Ergodic theory of expanding Thurston maps

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Thurston’s theorem on characterization of rational maps among topological self-branched covering of 2-sphere.

[Douady & Hubbard 1993]
Thurston’s Theorem

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Expanding Thurston maps


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Thurston maps

*Thurston map:* A non-homeomorphic branched covering map $f : S^2 \to S^2$ with $\text{card}(\text{post } f) < +\infty$.

Postcritical set:
$\text{post } f = \{ f^n(x) | n \in \{ 1, 2, \ldots \}, x \text{ is a critical point of } f \}$. 
Pillow: $f$, a Lattès map
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Example

Pillow: \( f^2 \)
Example

Pillow: $f^3$
A Thurston map $f : S^2 \to S^2$ is expanding if there exist
- a metric $d$ on $S^2$ that induces the standard topology on $S^2$,
- a Jordan curve $C \subseteq S^2$ containing post $f$

such that

$$\lim_{n \to +\infty} \max\{\text{diam}_d(X) \mid X \text{ is a conn. comp. of } S^2 \setminus f^{-n}(C)\} = 0.$$ 

Remark: the definition is independent of the choices of $d$ and $C.$
Expanding Thurston maps

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Remark: the definition is independent of the choices of \( d \) and \( C \).
**Proposition.** Let $R: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational Thurston map. Then the following are equivalent:

1. $R$ is expanding,
2. the Julia set of $R$ is $\hat{\mathbb{C}}$,
3. $R$ has no periodic critical points.
Invariant Jordan curves

Theorem (Cannon, Floyd, & Parry 01, Bonk & Meyer 10)

Let $f$ be an expanding Thurston map. For each $n \in \mathbb{N}$ sufficiently large, there exists an $f^n$-invariant Jordan curve $C \subseteq S^2$ containing post $f$. 
$f$ - expanding Thurston map
$\mathcal{C}$ - Jordan curve on $S^2$ containing post $f$

d - visual metric (with expansion factor $\Lambda > 1$)
characterized by

- $\text{diam}_d(X^n) \precsim \Lambda^{-n}$,
- $d(X^n, Y^n) \succeq \Lambda^{-n}$ if $X^n \cap Y^n = \emptyset$.

If $f$ is rational, then $d \overset{\text{q.s.}}{\simeq}$ spherical metric.

$\sim$ - quasisymmetrically equivalent
Visual metrics

\( f \) - expanding Thurston map
\( \mathcal{C} \) - Jordan curve on \( S^2 \) containing post \( f \)
\( d \) - visual metric (with expansion factor \( \Lambda > 1 \))

characterized by

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### Sullivan’s dictionary

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**Cannon’s Conjecture** The boundary at infinity of a Gromov hyperbolic group is homeomorphic to $S^2 \ q.s. \ S^2 \ q.s. \ \sim$ spherical metric.

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**Theorem (Bonk & Meyer 10, Haïssinsky & Pilgrim 09)**

An expanding Thurston map is conjugate to a rational map iff visual metric $q.s. \ \sim$ spherical metric.
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$f$ - expanding Thurston map

- Existence & uniqueness of the measure of maximal entropy ([Haïssinsky & Pilgrim 09], [Bonk & Meyer 10])
- Ergodic properties of the measure of maximal entropy
- Existence & uniqueness of the equilibrium states
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Theorem (L. 13)

Let $f$ be an expanding Thurston map, with $\mu_0$ its measure of maximal entropy, $p \in S^2$.

Choose $w_n(x)$ to be 1 or $\deg f^n(x)$. Then as $n \to +\infty$,

$$\frac{1}{(\deg f)^n} \sum_{x \in f^{-n}(p)} w_n(x) \delta_x \xrightarrow{w^*} \mu_0,$$

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Each expanding Thurston map $f$ has exactly $1 + \deg f$ fixed points (counted with local degree $\deg f(x)$).
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\[ f : X \to X, \mu \text{ Borel measure} \]

\[ J \text{ is the Jacobian function of } f \text{ w.r.t. } \mu \text{ if} \]

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Let \( f : X \to X \) be a continuous map. 

\( \phi : X \to \mathbb{R} \) is a Hölder-continuous function (called potential).

For an \( f \)-invariant Borel probability measure \( \mu \),

\[
  h_\mu(f) + \int \phi \, d\mu \quad \text{measure-theoretic pressure}
\]

\[
  P(f, \phi) \quad \text{(topological) pressure}
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  P(f, \phi) = \sup \left\{ h_\mu(f) + \int \phi \, d\mu \mid f \text{-invariant Borel prob. measure } \mu \right\}
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Existence & uniqueness of equilibrium states

Rational maps on \( \hat{\mathbb{C}} \) with Hölder continuous potentials:

[Denker & Urbański 1991] - potential \( \phi < P(f, \phi) \)

[Przytycki & Urbański 10], [Comman & Rivera-Letelier 11] - special classes of rational maps

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*Then there exists a unique equilibrium state \( \mu_\phi \) for \( f \) and \( \phi \).*

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Then there exists a unique equilibrium state \( \mu_\phi \) for \( f \) and \( \phi \).

Visual metric on \( S^2 \)
(f, µφ) is exact, i.e., for each Borel set \( E \subseteq S^2 \) with \( µφ(E) > 0 \),

\[
\lim_{n \to +\infty} µφ(f^n(E)) = 1.
\]

In particular, (f, µφ) is mixing, i.e., for any Borel sets \( A, B \subseteq S^2 \),

\[
\lim_{n \to +\infty} µφ(f^{-n}(A) \cap B) = µφ(A) \cdot µφ(B).
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Ergodic properties

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i.e., for any Borel sets $A, B \subseteq S^2$,

$$\lim_{n \to +\infty} \mu_\phi(f^{-n}(A) \cap B) = \mu_\phi(A) \cdot \mu_\phi(B).$$
The Ruelle operator $\mathcal{L}_\phi : C(S^2) \to C(S^2)$:

$$\mathcal{L}_\phi(u)(x) = \sum_{y \in f^{-1}(x)} \deg_f(y)u(y) \exp(\phi(y)).$$
Co-homologous potentials

$f$ - an expanding Thurston map  \( \phi, \phi' \) - Hölder cont. potentials
\( \mu_\phi, \mu_{\phi'} \) - the corresponding equilibrium states

Theorem (L. 14)
\[ \mu_\phi = \mu_{\phi'} \text{ if and only if there exists } K \in \mathbb{R} \text{ s.t. } \phi - \phi' \text{ and } K \text{ are co-homologous, i.e.,} \]
\[ \phi - \phi' - K = u \circ f - u \quad \text{for some } u \in C(S^2). \]
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Equidistribution w.r.t. equilibrium states

$f$ - an expanding Thurston map \hspace{1cm} \phi - a Hölder cont. potential

$\mu_\phi$ - the unique equilibrium state for $f$ and $\phi$

$w_n(x) = \deg_{f^n}(x) \exp \left( \sum_{i=0}^{n-1} \tilde{\phi}(f^i(x)) \right)$ - the weight

**Theorem (L. 14)**

For each $p \in S^2$, as $n \to +\infty$,

$$\frac{1}{Z_n} \sum_{x \in f^{-n}(p)} w_n(x) \delta_x \overset{w^*}{\to} \mu_\phi. \quad \text{(preimage pts)}$$

What about periodic points?
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\( \mu_\phi \) - the unique equilibrium state for \( f \) and \( \phi \)  
\( w_n(x) = \deg_{f^n}(x) \exp \left( \sum_{i=0}^{n-1} \tilde{\phi}(f^i(x)) \right) \) - the weight

**Theorem (L. 14)**

For each \( p \in S^2 \), as \( n \to +\infty \),

\[
\frac{1}{Z_n} \sum_{x \in f^{-n}(p)} w_n(x) \delta_x \xrightarrow{w^*} \mu_\phi. \quad (\text{preimage pts})
\]

What about periodic points?
Equidistribution w.r.t. equilibrium states

\( f \) - an expanding Thurston map \( \phi \) - a Hölder cont. potential
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What about periodic points?
Large deviation principles and equidistribution

$f$ - an expanding Thurston map   $\phi$ - a Hölder cont. potential

equidistribution (w.r.t. the equilibrium state $\mu_\phi$)

$\uparrow$

large deviation principles (w.r.t. $\mu_\phi$)

$\uparrow$ [Kifer 90, Comman & Rivera-Letelier 11]

(i) existence and uniqueness of the equilibrium state

(ii) certain characterization of the topological pressure $P(f, \phi)$

(iii) upper semi-continuity of $\mu \mapsto h_\mu(f)$
Large deviation principles and equidistribution

\[ f \] - an expanding Thurston map \quad \phi \] - a Hölder cont. potential

equidistribution \ (w.r.t. the equilibrium state \( \mu_\phi \))
\[ \uparrow \]

large deviation principles \ (w.r.t. \( \mu_\phi \))
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Large deviation principles and equidistribution

\( f \) - an expanding Thurston map  \( \phi \) - a Hölder cont. potential

\[
equidistribution \ (w.r.t. \ \text{the equilibrium state} \ \mu_\phi) \\
\uparrow \\
large \text{deviation principles} \ (w.r.t. \ \mu_\phi) \\
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Weak expansion properties

E.T.M.  ↗  ↓  ↘  ↘  yes  no  →  ↘  yes  no  →  ?

w/o periodic critical pts  \[\not\rightarrow\]  \[\rightarrow\]  h-expansive (Bowen '72)

w/ periodic critical pts  \[\not\rightarrow\]  \[\rightarrow\]  asymptotic h-expansive (Misiurewicz '73)

upper semi-continuity of \(\mu \mapsto h_\mu(f)\)

---

**Theorem (L. 14)**

Let \(f\) be an expanding Thurston map. Then \(f\) is asymptotic h-expansive iff \(f\) has no periodic critical pts. Moreover, \(f\) is never h-expansive.
Theorem (L. 14)

Let $f$ be an expanding Thurston map. Then $f$ is asymptotic $h$-expansive iff $f$ has no periodic critical points. Moreover, $f$ is never $h$-expansive.
Weak expansion properties

E.T.M.

w/o periodic critical pts
no

w/ periodic critical pts
yes

\begin{itemize}
\item expansive
\item \textit{h}-expansive (Bowen '72)
\item asymptotic \textit{h}-expansive (Misiurewicz '73)
\item upper semi-continuity of \( \mu \mapsto h_\mu(f) \)
\end{itemize}

\textbf{Theorem (L. 14)}

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Weak expansion properties

\[ \text{E.T.M.} \]

\[ \text{w/o periodic critical pts} \quad \text{no} \quad \Rightarrow \quad \text{expansive} \]
\[ \Downarrow \]
\[ h\text{-expansive (Bowen ’72)} \]
\[ \Downarrow \]
\[ \text{asymptotic } h\text{-expansive (Misiurewicz ’73)} \]
\[ \Downarrow \]
\[ \text{upper semi-continuity of } \mu \mapsto h_\mu(f) \]

\[ \text{w/ periodic critical pts} \quad \text{yes} \quad \Rightarrow \quad \text{?} \]

**Theorem (L. 14)**

*Let* \( f \) *be an expanding Thurston map.*

*Then* \( f \) *is asymptotic* \( h\)-*expansive iff* \( f \) *has no periodic critical pts.*

*Moreover,* \( f \) *is never* \( h\)-*expansive.*
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E.T.M.

\[\begin{array}{c}
\text{w/o periodic} \\
\text{critical pts}
\end{array}\quad \begin{array}{c}
\text{w/ periodic} \\
\text{critical pts}
\end{array}\]

\[\begin{array}{c}
\text{no} \\
\text{yes}
\end{array}\quad \begin{array}{c}
\text{yes} \\
? \\
\end{array}\]

expansive
\[\Downarrow\]
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\[\Downarrow\]
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\[\Downarrow\]
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E.T.M.

w/o periodic critical pts

no

 ↘

h-expansive (Bowen '72)

ﬀ

yes

 ↘

asymptotic h-expansive (Misiurewicz '73)

ﬀ

 ↘

upper semi-continuity of \( \mu \mapsto h_\mu(f) \)

w/ periodic critical pts

? ___

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Weak expansion properties

E.T.M.

w/o periodic critical pts  \rightarrow no \rightarrow h-expansive (Bowen '72) \rightarrow\downarrow

w/ periodic critical pts

\rightarrow yes \rightarrow asymptotic h-expansive (Misiurewicz '73) \rightarrow\downarrow

\rightarrow upper semi-continuity of \mu \rightarrow h_\mu(f)

Theorem (L. 14)

Let f be an expanding Thurston map. Then f is asymptotic h-expansive iff f has no periodic critical pts. Moreover, f is never h-expansive.
Corollary

\[ f \text{ - an expanding Thurston map without periodic critical points} \]
\[ \psi : S^2 \rightarrow \mathbb{R} \text{ continuous} \]

Then there exists at least one equilibrium state for \( f \) and \( \psi \).

Proof.

The space of \( f \)-invariant Borel probability measures is compact in the weak* topology.

\[ \mu \mapsto h_\mu(f) + \int \psi \, d\mu \quad \text{is upper semi-continuous}. \]
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Equidistribution revisited

\( f \) - an expanding Thurston map \textbf{without periodic critical points}
\( \phi \) - a Hölder continuous potential
\( \mu_\phi \) - the unique equilibrium state for \( f \) and \( \phi \)

\[ w_n(x, \phi) = \exp \left( \sum_{i=0}^{n-1} \phi(f^i(x)) \right) \text{ or } \deg f_n(x) \exp \left( \sum_{i=0}^{n-1} \phi(f^i(x)) \right) \]

**Theorem (L. 14)**

As \( n \to +\infty \),

\[
\frac{1}{Z_n} \sum_{x=f^n(x)} w_n(x, \phi) \delta_x \xrightarrow{w^*} \mu_\phi, \quad \text{(periodic pts)}
\]

\[
\frac{1}{Z_n'} \sum_{x \in f^{-n}(p)} w_n(x, \tilde{\phi}) \delta_x \xrightarrow{w^*} \mu_\phi, \quad \text{(preimage pts)}
\]

for \( p \in \mathbb{S}^2 \).
Equidistribution revisited

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Equidistribution revisited

\( f - \) an expanding Thurston map **without periodic critical points**

\( \phi - \) a Hölder continuous potential

\( \mu_\phi - \) the unique equilibrium state for \( f \) and \( \phi \)

\[
\begin{align*}
  w_n(x, \phi) &= \exp \left( \sum_{i=0}^{n-1} \phi(f^i(x)) \right) \text{ or } \deg_{f^n(x)} \exp \left( \sum_{i=0}^{n-1} \phi(f^i(x)) \right)
  
  \text{Theorem (L. 14)}
  
  &\text{As } n \to +\infty, \quad \frac{1}{Z_n} \sum_{x=f^n(x)} w_n(x, \phi) \delta_x \xrightarrow{w^*} \mu_\phi, \quad (\text{periodic pts})
  
  &\quad \frac{1}{Z'_n} \sum_{x \in f^{-n}(p)} w_n(x, \tilde{\phi}) \delta_x \xrightarrow{w^*} \mu_\phi, \quad (\text{preimage pts})
  
  \text{for } p \in S^2.
\end{align*}
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Equidistribution revisited

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Thank you!