Stochastic dynamics from a deterministic one

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in progress w. P. Bálint, IP. Tóth (both BUTE), P. Nándori (NYU & Maryland),

ICTP, Dynamical Systems

Heat Equation (no mass flow)

$$\frac{\partial T(x,t)}{\partial t} = \frac{1}{c}\nabla\left[\kappa\nabla T(x,t)\right]$$

c - specific heat/unit volume $(= 1)$
$\kappa = \kappa(T)$ - thermal conductivity

For a wide class of models: $\kappa(T) = \mathrm{const.}\sqrt{T}$
(insulating materials, or gas of weakly/rarely interacting particles)

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Gaspard-Gilbert, 2008–: model of localized hard disks (balls) - a two step approach:

1. From the microscopic kinetic equ. of the Hamiltonian model derive a mesoscopic master equ.
   in the **rare (but strong) interaction limit;**
   It is a **Markov jump process.**
2. Then from the mesoscopic master equ.
   derive the macroscopic heat equ.

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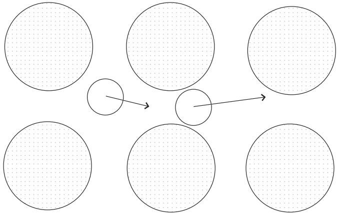

chain length $N = 2$
periodic scatterers (shaded disks)
confined moving disks (white circles)

Physical relevance: No mass transport
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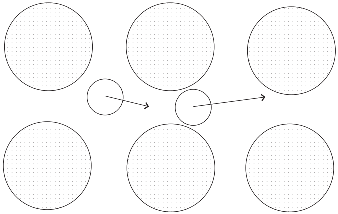

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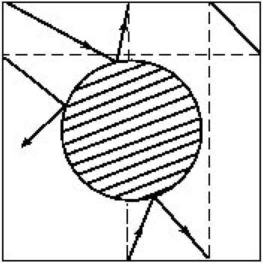
The chain is a semi-dispersing billiard

Definition. **Billiard flow** is a dynamical system $(\mathcal{M}, \{S^t | t \in \mathbb{R}\}, \tilde{\mu})$
where $\tilde{\mu}$ = Liouville-measure,
$\mathcal{M} = Q \times S_{d-1}$ (here $Q = \mathbb{T}^2 \setminus$ disk)
$\{S^t | t \in \mathbb{R}\}$: billiard dynamics = uniform motion in Q and elastic reflection at the scatterers ∂Q.



Billiard Ball Map: (M, T, μ), where $M = \partial Q \times S_{d-1}$

Definition. A billiard is

- a **dispersing** one (**Sinai-billiard**) if the scatterers are **strictly convex** (as on picture above)
- a **semi-dispersing** one if they are (only) **convex.**

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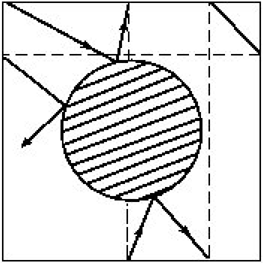

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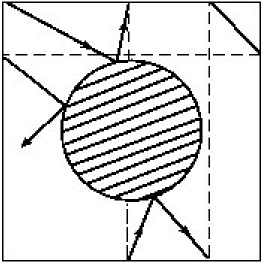

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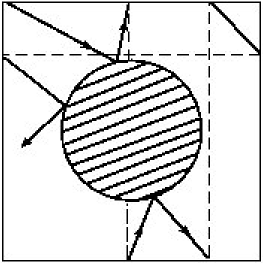

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Parameter choice of Gaspard-Gilbert,'08.

- box size: *1*; chain length $= N$;
- periodic b. c.'s along y-axis
- radius of fixed scatterers (shaded circles)$= \rho_f$
- radius of moving disks (empty circles) $= \rho_m$
- condition of confinement: $\rho_f + \rho_m > 1/2$
- condition of conductivity:
  $\rho_m > \rho_{crit} = \sqrt{(\rho_f + \rho_m)^2 - (1/2)^2}$
- small parameter $\varepsilon = \rho_m - \rho_{crit} > 0$

Rare interaction limit:

- Keep $\rho_f + \rho_m =: \rho$ fixed
- Then their phase spaces essentially only depend on ρ!
- If $\rho_m = \rho_{crit}$, then we have N non-interacting billiards.

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Liouville equation

Ernst-Dorfman, '89: The kinetic equ. for the N-particle density $p_N(q_1, v_1, \dots, q_N, v_N; t)$ is

$$\partial_t p_N = \sum_{j=1}^{N} \left(-v_j \partial_{q_j} + K_{wall,j} + C_{j,j+1}\right) p_N$$

- the first two terms on the RHS describe the billiard dynamics of each disk within its cell (denote wall collision rate by $\nu_{\mathrm{wall},\varepsilon}$)
- the third one: the binary interaction of neighboring disks provides energy transfer (denote binary collision rate by $\nu_{\mathrm{bin},\varepsilon}$)

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Scale separation as

$\varepsilon \searrow 0$, i. e. $\nu_{\text{wall},\varepsilon} (\sim \nu_{wall,crit} > 0) \gg \nu_{\text{bin},\varepsilon} \to 0$

1. **GG08 derives a master equation for the density** $P_N(E_1, \ldots, E_N; t)$ $(E_j = v_j^2 : 1 \leq j \leq N)$
2. HDL: **from the master equation they obtain the coefficient of heat conductivity**: $\kappa = \text{const.}\sqrt{T}$

1. Dynamical: Kinetic Equation $\Longrightarrow$ Markov Generator of a Markov jump process of energies
2. Probabilistic: Markov Jump Process $\Longrightarrow$ Heat Equation (Hydrodynamic Limit à la Varadhan)
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Actual challenge. Part 1: Rigorous theory for GG

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Mesoscopic equ. is a system of interacting stochastic differential equ.'s.

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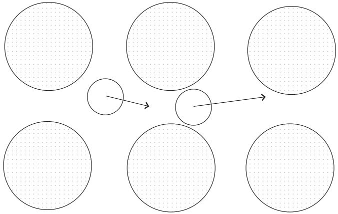
Dynamical approach for step 1. Warm-up result

By Hirata-Saussol-Vaienti, 1999:
If

- *a dynamical system* (M, T, μ) *is mixing in a controlled way (e. g.* α*-mixing)*
- *and* A_ε *is a sequence of nice subsets (e. g. to avoid e. g. neighborhoods of periodic points) with* $\lim_{\varepsilon\to 0} \mu(A_\varepsilon) = 0$

then the **successive entrance times of the dynamics into** A_ε **form a Poisson process on the time scale of** $\mu(A_\varepsilon)^{-1}$.

More quantitative conditions for entrance times to infinitesimal balls $B_r(x)$; $x \in M$: Chazotte-Collet, 2013



Free bry conditions along
x-axis.
$N = 2$
$\dim Q = 4; \quad \dim \mathcal{M} = 7;$
$\dim \gamma^u = \dim \gamma^s = 3$

The model is isomorphic to a 4D semi-dispersing billiard.
It is K-mixing (Bu-Li-Pe-Su, ’92), but no mixing rate is known.

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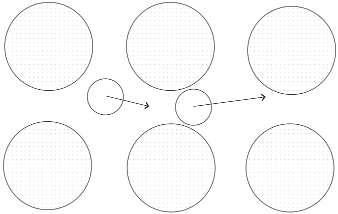

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Standard pairs and Markovity. $D = 2$

(M, T, μ) billiard ball map. dim $M = 2$.

Standard pair: $\ell = (W, \rho)$, where W is an unstable curve, ρ: nice probability density on W.

Decompose M into a proper family of standard pairs. Let $\ell = (W, \rho)$ be a standard pair, $A : M \to \mathbb{R}$. By law of total prob.

$$\mathbb{E}_\ell(A \circ T^n) = \sum_\alpha c_{\alpha n} \mathbb{E}_{\ell_{\alpha n}}(A)$$

where $c_{\alpha n} > 0$, $\sum_\alpha c_{\alpha n} = 1$. Moreover, $\ell_{\alpha n} = (W_{\alpha n}, \rho_{\alpha n})$ are standard pairs with $T^n W = \cup_\alpha W_{\alpha,n}$ where $\rho_{\alpha n}$ is the pushforward of ρ up to a multiplicative factor.

Theorem (Growth lemma $\sim$ Markov property)

$\exists, \beta_1, \beta_2 > 0$ *s. t.* $\forall \varepsilon > 0, \forall n \geq \log \frac{1}{|W|^{\beta_1}}$

$$\sum_{\text{length}(\ell_{\alpha n}) < \varepsilon} c_{\alpha n} \leq \beta_2 \varepsilon.$$

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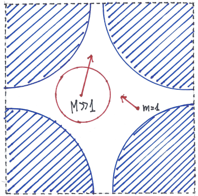
Chernov-Dolgopyat model

Disk of mass $M \gg 1$ and a point of mass $m = 1$
dim $Q = 4$; dim $\mathcal{M} = 7$
Since $M \gg 1$, the disk is *almost* still.
Consequently the 2-particle billiard is *almost* a 2D Sinai billiard.
One can *almost* use **its** standard pairs.



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Masses are equal, effect of collisions of the particles is drastic!
As opposed to Ch-D model, **unstable cones of the GG-model are far from those of the 2D Sinai billiard**

Problem at *certain* collisions of the two disks **an unstable direction (of the one-particle billiard!)** can go into a central-stable cone of the 4-D flow!
Added to that we can not characterize these unpleasant collisions in GG08.
However, we can do this for the piston model!!

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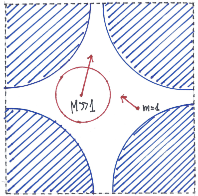

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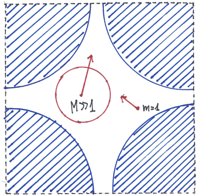


Disk of mass $M \gg 1$ and a point of mass $m = 1$
dim $Q = 4$; dim $\mathcal{M} = 7$
Since $M \gg 1$, the disk is *almost* still.
Consequently the 2-particle billiard is *almost* a 2D Sinai billiard.
One can *almost* use **its** standard pairs.

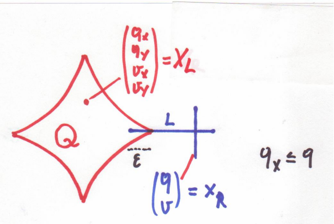
GG-model

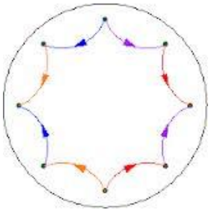
Masses are equal, effect of collisions of the particles is drastic!
As opposed to Ch-D model, **unstable cones of the GG-model are far from those of the 2D Sinai billiard**

Problem at *certain* collisions of the two disks **an unstable direction (of the one-particle billiard!**) can go into a central-stable cone of the 4-D flow!
Added to that we can not characterize these unpleasant collisions in GG08.
However, we can do this for the piston model!!

Dimension reduction: Billiard coupled to a piston

$\dim Q = 3$; $\dim \mathcal{M} = 5$; $\dim \gamma^u = \dim \gamma^s = 2$
Collision rule: $v_x^+ = v^-$, $v^+ = v_x^-$, $v_y^+ = v_y^-$. $(m_1 = m_2 = 1)$





Euclidean Model: S-billiard

Right corner of Q is at $(0, 0)$

$$(q_x, q_y) \in Q,$$
$$q \in [-\varepsilon, L - \varepsilon]$$

Hyperbolic Model:
Hyperbolic octagon
w. right angles
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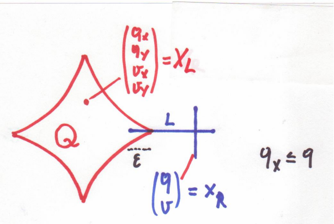

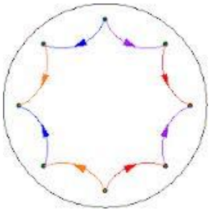

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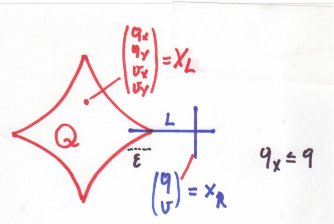

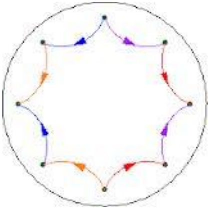

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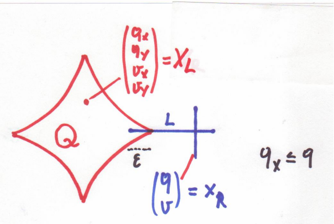

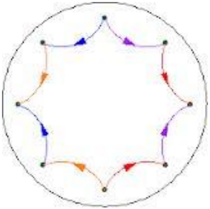

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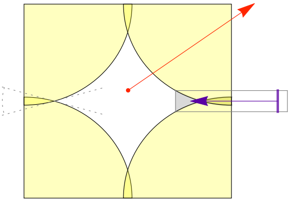
The piston model (Euclidean)



(Figure by Thomas Gilbert)

$$\mathcal{M}_{\text{bill}} = \{(q_x, q_y, v_x, v_y) \mid (q_x, q_y) \in Q, v_x^2 + v_y^2 = 1\}$$

$$\mathcal{M} = \{(q_x, q_y, v_x, v_y; q, v) | (q_x, q_y) \in Q, q \in [-\varepsilon, L - \varepsilon], \\ v_x^2 + v_y^2 + v^2 = 1\}$$

$\mathcal{M}^\varepsilon = \{(q_x, q_y, v_x, v_y; z, v) \in \mathcal{M} \mid q_x + \varepsilon \le z\}$
$\Phi^{\varepsilon,t} : \mathcal{M} \to \mathcal{M}$ – dynamics, describing the mechanical process

$E = v_x^2 + v_y^2 : \mathcal{M} \to \mathbb{R}_+$ – energy of left (billiard) particle
$\tilde{X}^\varepsilon(t) = \tilde{X}^\varepsilon(t, w) = E(\Phi^{\varepsilon,t}(w))$ – left particle energy process
initial state $w \in M$
$X^\varepsilon(t) = X^\varepsilon(t, w) = \tilde{X}^\varepsilon(\frac{t}{\varepsilon^2}, w)$ – properly rescaled energy process

$X(t)$ – Markov jump process w. kernel $K(E, E^+)$ (and rate $\Lambda(E)$)

Theorem (B-N-Sz-T)

Assume ν is the initial law for the energies of the billiard particle for both processes. For a wide class of ν, as $\varepsilon \to 0$,

$$X^\varepsilon(t) \overset{D[0,\infty)}{\Longrightarrow} X(t)$$

Proof so far for right-angled hyperbolic octagon; no corners!

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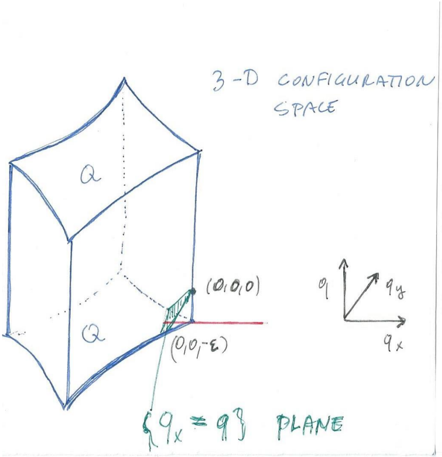
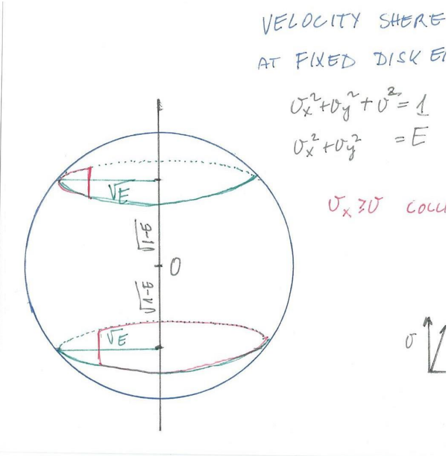
Calculation:

$$K(E,E^+) = \frac{\tan\beta}{2\pi L|Q|}\frac{\sqrt{1-\min\{E,E^+\}}}{\sqrt{1-E^+}\sqrt{E+E^+-1}}\mathbb{1}_{\{E+E^+>1\}}$$

$$\Lambda(E) = \begin{cases} \frac{\tan\beta}{2L|Q|}\sqrt{1-E} & \text{if } E<1/2 \\ \frac{\tan\beta}{\pi L|Q|}\left[\sqrt{2E-1}+\frac{\sqrt{1-E}}{2}\left(\frac{\pi}{2}-\arcsin\left(3-\frac{2}{E}\right)\right)\right] & \text{if } E>1/2 \end{cases}$$


3-D CONFIGURATION SPACE
Q
Q
(0,0,0)
(0,0,-ε)
q
q_y
q_x
{q_x = q} PLANE


VELOCITY SHERE
AT FIXED DISK E
$v_x^2 + v_y^2 + v^2 = 1$
$v_x^2 + v_y^2 = E$
$v_x \geq v$ COLL
$\sqrt{E}$
$\sqrt{1-E}$
0
$\sqrt{1-E}$
$\sqrt{E}$
v

Main components of proof

1. Let (W^u, ϕ) be a standard pair, i. e. assume W^u is an unstable curve in $\mathbb{R}^3$ - with a prob. density ϕ on it - satisfying
   - $\forall x \in W^u \qquad dW^u(x) \subset C^u_x$, the unstable cone at x
   - its curvature $\leq \Gamma_{\max} < \infty$.
2. Upper bd for 2D Sinai billiard flow correlation decay rate

Definition

Given $F : \mathcal{M} \to \mathbb{R}$, $x \in \mathcal{M}$, $\quad r > 0$, denote $\operatorname{osc}_r(F, x) = \sup_{B_r} F - \inf_{B_r} F$, where $B_r(x) \in \mathcal{M}$ is the ball of radius r centered at x.
F is **generalized Hölder** if $\exists \alpha > 0$, s. t.

$$\|F\|_\alpha := \sup_r r^{-\alpha} \int_{\mathcal{M}} \operatorname{osc}_r(F, x) d\tilde{\mu}(x) < \infty.$$

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$$\operatorname{var}_\alpha(F) = ||F||_\alpha + \sup_{\mathcal{M}} F - \inf_{\mathcal{M}} F$$

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Theorem (Chernov (2007))

Let $\Phi^t : \mathcal{M} \to \mathcal{M}$ be a Lorentz process billiard flow (finite horizon, no corners) and $F, G : \mathcal{M} \to \mathbb{R}$ two generalized Hölder continuous functions with $\int_{\mathcal{M}} F d\mu = 0$. Then

$$|\int_{\mathcal{M}} (F \circ \Phi^t) G d\tilde{\mu})| \leq c \mathrm{var}_\alpha(F) \mathrm{var}_\alpha(G) e^{-a\sqrt{|t}}$$

Here $c, a > 0$ depend on α and the billiard table only.

Theorem (B-N-Sz-T, asymptotic equidistribution)

Let $\ell = (W, \phi)$ be a standard pair and let $\tilde{\phi}$ be the measure on $\mathcal{M}$ given by it. Let $F : \mathcal{M} \to \mathbb{R}$ be a generalized Hölder continuous function with $\int_{\mathcal{M}} F d\tilde{\mu}(x) = 0$. Then for all $t > 0$

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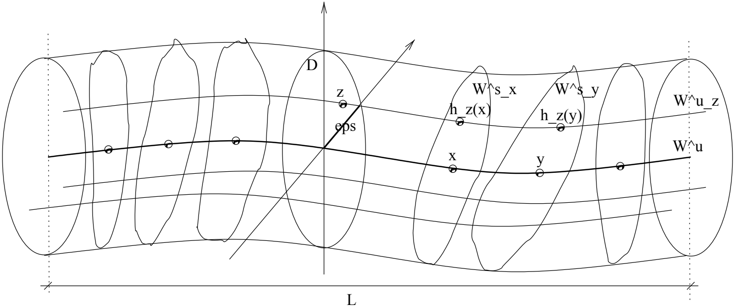
Proof of asymptotic equidistribution

Idea: blow up the standard pair i. e. take a small neighborhood U of size $\varepsilon = \varepsilon(t)$ of product structure around the unstable curve W^u and define a nice density function $G = G_t$ on it. Will apply Chernov's Thm to F and to this particular G.



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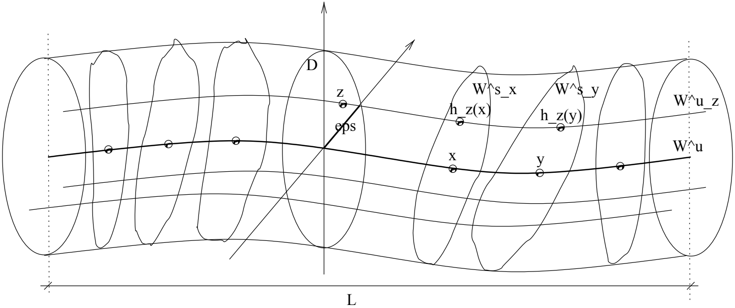

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$$\tilde{\mu}(B) = \int_D \mu^{\text{cond}}_{W^u_z}(B) d\mu^{\text{factor}}_D(z)$$

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(For $x \in H \quad Area(W^{cs}_x) \sim \varepsilon^2 \pi$.)

Lemma

$$\mu_{W^u}(W^u \setminus H) \leq C\varepsilon$$

Denote: $U_0 = \cup_{x \in H}(W^{cs_x} \cap U) \subset U$

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For $z \in D$ let $H_z = W^u_z \cap U_0$. Then $\exists C < \infty$, s. t. $\forall z \in D$

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$\{W_j^{cs}\}_{j\in I}$ is set of *all* W^{cs}'s inside U. Denote $J = I \setminus H$.

Measurable partition of U:

$\tilde{\mu}_U = \mu_I^{\text{factor}} \times \nu$ where the ν-s are the conditional measures on each W_j^{cs}.

I. e. $\nu : I \times \mathcal{B}(U) \to [0,1]$

Let further $\gamma = \mu_I^{\text{factor}}\Big|_H = \mu_H^{\text{factor}}$ and $m = \text{Leb}\Big|_H$.

Choose prob. density $q = q_\varepsilon \in C^2(D \to \mathbb{R}_+)$ s. t.

$$||q||_{\sup} \le \frac{1000}{\varepsilon^2} \qquad ||\nabla q||_{\sup} \le \frac{1000}{\varepsilon^3}$$

with both $q, \nabla q$ vanishing outside ∂D.

Define the measure ρ on U_0 as: $\rho(A \times B) = \int_A \phi dm \int_B q dm_D$.

Let $G_0 : U_0 \to \mathbb{R}_+$ be defined as $G_0 = \frac{d\rho}{d\mu}$.

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$$||q||_{\sup} \le \frac{1000}{\varepsilon^2} \qquad ||\nabla q||_{\sup} \le \frac{1000}{\varepsilon^3}$$

with both $q, \nabla q$ vanishing outside ∂D.
Define the measure ρ on U_0 as: $\rho(A \times B) = \int_A \phi dm \int_B q dm_D$.
Let $G_0 : U_0 \to \mathbb{R}_+$ be defined as $G_0 = \frac{d\rho}{d\mu}$.

$\{W_j^{cs}\}_{j\in I}$ is set of *all* W^{cs}'s inside U. Denote $J = I \setminus H$.
Measurable partition of U:
$\tilde{\mu}_U = \mu_I^{\text{factor}} \times \nu$ where the ν-s are the conditional measures on each W_j^{cs}.
I. e. $\nu : I \times \mathcal{B}(U) \to [0, 1]$
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Proposition

For a. e. $x \in H$

$$\int_{W_x^{cs}} G_0(r) d\nu_x(r) = \frac{\phi(x)}{\beta(x)}$$

Reminder: ϕ - st. pair density on H, β - density of $\gamma = \mu_H^{\text{factor}}$.
($\beta := \frac{\mathrm{d}\gamma}{\mathrm{d}m} : H \to \mathbb{R}^+$.)

Final step: from G_0 uniformly *dynamically* Hölder on U_0 to G Hölder on U.

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Main components of proof continued

3. Proof of Markovity
4. Establish the kernel $K(x, A)$
5. Handling unstable direction turning into central-stable one.