Invariant pseudo-foliations for minimal homeomorphisms in dimension 2

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Describe the topological dynamics of **minimal** homeos/diffeos of $\mathbb{T}^2$

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Problem: C^r-closing lemma (Very very weak version)

Does

$$\{f \in \mathrm{Diff}^r(\mathbb{T}^2) : f \text{ minimal}\}$$

have empty interior in $\mathrm{Diff}^r(\mathbb{T}^2)$, $r \geqslant 2$?

Examples

1. **Ergodic translations:** $(\alpha, \beta) \in \mathbb{R}^2$,

$$R_{\alpha,\beta} : \mathbb{T}^2 \ni (x, y) \mapsto (x + \alpha, y + \beta),$$

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2. **Time-t of reparametrizations of linear flows:** $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\psi \in C^\infty(\mathbb{T}^2, \mathbb{R}_+)$, $\Phi_X : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2$ induced flow by $X := \psi \cdot (1, \alpha)$.

   Φ_X^t is minimal for generic t and is conjugate to a translation iff $\exists c \in \mathbb{R}$, $\exists u : \mathbb{T}^2 \to \mathbb{R}$ such that $\partial_x u + \alpha\partial_y u = \phi - c$

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3. [Furstenberg, 1961]: $\alpha \in \mathbb{R} \backslash \mathbb{Q}$, $\phi \colon \mathbb{T}^1 \to \mathbb{R}$ t.q. $\nexists u \in C^0(\mathbb{T}^1, \mathbb{R})$, $u \circ R_\alpha - u = \phi$:

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4. **Irrational Dehn twists:** $\alpha \in \mathbb{R} \backslash \mathbb{Q}$, $m \in \mathbb{Z} \backslash \{0\}$

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Generic diffeomorphisms in

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Question 2

What about minimal diffeomorphisms isotopic to Dehn twists?

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$$\frac{\tilde{f}^n - id}{n} = \frac{1}{n} \sum_{j=0}^{n-1} \phi \circ f^j \to \rho = \int_{\mathbb{T}} \phi \, \mathrm{d}\mu, \ \forall \mu \in \mathfrak{M}(f)$$

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- Definition: f **pseudo-rotation** if $\sharp \rho(\tilde{f}) = 1$

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So, if $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is minimal, then $\rho(\tilde{f}) \cap \mathbb{Q}^2 = \varnothing$ and is either

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Counterexample to Franks-Misiurewicz conj. [Avila, 2013]

$\exists f \in \mathrm{Diff}_0^\infty(\mathbb{T}^2)$ minimal s.t. $\rho(\tilde{f})$ segment irrational slope

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$f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is minimal and **not a pseudo-rotation**, then there exists **f-invariant (pseudo)-foliation.**

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Rotational deviations

1. **Circle homeomorphisms:** If $f \in \mathrm{Homeo}_0(\mathbb{T}^1)$ and $\tilde{f} \colon \mathbb{R} \circlearrowleft$ lift, then

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Theorem [Folklore?]

If f is minimal and $\rho(\tilde{f}) \subset \ell_\alpha^v := \alpha v + \mathbb{R}v^\perp$, then
$\exists$ **invariant pseudo-foliation** $\iff$ **uniformly bounded** v**-deviations** i.e.

$$\sup_{z,n} \left|\langle \tilde{f}^n(z) - z - n\rho, v\rangle\right| < \infty$$

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- $\Lambda_r^{\pm}$ are **"just hairs"** if f exhibits **horizontal unbounded deviations.**
- [Guelman-Koropecki-Tal, 2014]: If f is area-preserving and $\rho(\tilde{f}) = \{0\} \times [a,b]$, then $\sup_{n,z} \left|\langle \tilde{f}^n(z) - z, (1,0)\rangle\right| < \infty$.

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but in such a case Λ_r^+ is not dynamically defined

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The following short exact sequence *splits*:

$$0 \to \widetilde{\mathrm{Ham}}(\mathbb{T}^2) \to \widetilde{\mathrm{Symp}_0}(\mathbb{T}^2) \xrightarrow{\rho_L} \mathbb{R}^2 \to 0$$

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$$\rho_L(\tilde{f}) := \int_{\mathbb{T}^2} \tilde{f} - id \, \mathrm{dLeb}$$

  is a group homomorphism
- $\widetilde{\mathrm{Ham}}(\mathbb{T}^2) := \ker \rho_L$

The following short exact sequence *splits*:

$$0 \to \widetilde{\mathrm{Ham}}(\mathbb{T}^2) \to \widetilde{\mathrm{Symp}_0}(\mathbb{T}^2) \xrightarrow{\rho_L} \mathbb{R}^2 \to 0$$

Then, $\widetilde{\mathrm{Symp}_0}(\mathbb{T}^2) = \mathbb{R}^2 \ltimes \widetilde{\mathrm{Ham}}(\mathbb{T}^2)$

Hamiltonian skew-product

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and hence,

$$F^n(0,z) = (n\rho, \tilde{f}^n(z) - n\rho), \quad \forall n, z$$

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and,

$$\Lambda_r^{\pm} := \bigcup_{t \in \mathbb{T}^2} \Lambda_r^{\pm}(t) \subset \mathbb{T}^2 \times \mathbb{R}^2$$

is F-invariant!

Some properties of $\Lambda_r^{\pm}$

Suppose f is minimal, $\rho(\tilde{f}) = \{\alpha\} \times [a, b]$ and there exists $z \in \mathbb{R}^2$ such that

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3. By (1), $\Lambda_r^{\pm}(t) \cap \Omega_{r'}^{\pm}(t) = \varnothing$ and this produces a contradiction

Minimal homeos isotopic to Dehn tiwsts

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$$\rho_v(\tilde{f}) := \left\{ \int_{\mathbb{T}^2} \phi_v \, \mathrm{d}\mu : \mu \in \mathfrak{M}(f) \right\} \subset \mathbb{R}$$

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- [Addas-Zanata-Garcia-Tal, 2014]: If f is minimal, then $\rho(\tilde{f}) = \{\alpha\}$ with $\alpha \in \mathbb{R} \backslash \mathbb{Q}$

The Hamiltonian skew-product (Dehn twist case)

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5. Define, for each $t \in \mathbb{T}$ and each $r \in \mathbb{R}$,

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5. A homeomorphism isotopic to Dehn twist may be a non-twist map, but it is at large scale!

Thanks!