Good universal weights for non-conventional ergodic averages

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ICTP School and Conference in Dynamical Systems, 2015

The Erdös-Turan conjecture

The Erdös-Turan conjecture (1936) states that if $A \subset \mathbb{N}$ such that

$$\sum_{n \in A} \frac{1}{n} = \infty,$$

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then A contains arbitrarily long arithmetic progressions.
The converse of the statement needs not be true. For instance, the set

$$\{1, 10, 11, 100, 101, 102, 1000, 1001, 1002, 1003, 10000, \ldots\}$$

has arbitrarily long arithmetic progressions, although the sum of the reciprocals of the elements of this set is finite.

The E.-T. conjecture: Sets with positive upper density

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It is known that a set with positive upper density satisfies the hypothesis of the Erdös-Turan conjecture, and we know from **Szemerédi's Theorem** that the set has arbitrarily long arithmetic progressions.

The E.-T. conjecture: The set of prime numbers

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Theorem (Euler, 1737)

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In fact, it is also known that

$$\sum_{n \in P \cap [N]} \frac{1}{n} \geq \log\log(N+1) - \log\frac{\pi^2}{6}$$

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We also know that the set of primes P contains arbitrarily long arithmetic progressions. This is shown by Green and Tao [19].

Multiple Recurrence (Furstenberg)

In 1977 [17], H. Furstenberg showed that for any probability measure-preserving system $(Y, \mathcal{G}, \nu, S)$, and $E \in \mathcal{G}$ with $\nu(E) > 0$, then for any positive integer $k \geq 1$,

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \nu\left(\bigcap_{i=0}^{k-1} S^{-in} E\right) > 0.$$

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Furstenberg used this to provide an ergodic theoretic proof of Szemerédi's theorem: If a set $\tilde{E} \subset \mathbb{Z}$ has a positive upper-density, then $\tilde{E}$ contains an arbitrary long arithmetic progression.

Multiple Recurrence (Furstenberg-Katzlenson)

Later, H. Furstenberg and Y. Katznelson (1978 [18]) showed that for commuting measure-preserving transformations $S_1, S_2, \ldots, S_k$, we have

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \nu\left(\bigcap_{i=1}^{k} S_i^{-n} E\right) > 0$$

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Question: Can we get a better understanding of the structure of set of positive upper density in $\mathbb{N}$?

Motivation

One way of obtaining more information would be to look at these averages with weights.

Example: Can we show that for weights $(c_n)_n$ of nonnegative numbers, we still have

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} c_n \cdot \nu \left(\bigcap_{i=1}^{k} S_i^{-n} E \right) > 0?$$

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The first natural case is to look at a randomized weight, e.g. given a measure-preserving (ergodic) system $(X, \mathcal{F}, \mu, T)$ with a set $A \in \mathcal{F}$ of positive measure, does there exists a set of full-measure $X_A \subset X$ such that for any $x \in X_A$,

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_A(T^n x) \cdot \nu\left(\bigcap_{i=1}^{k} S_i^{-n} E\right) > 0?$$

Motivation: Return Times

Since

$$\mathbf{1}_A(T^n x) = \begin{cases} 1 & \text{if } T^n x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

we can look at the subsequences $\{n_j(x)\}$ of the multiple recurrent averages, where $n_j(x)$ is the j-th return time of $T^n x$ to A (i.e. $\mathbf{1}_A(T^{n_j(x)}x) = 1$ for all $j \in \mathbb{N}$). So the previous question would be equivalent of asking whether

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n_j(x)<N} \nu\left(\bigcap_{i=1}^{k} S_i^{-n_j(x)} E \right) > 0.$$

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Behaviors of averages with random weights have been observed extensively in the study of the **return times**, which was initiated by A. Brunel in his Ph.D. thesis from 1966 [10].

Convergence of multiple recurrent averages

We note that

$$\frac{1}{N}\sum_{n=0}^{N-1} c_n \cdot \nu\left(\bigcap_{i=1}^{k} S_i^{-n}E\right) = \int \frac{1}{N}\sum_{n=0}^{N-1} c_n \prod_{i=1}^{k} \mathbf{1}_E(S_i^n y)\,d\nu(y).$$

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We can get more information about the multiple recurrent averages by studying the following: Given $g_1, g_2, \ldots, g_k \in L^\infty(\nu)$, do the averages

$$\frac{1}{N}\sum_{n=0}^{N-1} c_n \prod_{i=1}^{k} g_i \circ S_i^n$$

converge weakly? In $L^2(\nu)$-norm? Almost everywhere?

Non-conventional ergodic averages

The study of non-conventional ergodic averages, i.e.

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For the pointwise convergence, we have the double recurrence result by Bourgain, for the case $k = 2$ and $S_i = S^{a_i}$ (1990 [8]).

Good universal weights

Definition

We say $(X_n)_n$ *is a* **process** *if for all nonnegative integers* $n \geq 0$, X_n *is a bounded and measurable function on some probability measure space* $(\Omega, \mathcal{S}, \mathbb{P})$.

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Definition

We say the sequence $(a_n)_n$ is a good universal weight for a process $(X_n)_n$ pointwise (resp. in norm) if for every probability measure-preserving space $(\Omega, \mathcal{S}, \mathbb{P})$ for which the process $(X_n)_n$ is defined, then the averages

$$\frac{1}{N}\sum_{n=0}^{N-1} a_n X_n(\omega)$$

converges for $\mathbb{P}$-a.e. $\omega \in \Omega$ (resp. in $L^2(\mathbb{P})$).

Good universal weights can be randomized

Brunel and Keane in 1969 [11] showed that measure-preserving system $(X, \mathcal{F}, \mu, T)$ from a certain class and a set of positive measure $A \subset \mathcal{F}$, there exists a set of full-measure $X_A \subset X$ such that for any $x \in X'$ and any other measure-preserving system $(Y, \mathcal{G}, \nu, S)$ and any $g \in L^1(\nu)$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_A(T^n x) g(S^n y)$$

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exists for ν-a.e. $y \in Y$.
Later, Bourgain showed [7] that for any given measure-preserving system $(X, \mathcal{F}, \mu, T)$ and $f \in L^\infty(\mu)$, $a_n = f(T^n x)$ is also a good universal weight for the pointwise ergodic theorem. A simpler proof was later provided by Bourgain, Furstenberg, Katznelson, and Ornstein [9].

Extensions of the return times theorem

The return times theorem has been extended in multiple directions. For example...

- Rudolph (Multi-term return times theorem, 1998, [24])
- A. (Multiple recurrence and the multi-term return times theorem, 2000, [1])
- Host and Kra (Good universal weight for the norm convergence of non conventional ergodic averages, 2009, [21])

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More precisely, Host and Kra showed that given an ergodic dynamical system $(X, \mathcal{F}, \mu, T)$ and a function $f \in L^\infty(\mu)$, there exists a set of full-measure $X_f \subset X$ such that for any $x \in X_f$, for any positive integer k, and for any other measure-preserving system $(Y, \mathcal{G}, \nu, S)$ with $g_1, \ldots, g_k \in L^\infty(\nu)$, the averages

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) g_1 \circ S^n \cdot g_2 \circ S^{2n} \cdots g_k \circ S^{kn}$$

converge in $L^2(\nu)$.

Universal Weights

The result by Host and Kra shows that if $c_n = f(T^n x)$, then for μ-a.e. $x \in X$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} c_n \cdot \nu(E \cap S^{-n}E \cap \cdots \cap S^{-(k-1)n}E)$$

exists for any other measure-preserving system $(Y, \mathcal{G}, \nu, S)$.

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exists for any other measure-preserving system $(Y, \mathcal{G}, \nu, S)$.
Question: Can c_n be generalized? For instance, can we have $c_n = f_1(T^{an}x)f_2(T^{bn}x)$ for some $f_1, f_2 \in L^\infty(\mu)$?

Double Recurrence Wiener-Wintner averages

Recall that given a measure-preserving system $(X, \mathcal{F}, \mu, T)$, the Cesaro averages of $f_1(T^{an}x)f_2(T^{bn}x)$ converge for μ-a.e. $x \in X$ due to Bourgain [8].

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Later in 2014, this result was extended further in a following way:

Double Recurrence Wiener-Wintner averages

Theorem 1 (A., Duncan, Moore 2014, [3])

Let $(X, \mathcal{F}, \mu, T)$ be a standard ergodic dynamical system. Let $f_1, f_2 \in L^\infty(X)$. Let $W_N(f_1, f_2, x, t) = \frac{1}{N}\sum_{n=0}^{N-1} f_1(T^{an}x)f_2(T^{bn}x)e(nt)$. Then there exists a set of full-measure X_{f_1,f_2} such that for all $x \in X_{f_1,f_2}$ and for any $t \in \mathbb{R}$, the sequence $W_N(f_1, f_2, x, t)$ converges.
Also, if either f_1 or f_2 belongs to $\mathcal{Z}_2^\perp$, then for any $x \in X_{f_1,f_2}$,

$$\limsup_{N\to\infty} \sup_{t\in\mathbb{R}} |W_N(f_1, f_2, x, t)| = 0,$$

where $\mathcal{Z}_k$ is the k-th Host-Kra-Ziegler factor.

Remarks on the D.R.W.W. result

- This Wiener-Wintner result already shows that the sequence $c_n = f_1(T^{an}x)f_2(T^{bn}x)$ is μ-a.e. a good universal weight for the mean ergodic theorem, i.e. for any other dynamical system $(Y, \mathcal{G}, \nu, S)$ and $g \in L^\infty(\nu)$, the averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^{an}x)f_2(T^{bn}x)g \circ S^n$$

converge in $L^2(\nu)$.

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  converge in $L^2(\nu)$.
- The l-th Host-Kra-Ziegler factor $\mathcal{Z}_l$ of X is the inverse limit of l-step nilsystems of X.

Definition

Let G be a l-step nilpotent Lie group, and Γ be a discrete co-compact subgroup of G. Then G/Γ is called an l-step nilmanifold. A measure-preserving system $(X, \mathcal{F}, \mu, T)$, where $X = G/\Gamma$, $\mathcal{F}$ a Borel sigma-algebra with Haar measure μ, and $Tx = g \cdot x$ for some $g \in G$, is called an l-step nilsystem.

Double recurrence and nilsequence Wiener-Wintner

In 2014 Ergodic Theory Workshop at UNC-Chapel Hill, B. Weiss asked whether Theorem 1 can be extended to nilsequences.

Definition

Let $(X = G/\Gamma, \mathcal{F}, \mu, T)$ *be an l-step nilsystem. If* $F \in \mathcal{C}(X)$ *and* $g \in G$*, we say the sequence* $a_n = F(g^n x)$ *is a basic l-step nilsequence. An l-step nilsequence is a uniform limit of basic l-step nilsequences.*

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We note that $e(nt) = e^{2\pi int}$ is a 1-step nilsequence, and for any real polynomial p of degree l, $e(p(n))$ is an l-step nilsequence.

Double recurrence and nilsequence Wiener-Wintner

We answered to this question positively.

Theorem 2 (A. 2015 [2])

Let $(X, \mathcal{F}, \mu, T)$ be a measure-preserving system, and $f_1, f_2 \in L^\infty(\mu)$. Then there exists a set of full-measure $X_{f_1,f_2} \subset X$ such that for any $x \in X_{f_1,f_2}$ and for any nilsequence $(b_n)_n$, the averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^{an}x)f_2(T^{bn}x)b_n$$

converge.

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converge.

This result was also obtained by Zorin-Kranich independently [27].

Remark: Polynomial Wiener-Wintner

If $b_n = e(p(n))$ for some real polynomial p, we would have the polynomial Wiener-Wintner result for the double recurrence. This result was obtained prior to the nilsequence result by A. and Moore in 2014 [6].

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- This would imply that the sequence $c_n = f_1(T^{an}x)f_2(T^{bn}x)$ is a μ-a.e. good universal weight for the process $X_n(y) = g(S^{p(n)}y)$ in norm for any measure-preserving system $(Y, \mathcal{G}, \nu, S)$, $g \in L^\infty(\nu)$ and any polynomial $p : \mathbb{Z} \to \mathbb{Z}$

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- The polynomial Wiener-Wintner averages (with a single function) were studied previously by E. Lesigne (1990 [22], 1993 [23]), N. Frantzikinakis (2006 [16]), and recently by T. Eisner and B. Krause (polynomial power of T, 2014 [15]).

Double recurrence and Furstenberg averages

Recently, we have shown that $a_n = f_1(T^{an}x)f_2(T^{bn}x)$ for any $a, b \in \mathbb{Z}$ distinct is a good universal weight for the Furstenberg averages.

Double recurrence and Furstenberg averages

Recently, we have shown that $a_n = f_1(T^{an}x)f_2(T^{bn}x)$ for any $a, b \in \mathbb{Z}$ distinct is a good universal weight for the Furstenberg averages.

Theorem 3 (A., Moore 2015, [5])

Let $(X, \mathcal{F}, \mu, T)$ be a measure-preserving system, and suppose $f_1, f_2 \in L^\infty(\mu)$. Then there exists a set of full-measure $X_{f_1,f_2} \subset X$ such that for any $x \in X_{f_1,f_2}$, for any $a, b \in \mathbb{Z}$ distinct, for any positive integer k, and for any other dynamical system $(Y, \mathcal{G}, \nu, S)$ with functions $g_1, \ldots, g_k \in L^\infty(\nu)$, the averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^{an}x)f_2(T^{bn}x)\prod_{i=1}^{k} g_i \circ S^{in}$$

converge in $L^2(\nu)$.

Remark

- This is a strengthening of the results of Host and Kra and of Bourgain's double recurrence theorem. In fact, we can show that for any $A, B \in \mathcal{F}$, we know that there exists a set of full-measure $X_{A,B} \subset X$ such that for any $x \in X_{A,B}$, for any other measure-preserving system $(Y, \mathcal{G}, \nu, S)$ and $E \in \mathcal{G}$, the limit

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_A(T^{an}x)\mathbf{1}_B(T^{bn}x)\nu\left(\bigcap_{i=1}^{k} S^{-in}E\right)$$

exists.

Commuting Case

Furthermore, Theorem 3 can be extended to the case with commuting transformations. This result combines and extends the previous work of Bourgain and Tao.

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Theorem 4 (A., Moore 2015, [4])

Let $(X, \mathcal{F}, \mu, T)$ be a measure-preserving system, and suppose $f_1, f_2 \in L^\infty(\mu)$. Then there exists a set of full-measure $X_{f_1,f_2} \subset X$ such that for any $x \in X_{f_1,f_2}$, for any $a, b \in \mathbb{Z}$ distinct, for any positive integer k, and for any other dynamical system with commuting transformations $(Y, \mathcal{G}, \nu, S_1, \ldots, S_k)$ with functions $g_1, \ldots, g_k \in L^\infty(\nu)$, the averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^{an}x) f_2(T^{bn}x) \prod_{i=1}^{k} g_i \circ S_i^n$$

converge in $L^2(\nu)$.

Outline of the Proof of Theorem 3

The idea of the proof is to look at whether either f_1 or f_2 belongs to the orthogonal complement of $\mathcal{Z}_{k+1}(T)$, or if they are both in this factor (where k is the number of transformations from the Furstenberg averages).

Outline of the Proof of Theorem 3

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- If one of the functions $g_1, \ldots, g_k$ belongs to $\mathcal{Z}_k(S)^\perp$, then the averages converge to 0 in norm.
- If all of them belong to $\mathcal{Z}_k(S)$, we apply Leibman's convergence theorem.

Next Steps

1. Nilpotent case. Can the double recurrence good universal weight results be extended to the systems $(Y, \mathcal{G}, \nu, S_1, \ldots, S_k)$, where the transformations $S_1, \ldots, S_k$ are generating a nilpoteng group?

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2. Positivity. Given a measure-preserving system $(X, \mathcal{F}, \mu, T)$, with some $f_1, f_2 \in L^\infty(\mu)$, does there exist a set of full-measure $X_{f_1,f_2} \subset X$ such that for any $x \in X_{f_1,f_2}$ and for any other measure-preserving system $(Y, \nu, S_1, \dots, S_k)$ with any $E \in \mathcal{G}$ a set with positive measure, we have

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{an}x) f_2(T^{bn}x) \nu \left(\bigcap_{i=1}^{k} S_i^{-n} E \right) > 0?$$

Any properties on f_1 and f_2 (beyond the clear requirement that $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{an}x) f_2(T^{bn}x) > 0$)? What about a positive lower bound? Syndeticity?

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Thank you for the invitation!