Scaling mean and a Law of Large Permanents

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I.C.T.P. - July 2015

Symmetric Sums and Means

$$a_1 a_2 \cdots a_k + a_1 a_3 \cdots a_{k+1} + a_2 a_3 \cdots a_{k+1} + \cdots + a_{n-k+1} \cdots a_n$$

$$= \sum_{1 \le i_1 < i_2 \cdots < i_k \le n} a_{i_1} a_{i_2} \cdots a_{i_k} := S(n,k)$$

$$s(n,k) = \left(\frac{S(n,k)}{\binom{n}{k}} \right)^{1/k}$$

$$s(n,1) \geq s(n,2) \geq \cdots \geq s(n,n) \quad \text{McLaurin}$$

$$\overbrace{}^{(AM \geq AG)}$$

$T : X \to X$, μ-ergodic, $f \wedge \log f \in L^1_\mu$

$a_i = f(T^{i-1}x)$

- $s(n,1) \longrightarrow \int f \, d\mu$
- $s(n,n) \longrightarrow \exp\left(\int \log f \, d\mu\right)$

$$\boxed{\lim_{n \to \infty} s(n,k) \ ?}$$

What if $k = k(n)$?

Example
$k(n) = \frac{n}{2}$

Solved in 1976!!
using complex integral
and saddle point
method.

$$
(b_{ij}) = \begin{array}{c}
\left[\begin{array}{cccc}
a_1 & a_2 & \cdots & a_n \\
a_1 & a_2 & \cdots & a_n \\
\vdots & & & \\
a_1 & a_2 & \cdots & a_n \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & & \vdots \\
1 & 1 & \cdots & 1
\end{array}\right]
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} k \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} n-k
\end{array}
\end{array}
$$

$$
S(n,k) = \sum_{\sigma \in S_n} b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{n\sigma(n)}
$$

Permanent of a square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix}$$

$$\mathrm{per}(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

Introduced by Cauchy and Binet in 1812!!

Compare with Determinant (no (-1)!!)

Cons for Permanent

(not geometric)

- Not related to the linear operator $A: \mathbb{R}^n \circlearrowleft$
- Not invariant under Gauss ellimination

Hard to compute!! ← active research

- $\operatorname{per}(A \cdot B) \neq \operatorname{per}(A) \cdot \operatorname{per}(B)$, nor similar.
- There is no transformation $\mathcal{D}: M_n \to M_n$

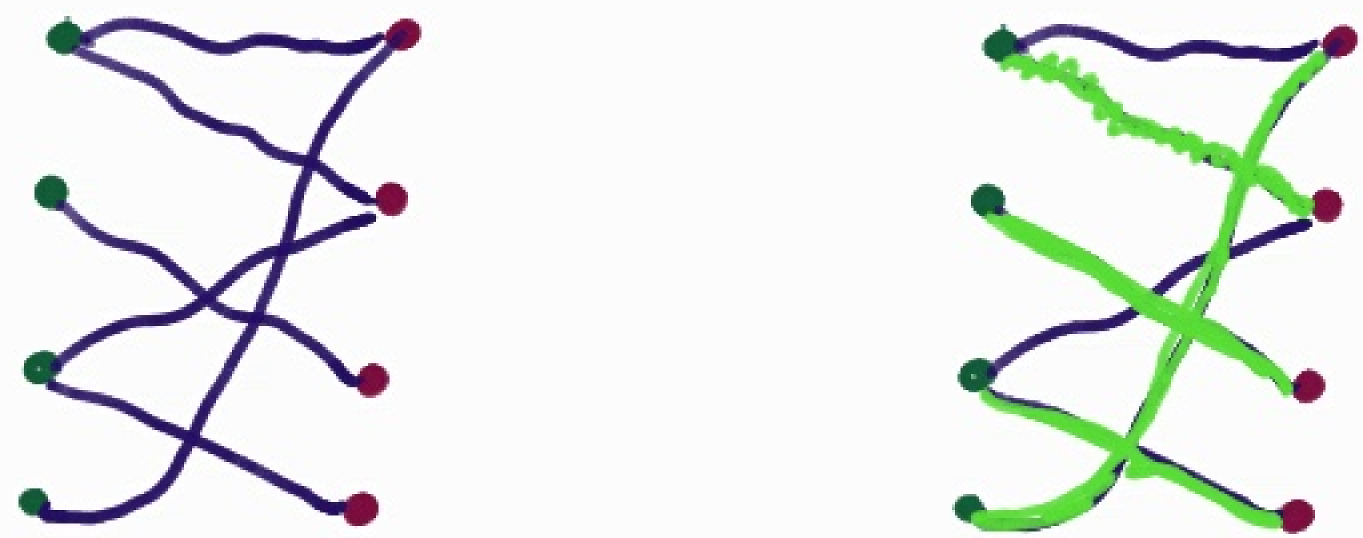
such that

$$\operatorname{per}(A) = \det(\mathcal{D}A)$$

Det and per are really different!!

Pros: Counting in Combinatorics

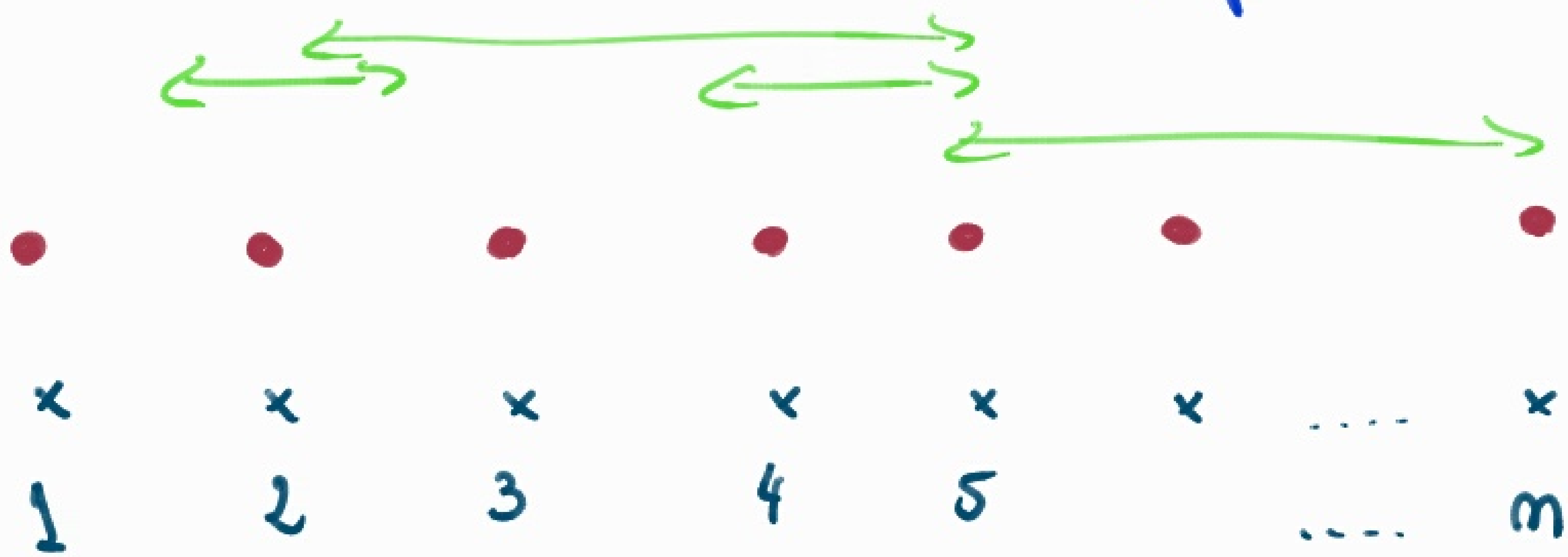
Perfect Matchings



$$\#\text{perf. match} = \operatorname{per}(\text{incidence Matrix})$$

+ many other classical problems of counting

Simultaneous transition of particles



p_{ij} – prob. ball i moves to box j

$\mathrm{per}(p_{ij})$ = prob. after simultaneous transition one ball in each box.

L. Harper.

Double stochastic Matrices Ω_n

$A = (a_{ij})_n$ $\qquad$ $\sum_i a_{ij} = \sum_j a_{ij} = 1$

$a_{ij} \geqslant 0$

$0 \leq per(A) \leq 1$ $\qquad$ Any better?

- Ω_n is convex with permutations as vertices.
- $J_n = \left(\frac{1}{n}\right)_{ij}$ at the "center"

Van der Waerden Conjecture (1926, solved in Falikman - Egorichev 1981) Independently!!

$A \in \Omega_n$

$$\boxed{\operatorname{per}(A) \geqslant \operatorname{per}(J_n) = \frac{n!}{n^n}}$$

Sinkhorn decomposition

$B = (b_{ij}),\ b_{ij} > 0$

$A = D B E \in \Omega_n$

D, E diagonals "unique"

$\operatorname{per}(A) =$

$\operatorname{per}(D)\operatorname{per}(B)\operatorname{per}(E)$

Back to means

$A = (a_{ij})$, $a_{ij} > 0$

$$\underbrace{pm(A)}_{\text{permanental mean}} = \left(\frac{per(A)}{n!}\right)^{1/n}$$

Good properties of a mean !!

Example: $a_{ij} \equiv c$, $pm(A) = c$

What about infinite matrices?
Dynamically defined entries?
Random entries?

our results !!

Obs: these are not artificial problems !!

Let (T, X, μ) and (S, Y, ν) ergodic

$f: X \times Y \to \mathbb{R}^+$. Consider the matrix

$$\boxed{A(x,y) = \left[\left(f(T^i x, S^j y) \right)_{ij} \right]_{\mathbb{N}^2}}$$

For $n \geq 1$, consider the $n \times n$ matrix

$A^{(n)}(x,y) :=$ the $n \times n$ truncation of $A(x,y)$

Two questions:

$$\lim_{n \to \infty} \mathrm{pm}\left(A^{(n)}(x,y)\right) = \text{value depending on } f, \mu \times \nu$$

Scaling Mean

$$f : X \times Y \longrightarrow \mathbb{R}^+$$

$$sm(f) = \inf_{\varphi, \psi} \frac{1}{gm(\varphi)\, gm(\psi)} \int \varphi(x)\, f(x,y)\, \psi(y)$$

where

$$gm(\varphi) = \exp\left(\int \log \varphi \, d\mu \right)$$ the geometric mean.

Think about matrices $\frac{1}{n^2} \frac{x^T F y}{gm(x)\, gm(y)}$

Properties

- $\mathrm{sm}(f) \leq \mathrm{sm}(g)$ if $f \leq g$. Monotone
- $f \equiv c \Rightarrow \mathrm{sm}(f) = c$
- $\mathrm{sm}(\tilde{\varphi} f \tilde{\psi}) = \mathrm{gm}(\tilde{\varphi})\, \mathrm{sm}(f)\, \mathrm{gm}(\tilde{\psi})$ for $\tilde{\varphi} : X \to \mathbb{R}^+$, $\tilde{\psi} : Y \to \mathbb{R}^+$ } well behaved for scalings

Double Stochastic functions $\tilde{f} \in DS$

$$\int \tilde{f}(\cdot, y)\, d\mu = \int \tilde{f}(x, \cdot)\, d\nu = 1$$

<u>Prop</u>: $\mathrm{sm}(\tilde{f}) = 1$ "Jensen Innequality AM $\geq$ GM".

Functional Sinkhorn Decomposition

Thm (B.I.P.) If $\log f \in L^\infty_{\mu \times \nu}$ then there exists "unique" scalings $\varphi : X \to \mathbb{R}^+$, $\psi : Y \to \mathbb{R}^+$ such that $\tilde{f} = \varphi f \psi$ is double-stochastic.

"Proof": Ideas by Menon ... Nussbaum

- Reduce to a fixed point problem
- Use projective Hilbert metric
- Show the operator has finite diameter and hence is contracting.

The solution is an attracting global fixedpoint.

Thm. Law of Large Permanents (BIP)

If (X, T, μ), (Y, S, ν) are ergodic and $f : X \times Y \to \mathbb{R}^+$ has $\log f \in L^\infty_{\mu \times \nu}$ then

$$\lim_{n \to \infty} \operatorname{pm}\left(A^n(x,y)\right) = \operatorname{sm}(f)$$

for $\mu \times \nu$ - almost every $(x,y) \in X \times Y$.

Proof for periodic points

$f \rightsquigarrow$ matrix M_m

$$f = \left[\left(f_{ij} \right)_{m \times m} \right] \qquad A^{km} = f \otimes \mathbb{1}_k$$

Kronecker product

$$\begin{bmatrix} \overbrace{[f]\ [f] \cdots [f]}^{k \text{ times}} \\ \vdots \\ \qquad\qquad [f] \end{bmatrix}$$

Thm Friedland '79

$$\lim_{k \to \infty} \left(\operatorname{per} \left(f \otimes J_k \right) \right)^{1/k} = \frac{1}{e^m}$$

for $f \in \Omega_m$

Sinkhorn + matrix decomp.

"General Proof"

- Use functional Sinkhorn → "reduce double stoch"
- Sharp estimates of permanents
- Birkhoff ergodic theorem to compare A^n with a Kronecker product
- $\mathbb{N}^2$-action ergodic theorem

- To do → The same thm for general $\mathbb{N}^2$-action
- iid entries

Recover Halász-Székely '76 thm

let (X, T, μ) ergodic, $\log f \in L^\infty_\mu$, $0 < \lambda < 1$.

Suppose $k(n)$ is a sequence of integers satisfying

$$\lim \frac{k(n)}{n} = \lambda$$

$$\boxed{f(T^i x) = a_i}$$

Then

$$\lim_{n \to \infty} s(n, k(n)) = \lambda \left(\frac{1-\lambda}{r} \right)^{\frac{1-\lambda}{\lambda}} \exp\left(\frac{1}{\lambda} \int \log(f+r)\, d\mu \right)$$

where r is the unique positive root of

$$\int \frac{f}{f+r}\, d\mu = \lambda$$

this was "obscure" in H-S.
Now it appears when computing Sinkhorn

Muirhead means

$$\alpha = (\alpha_1 \ldots\ldots \alpha_n) \quad , \quad z = (z_1 \ldots\ldots z_n)$$

$$\mathrm{mum}_\alpha(z) = \left(\frac{1}{n!} \sum_{\sigma \in S_n} z_{\sigma(1)}^{\alpha_1} z_{\sigma(2)}^{\alpha_2} \cdots z_{\sigma(n)}^{\alpha_n} \right)^{\frac{1}{\alpha_1+\alpha_2+\ldots+\alpha_n}}$$

- $\alpha = (1, 0, \ldots, 0)$ Arithmetic mean
- $\alpha = (1, 1, \ldots 1)$ Geometric mean
- $\alpha = (1 \ldots 1, 0 \ldots 0)$ Symmetric mean