Singular vector fields far away from horseshoe

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Framework and notions

M : d-dim, compact boundaryless Riemannian manifold

$\mathcal{X}^r(M)$: C^r vector fields on M, $r \geq 1$

Given a C^1 vector field $X : M \to TM$, X generates a C^1 flow $\phi : \mathbb{R} \times M \to M$, i.e.,

- $\phi_0 = \mathrm{id} : M \to M$,
- $\phi_{t+s} = \phi_t \circ \phi_s$, $\forall t, s \in \mathbb{R}$.

Denote $\Phi_t = \mathrm{d}\phi_t : TM \to TM$, tangent flow.

$\mathrm{Sing}(X) = \{x \in M : X(x) = 0\}$: the set of singularities of X.

x is a **periodic point** if $X(x) \neq 0$ and $\exists T > 0$ s.t. $\phi_{t+T}(x) = \phi_t(x)$, $\forall t \in \mathbb{R}$. $\mathrm{Orb}(x)$ is **periodic orbit**.

"Most" dynamics

1. open and dense subset
2. generic: countable intersection of open and dense subsets

$r = 1$ Franks lemma, closing lemma, connecting lemmas

$r > 1$???

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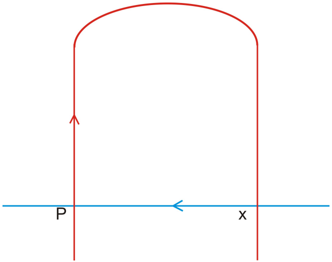
Morse-Smale system, homoclinic orbit and horseshoe

Morse-Smale system: $\Omega = \{C_1, \cdots, C_k\}$ + transversality

p : a hyperbolic fixed (periodic) point

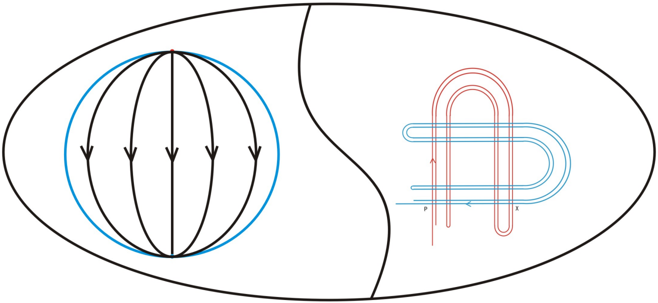
x is a homoclinic point (w.r.t p) if $W^s(p)$ intersects $W^u(p)$ transversely at x



Birkhoff-Smale Theorem: Transverse homoclinic orbit leads to horseshoe.

Palis (Weak) Density Conjecture

Palis (Weak) Density Conjecture (PWDC): The union of Morse-Smale systems and systems with horseshoe forms an open and dense subset.



Palis (Strong) Density Conjecture (PSDC): Generic system far away from homoclinic bifurcations is (**singular?**) hyperbolic.

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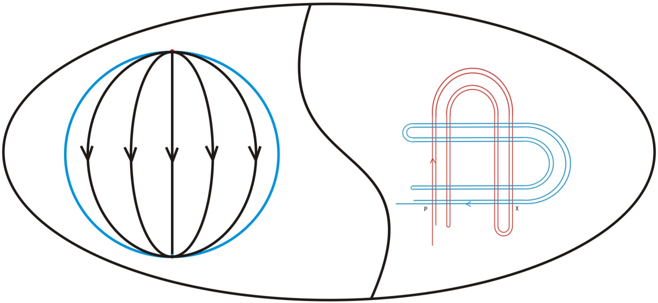

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Progress

For diffeomorphisms:

- Pujals-Sambarino, 2000, 2 dim, PSDC
- Bonatti-G-Wen, 2007, 3 dim, PWDC
- Crovisier, 2010, n dim, PWDC

For non-singular vector fields:

- Arroyo-Hertz, 2003, 3 dim, PSDC
- Xiao-Zheng, 2015, any dim, PWDC

For singular vector fields:

- G-D.Yang, 2014, arXiv, 3 dim, PWDC
- Crovisier-D.Yang, 2015, arXiv, 3 dim, PSDC

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Chain recurrent class

Chain equivalence:

$x \sim y \Leftrightarrow \forall \epsilon > 0$, $\exists \epsilon$-chains from x to y and from y to x.

$$\mathrm{CR} = \{x : x \sim x\}$$

$\sim$: closed equivalence relation over CR.

Chain recurrent class: Equivalent class of CR under $\sim$

Strategy for proving PWDC

MS: set of Morse-Smale systems

HS: set of systems with a horseshoe

If PWDC were not satisfied, take a generic system $\notin \overline{MS \cup HS}$.
Let C be a nontrivial chain class C. (for diffeo.)

Step 1. $\exists$ nontrivial minimal set $\Lambda \subset$ with partially hyperbolic splitting

$$T_\Lambda M = E^s \oplus E^c \oplus E^u, \quad \dim E^c = 1.$$

Step 2. Analyze the dynamics of center leaves to get a contradiction.

C contains a singularity! What does "partially hyperbolic splitting" look like?

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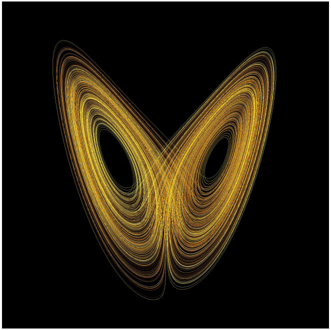
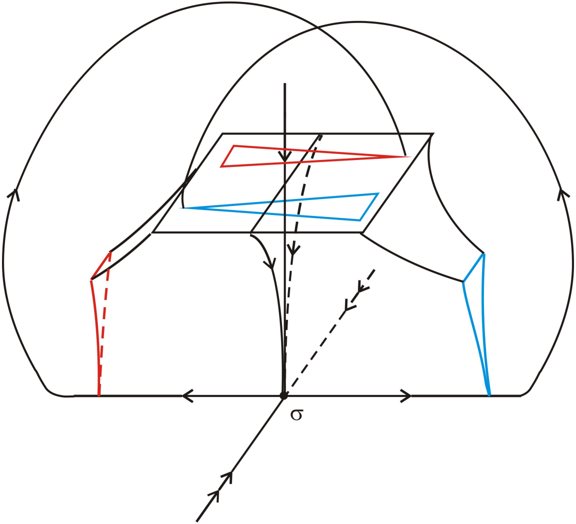
Lorenz attractor

$$\begin{aligned} x' &= \sigma(y - x), & \sigma &= 10, \\ y' &= \rho x - y - xz, & \rho &= 28, \\ z' &= xy - \beta z, & \beta &= 8/3. \end{aligned}$$



Geometric Lorenz flow



Guckenheimer-Williams, Afraimovich-Bykov-Shilnikov

Main result

Theorem 1. (G-Zheng) For C^1 generic v.f. $X \in \mathcal{X}^1(M^d) - \overline{\mathrm{MS} \cup \mathrm{HS}}$, X (or $-X$) has a singularity σ s.t. the chain recurrent class $C(\sigma)$ is nontrivial and

1. Every singularity in $C(\sigma)$ has the same index, i.e., $\mathrm{Ind}(\sigma)$.
2. Every singularity in $C(\sigma)$ is **Lorenz-like**.
3. $C(\sigma)$ admits a partially hyperbolic splitting.

Lorenz-like singularity

Let σ be a hyperbolic singularity of a v.f. X.

If the Lyapunov exponents of $\Phi_t(\sigma)$ is

$$\lambda_1 \leq \cdots \leq \lambda_s < 0 < \lambda_{s+1} \leq \cdots \leq \lambda_d,$$

the **saddle value** of σ is

$$\mathrm{sv}(\sigma) = \lambda_s + \lambda_{s+1}.$$

σ is **Lorenz-like** if $\mathrm{sv}(\sigma) \neq 0$, say $\mathrm{sv}(\sigma) > 0$, and

- $E^s(\sigma) = E^{ss}(\sigma) \oplus E^{cs}(\sigma)$ with $\dim E^{cs}(\sigma) = 1$,
- $W^{ss}(\sigma) \cap C(\sigma) = \{\sigma\}$.

Main result-restate

Theorem 1. (G-Zheng) For C^1 generic v.f. $X \in \mathcal{X}^1(M^d) - \overline{\mathrm{MS} \cup \mathrm{HS}}$, X (or $-X$) has a singularity σ s.t. the chain recurrent class $C(\sigma)$ is nontrivial and

1. Every singularity in $C(\sigma)$ has the same index, i.e., $\mathrm{Ind}(\sigma)$.
2. Every singularity in $C(\sigma)$ is **Lorenz-like**.
3. $C(\sigma)$ admits a partially hyperbolic splitting. Precisely,
   - if $\mathrm{sv}(\sigma) > 0$,
   $$T_{C(\sigma)}M = E^{ss} \oplus E^{cu}, \quad \dim E^{ss} = \dim E^{s}(\sigma) - 1.$$
   - if $\mathrm{sv}(\sigma) < 0$,
   $$T_{C(\sigma)}M = E^{cs} \oplus E^{uu}, \quad \dim E^{uu} = \dim E^{u}(\sigma) - 1.$$

   Especially, if $C(\sigma)$ contains a singularity ρ s.t. $\mathrm{sv}(\sigma)\mathrm{sv}(\rho) < 0$, then
   $$T_{C(\sigma)}M = E^{ss} \oplus E^{c} \oplus E^{uu}, \quad \dim E^{c} = 2.$$

4-dimensional case

Theorem 2. (G-Zheng) For C^1 generic v.f. $X \in \mathcal{X}^1(M^{\mathbf{4}}) - \overline{\mathrm{MS} \cup \mathrm{HS}}$, X (or $-X$) has a singularity σ s.t. $C(\sigma)$ is nontrivial and

1. Every singularity in $C(\sigma)$ has the same index, i.e., $\mathrm{Ind}(\sigma)$.
2. Every singularity in $C(\sigma)$ is **Lorenz-like**.
3. $C(\sigma)$ admits a partially hyperbolic splitting.Precisely,
   - if $\mathrm{Ind}(\sigma) = 3$ then $\mathrm{sv}(\sigma) > 0$, $C(\sigma)$ is Lyapunov stable and $T_{C(\sigma)}M = E^{ss} \oplus E^{cu}$ with $\dim E^{ss} = 2$.
   - if $\mathrm{Ind}(\sigma) = 1$ then $\mathrm{sv}(\sigma) < 0$, $C(\sigma)$ is Lyapunov stable for $-X$, $T_{C(\sigma)}M = E^{cs} \oplus E^{uu}$ with $\dim E^{uu} = 2$.
   - if $\mathrm{Ind}(\sigma) = 2$ and $\mathrm{sv}(\sigma) > 0$, $T_{C(\sigma)}M = E^{ss} \oplus E^{cu}$ with $\dim E^{ss} = 1$. Moreover, E^{cu} is **volume-expanding** and $C(\sigma)$ is **NOT** Lyapunov stable. Especially, if $C(\sigma)$ contains a singularity ρ s.t. $\mathrm{sv}(\sigma)\mathrm{sv}(\rho) < 0$, then $T_{C(\sigma)}M = E^{ss} \oplus E^{c} \oplus E^{uu}$ with $\dim E^{c} = 2$.

Corollary. (G-Zheng) For C^1 generic v.f. $X \in \mathcal{X}^1(M^{\mathbf{4}})$, if X (or $-X$) has a singularity σ with index 2, $C(\sigma)$ is Lyapunov stable and **singular hyperbolic**, then $C(\sigma)$ contains periodic orbits with **complex eigenvalues**.

Following Morales-Pacifico-Pujals, a compact invariant set Λ of X is **singular hyperbolic** if $\exists$ a Φ_t-invariant partially hyperbolic splitting $T_\Lambda M = E^{ss} \oplus E^{cu}$ s.t. E^{ss} is uniformly contracting, and E^{cu} is **sectional-expanding**, i.e., for any 2-dim subspace $L \subset E^{cu}$, $\Phi_t|L$ is uniformly **area-expanding**.

Bonatti-Pumariño-Viana, 1997: attractors in Corollary exist.

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4-dimensional case-restriction on splitting

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Bonatti's conjecture

Bonatti's conjecture: C^1 generically, every non-trivial singular chain class contains periodic orbits.

Remark: Bonatti's conjecture implies PWDC.

Open for $\dim = 3$ (G-D.Yang for $E^{ss} \oplus E^{cu}$)

Singular hyperbolic case:

- If the class is **Lyapunov stable**, **YES**:
  Morales-Pacifico, 3-dim, 2003,
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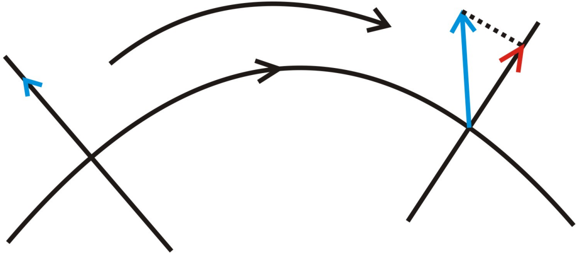
Linear Poincaré flow

$\Phi_t = \mathrm{d}\phi_t : TM \to TM$: tangent flow

$$N = \bigcup_{x \notin \mathrm{Sing}(X)} N_x, \quad N_x = \{v \in T_x M : v \perp X(x)\}$$

Linear Poincaré flow $\psi_t : N \to N$,

$\psi_t(v) =$ the orthogonal projection on N of $\Phi_t(v)$,



Basic property for systems far away from tangencies

HT: set of v.f. with a tangency associated to a **periodic orbit**

Theorem: (Wen, 2002) Let $X \notin \overline{HT}$. Then $\exists$ nbhd $\mathcal{U}$ of X and $T > 0$ s.t. $\forall Y \in \mathcal{U}$ and any hyperbolic periodic point p of Y with period $\geq T$,

$$\frac{\|\psi_T|N^s(p)\|}{m(\psi_T|N^u(p))} \leq \frac{1}{2},$$

where, $N(p) = N^s(p) \oplus N^u(p)$ is the hyperbolic splitting at p.

Pujals-Sambarino, 2000, 2-dim

ψ_t can only be defined on $M - \mathrm{Sing}(X)$ which is **NOT** compact.

ψ_t has a natural compactification: **extended linear Poincaré flow**

Let $G^1(M)$ be the projective bundle of TM,

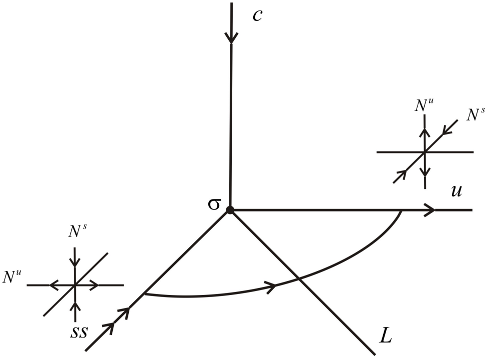
$\beta : G^1(M) \to M$ the bundle projection.

$$N = \{(L, v) \in \beta^*(TM) \subset G^1(M) \times TM : v \perp L\}.$$

Then we can define **extended linear Poincaré flow** $\psi_t : N \to N$, $\psi_t(L, v) = (\Phi_t(L), \pi\Phi_t(v))$, where π is the orthogonal projection along L.

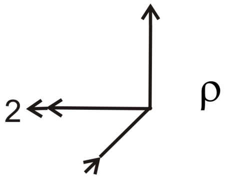
Lorenz-like

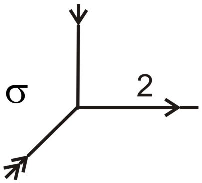
Lemma: Assume $X \notin \overline{HT}$. If $C(\sigma)$ is nontrivial, then σ is **Lorenz-like**.



Key: Assume $\mathrm{sv}(\sigma) > 0$. Find a $(\mathrm{Ind}(\sigma) - \mathbf{1})$-domination over the homoclinic loop. Then, use the observation in Li-G-Wen to get a contradiction.

Assumption:
$\dim M = 4, \mathrm{Ind}(\sigma) = 2, \mathrm{Ind}(\rho) = 1, \mathrm{sv}(\sigma) > 0, \mathrm{sv}(\rho) < 0.$





Homogeneity of singularity

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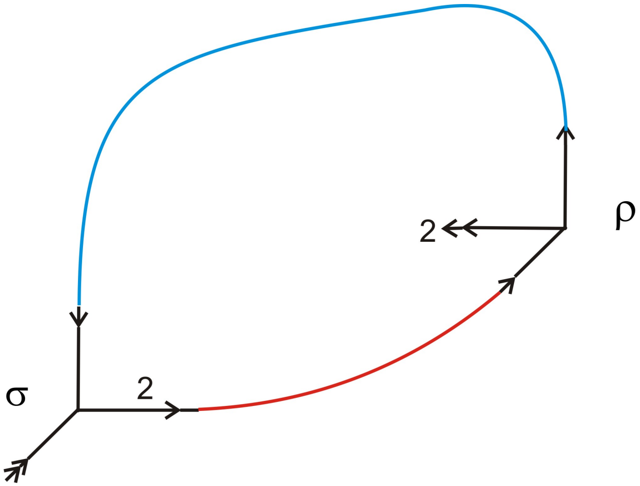

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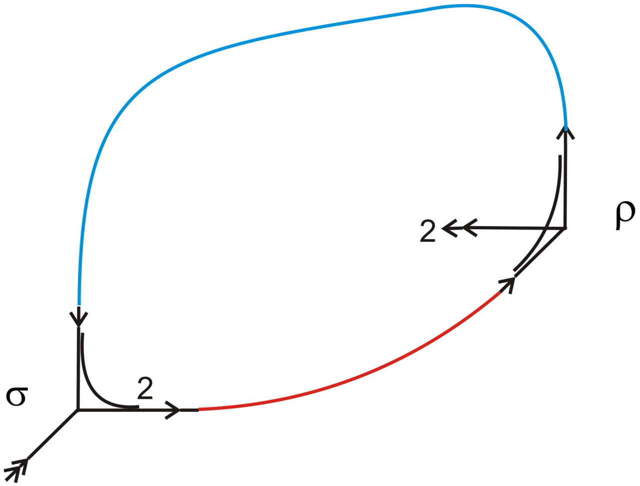

Local star property

$\dim M = 4, \mathrm{Ind}(\sigma) = 2$, $C(\sigma)$ is Lyapunov stable. Then $\forall \rho \in \mathrm{Sing}(x) \cap C(\sigma)$, $\mathrm{Ind}(\rho) = 2$ and $\mathrm{sv}(\rho) > 0$.

Local Star Property: $C(\sigma)$ is **local star**: $\exists$ nbhd $\mathcal{U}$ of X and U of $C(\sigma)$ s.t. any periodic orbit in U of any $Y \in \mathcal{U}$ is hyperbolic.

According to Shi-G-Wen or Arbieto-Morales-Santiago, $C(\sigma)$ is **singular-hyperbolic**. By Viana-J.Yang (2013, IMPA) or Arbieto-Lopez-Morales (2014, arXiv), $C(\sigma)$ contains periodic orbits. This proves that $\mathrm{Ind}(\sigma) = 3$.

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Volume expanding

If local star property is not satisfied, then $\exists \Lambda \subset C(\sigma)$ s.t. N_Λ has $(2,1)$ domination. This gives $E^u(\rho) = E^c(\rho) \oplus E^{uu}(\rho)$ and $\mathbf{W^{uu}}(\rho) \cap \mathbf{\Lambda} = \{\rho\}$ for any $\rho \in \mathrm{Sing}(X) \cap \Lambda$. And

$$T_\Lambda M = E^{ss} \oplus E^c \oplus E^{uu}.$$

Lemma. Every nontrivial invariant measure supported on $C(\sigma)$ should be supported on Λ. And

$$T_{C(\sigma)} M = E^{ss} \oplus E^{cu},$$

where E^{cu} is **volume expanding**.

SRB-like measure and entropy

Theorem. (Catsigeras-Enrich, 2011) Let $f : M \to M$ be a homeo on a cpt manifold M. $\exists$ the smallest compact set $K \subset \mathcal{M}_{inv}(f)$ s.t. for Lebesgue a.e. x, the limit points of

$$\frac{1}{n}(\delta_x + \delta_{fx} + \cdots + \delta_{f^{n-1}x})$$

are contained in K. Measures in K are called **SRB-like** measures.

Theorem. (Catsigeras-Cerminara-Enrich, 2015; Viana-J.Yang, 2013 IMPA) Let $f : M \to M$ be C^1 **diffeo** with a domination $TM = E \oplus F$. Then for every SRB-like measure μ,

$$h_\mu(f) \geq \int \log|\det(Tf|F(x))| \mathrm{d}\mu(x).$$

Sun-Tian, 2012, for $\mu \ll Leb$.

Proof continued for 4-dim.

Lemma. Let $Y \in \mathcal{X}^{\mathbf{2}}(M)$ be C^1 close to X s.t. $T_{C(\sigma_Y)}M = E^{ss} \oplus E^{cu}$. Then $\exists$ **non-hyperbolic** ergodic μ s.t.

$$h_\mu(\phi_1^Y) = \int \log|\det(\Phi_1|E^{cu}(x))|\mathrm{d}\mu(x) > 0.$$

According to Ledrappier-Young's characterization for measures satisfying Pesin's formula, the conditional measure along strong unstable manifolds W^{uu} is absolutely continuous w.r.t Lebesgue. Hence, $W^{uu}(\rho) \subset \Lambda$, which contradicts our previous conclusion.

Thanks!