,1]$, be a C^3 local diffeomorphism with negative Schwarzian derivative in the whole interval, except for a finite set $\mathcal{C}_f \subset (0,1)$. Then, there is a finite collection of attractors $A_1, \cdots, A_n$, such that

$$Leb(\beta_f(A_1) \cup \cdots \cup \beta_f(A_n)) = 1.$$

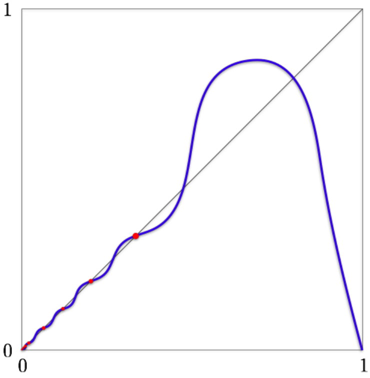
Furthermore, for almost all points x, we have $\omega_f(x) = A_j$ for some $j = 1, \cdots, n$.

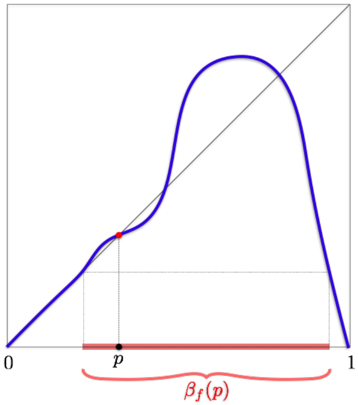


Maps without negative Schwarzian derivative condition

WARNING: non-flat C^∞ maps of the interval

1. may not have attractors (ex.: the identity map $f(x) = x$);
2. may have infinitely many attractors (picture on the left).





It can also occur that $0 < \text{Leb}\left(\bigcup_{A \in \text{attractors}} \beta_f(A)\right) < 1$ (picture on the right).

$\mathbb{B}_0(f)$

Let $\mathbb{B}_0(f)$ be the union of the basins of attraction of all attracting periodic orbits.

$\mathrm{Per}(f)$

Let $\mathrm{Per}(f)$ be the set of periodic points of f and $\mathcal{O}_f^-(\mathrm{Per}(f))$ the set of all *pre-periodic* points, i.e.,

$$\mathcal{O}_f^-(\mathrm{Per}(f)) = \bigcup_{n \geq 0} f^{-n}(\mathrm{Per}(f)).$$

Theorem (Vargas and van Strien, 2004)

If $f : [0,1] \to [0,1]$ is a non-flat C^3 map then f has at most $\#\mathcal{C}_f$ non-periodic attractors $A_1, ..., A_s$, $0 \le s \le \#\mathcal{C}_f$,

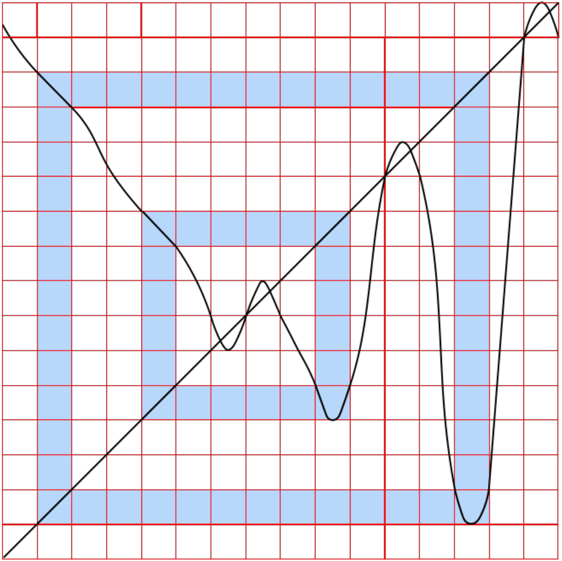
$$Leb\left(\mathbb{B}_0(f) \cup \mathcal{O}_f^-(\mathrm{Per}(f)) \cup \bigcup_{j=1}^{s} \beta_f(A_j)\right) = 1$$

and $\omega_f(x) = A_j$ for almost every $x \in \beta_f(A_j)$ and $0 \le j \le s$.

Furthermore, for each $0 \le j \le s$

1. either A_j is a cycle of intervals
2. or A_j is a Cantor set (and a minimal set) with $A_j = \omega_f(c)$, for some $c \in \mathcal{C}_f$.

Maps with the number of non periodic attracts equal to $\#\mathcal{C}_f$.



Non-flat

$f : [0,1] \to \mathbb{R}$ is called *non-flat* at $c \in [0,1]$ if $\exists \varepsilon > 0,\ \alpha, \beta \geq 1$ and C^2 diffeomorphisms $\phi_0 : [c-\varepsilon, c] \to \mathrm{Im}(\phi_0)$ and $\phi_1 : [c, c+\varepsilon] \to \mathrm{Im}(\phi_1)$ such that

$$f(x) = \begin{cases} a + \big(\phi_0(x-c)\big)^{\alpha} & \text{if } x \in (c-\varepsilon, c) \cap (0,1) \\ b + \big(\phi_1(x-c)\big)^{\beta} & \text{if } x \in (c, c+\varepsilon) \cap (0,1) \end{cases},$$

where $a = \lim_{0<\varepsilon\to 0} f(c-\varepsilon)$ and $b = \lim_{0<\varepsilon\to 0} f(c+\varepsilon)$.

Critical/non-regular set

If $f : [0,1] \to [0,1]$ is a non-flat piecewise C^2 map, then $\exists$ a finite set $\mathcal{C}_f$ (we may assume that $\mathcal{C}_f \subset (0,1)$) s.t. f is a local C^2 diffeomorphism on $[0,1] \setminus \mathcal{C}_f$.

The set $\mathcal{C}_f$ is called *Critical/non-regular set*.

Critical/non-regular values

$$\mathcal{V}_f = \{f(c_\pm)\,;\, c \in \mathcal{C}_f\}.$$

Theorem (Brandão, Palis, P, 2015)

If $f : [0,1] \to [0,1]$ is a non-flat piecewise C^2 map then f has at most $\#\mathcal{V}_f$ non-periodic attractors $A_1, ..., A_s$, $0 \le s \le \#\mathcal{V}_f$,

$$Leb\left(\mathbb{B}_0(f) \cup \mathcal{O}_f^-(\mathrm{Per}(f)) \cup \bigcup_{j=1}^{s} \beta_f(A_j)\right) = 1$$

and $\omega_f(x) = A_j$ for almost every $x \in \beta_f(A_j)$ and $0 \le j \le s$.

Furthermore, for each $0 \le j \le s$

1. either A_j is a cycle of intervals
2. or A_j is a Cantor set with $A_j = \omega_f(c_-)$ or $\omega_f(c_+)$, for some $c \in \mathcal{C}_f$.

Main steps of the proof

Step 1 (Dichotomy)

If $I = (a, b) \subset [0, 1]$ is such that $I \cap \mathcal{O}_f^+(\mathcal{V}_f) = \emptyset$ then

1. either $\omega_f(x) \cap I = \emptyset$ for almost all $x \in I \setminus \mathbb{B}_0(f)$
2. or $\omega_f(x) \supset I$ for almost every $x \in I$.

Step 2

$$\omega_f(x) \subset \overline{\mathcal{O}_f^+(\mathcal{V}_f)},$$

for almost every $x \in [0, 1] \setminus \left(\mathbb{B}_0(f) \cup \mathbb{B}_1(f) \cup \mathcal{O}_f^-(\mathrm{Per}(f)\right)$.

$\mathbb{B}_1(f)$ is the set of $x \in [0, 1]$ such that $\omega_f(x)$ is a cycle of intervals.

Step 3

$$\omega_f(x) = \bigcup_{\substack{c_\pm \in \omega_f(x) \\ c \in \mathcal{C}_f}} \overline{\mathcal{O}_f^+(c_\pm)},$$

for almost every $x \in [0,1] \setminus \left(\mathbb{B}_0(f) \cup \mathbb{B}_1(f) \cup \mathcal{O}_f^-(\mathrm{Per}(f))\right)$.

Step 4

If $\mathcal{U} \subset \{c_\pm\,;\, c \in \mathcal{C}_f\}$ such that

$$\mathrm{Leb}\left(\left\{x \in [0,1] \setminus \mathbb{B}_0(f)\,;\, \omega_f(x) = \bigcup_{u \in \mathcal{U}} \overline{\mathcal{O}_f^+(u)}\right\}\right) > 0$$

then $\exists\, u_0 \in \mathcal{U}$ such that

$$\omega_f(u_0) = \overline{\mathcal{O}_f^+(u_0)} = \bigcup_{u \in \mathcal{U}} \overline{\mathcal{O}_f^+(u)}.$$

Thanks!