Ergodic Properties of the Kusuoka measure

Mark Pollicott
Warwick University
Trieste, 6 August 2015
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Overview

- We want to consider certain measures that were originally introduced in the context of Fractals, such as the Sierpiński Triangle. These are the classical Kusuoka measures.

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A little context

In the first talk of the conference by Keith Burns on Monday morning, he spoke about Gibbs measures (equilibrium states) for non-uniformly hyperbolic systems (geodesic flows) and regular (Hölder continuous) potential.
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In this talk we will discuss Gibbs measures for a very simple uniformly hyperbolic system and non-Hölder continuous (discontinuous) potentials.
Introduction

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- Replace a triangle in the plane by the three triangles of half the size in the corners.
- We can then do the same for each of these three triangles.
- We continue this process iteratively to get the “fractal” $X$. 
Waclaw Sierpiński (1882-1969) was a distinguished polish number theorist and set theorist. In 1951 the Warsaw Scientific Society issued a medal in his honour.
Replacing the Sierpiński Triangle $X$ by a space of sequences $\Sigma$

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- We can define a metric on $\Sigma$ by

$$d\left((x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty}\right) = \sum_{n=0}^{\infty} \frac{e(x_n, y_n)}{2^n} \quad \text{where} \quad e(i, j) = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$
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- We can introduce some dynamics by $\sigma : \Sigma \to \Sigma$ the usual shift map given by

$$\sigma(x_n)_{n=0}^\infty = (x_{n+1})_{n=0}^\infty.$$ (This essentially corresponds to a map on $X$ which doubles distances).
Measures

The coding gives a convenient viewpoint for studying probability measures on $X$, by considering measures on $\Sigma$. Recall that $\mu$ is $\sigma$-invariant if

$$\mu([x_0, \ldots, x_{n-1}]) = \sum_{i=1}^{3} \mu([i, x_0, \ldots, x_{n-1}])$$

for all cylinders $[x_0, \ldots, x_{n-1}] := \{ y = (y_n)_{n=0}^{\infty} \in \Sigma : x_j = y_j \text{ for } 0 \leq j \leq n - 1 \}$ where $x_0, \ldots, x_{n-1} \in \{1, 2, 3\}$. 

Example (Most obvious example) The $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$-Bernoulli measure on $\Sigma$ satisfies $\mu([x_0, \ldots, x_{n-1}]) = \frac{1}{3^n}$ and corresponds to the "natural" measure on $X$. Similarly, one could take Gibbs measures (for Hölder potentials $\psi : \Sigma \to \mathbb{R}$).
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Similarly, one could take Gibbs measures (for Hölder potentials $\psi : \Sigma \to \mathbb{R}$).

Definition

A $\sigma$-invariant measure $\mu$ is a Gibbs measure (for the potential $\log \psi$) if

$$\psi(x) = \lim_{n \to +\infty} \frac{\mu[x_0, \cdots, x_n]}{\mu[x_1, \cdots, x_n]}$$

satisfies that $\psi : \Sigma \to \mathbb{R}$ is Hölder continuous.
Example (Obvious example revisited)
If $\mu$ is the $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$-Bernoulli measure on $\Sigma$ then $\mu[x_0, \cdots, x_{n-1}] = \frac{1}{3^n}$ and $\log \psi(x) = -\log 3$ is a constant function.

Example (Next most obvious example)
If $\mu$ is the $\left(p_1, p_2, p_3\right)$-Bernoulli measure on $\Sigma$ (with $p_1 + p_2 + p_3 = 1$) then $\mu(x_0, \cdots, x_{n-1}) = p_{x_0}p_{x_1}\cdots p_{x_{n-1}}$ and for $x = (x_n)_{n=0}^{\infty}$:

$$\log \psi(x) = \begin{cases} 
\log p_1 & \text{if } x_0 = 1 \\
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is a locally constant function.

More generally, in the standard Gibbs theory approach one likes the potential $\log \psi$ to be Hölder continuous. However, the Kusuoka measure is defined in a different sort of way and has a different kind of potential...
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Kusuoka measure

The Kusuoka measure was originally defined on the Sierpiński triangle $X$, but to describe the corresponding measure $\mu$ on $\Sigma$ we want to specify the measure of cylinder sets

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For concreteness, let us begin with the classical case.

**Definition (Classical Kusuoka measure)**

Let $A_1 = \begin{pmatrix} 3 \sqrt{15} & 0 \\ 0 & 1 \sqrt{15} \end{pmatrix}$, $A_2 = \frac{\sqrt{5}}{2} \begin{pmatrix} \sqrt{3} & 1 \\ 5 & \frac{1}{\sqrt{3}} \end{pmatrix}$ and $A_3 = \frac{\sqrt{5}}{2} \begin{pmatrix} \sqrt{3} & -1 \\ 5 & \frac{1}{\sqrt{3}} \end{pmatrix}$.

Let $E = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

We define

$$\mu([[i_0, \cdots, i_{n-1}])] = \text{trace} \left( (A_{i_0} \cdots A_{i_{n-1}})^T E (A_{i_0} \cdots A_{i_{n-1}}) \right) \text{ for } i_0, \cdots, i_{n-1} \in \{1, 2, 3\}$$
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- The measure $\mu$ is $\sigma$-invariant (by explicit computation).
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- The measure \( \mu \) is well defined (by explicit computation).
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Theorem (Kusuoka, 1989)

The measure \( \mu \) is ergodic.
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**Theorem (Kusuoka, 1989)**

*The measure \( \mu \) is ergodic.*

The corresponding measure on \( X \) is important in defining the “Laplacian” on the fractal.
The potential for the Kusuoka measure

We can attempt to define the “potential”

$$\psi(x) = \lim_{n \to +\infty} \frac{\mu[x_0, \cdots, x_n]}{\mu[x_1, \cdots, x_n]}.$$ 

If $\psi : \Sigma \to \mathbb{R}$ were Hölder continuous then we could apply general ideas from “theormodynamical formalism”.

Theorem (Bell-Ho-Strichartz, 2014)

There exist a dense set of discontinuities for $\psi(x)$. Despite this, it is possible to establish familiar ergodic properties.
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Main result

We can prove stronger ergodic results, such as exponential mixing:

**Theorem (Johansson-Öberg-P.)**

The measure \( \mu \) mixed exponentially fast, i.e., there exists \( 0 < \alpha < 1 \) such that for Lipschitz \( f_1, f_2 : \Sigma \to \mathbb{R} \) we can find \( C > 0 \) with

\[
\left| \int f_1 \circ \sigma^n f_2 \, d\mu - \int f_1 \, d\mu \int f_2 \, d\mu \right| \leq C \alpha^n
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In fact, \( \alpha \) isn’t very mysterious - we can take any value \( \frac{5}{7} < \alpha < 1 \).
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Anders Öberg and Anders Johansson (explaining something very patiently to a coauthor)
The ergodicity of the Kusuoka measure gives the

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The Birkhoff Ergodic Theorem was mentioned in the talk of Corinna Ulcigrai.
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The Birkhoff Ergodic Theorem was mentioned in the talk of Corinna Ulcigrai. Our result leads to various strengthenings via stronger statistical results under the stronger assumption that $f : \Sigma \to \mathbb{R}$ is Lipschitz.
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1. Central Limit Theorems
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3. Pointwise error terms in the Birkhoff ergodic theorem.
1. Central Limit Theorems

As we observed, ergodicity of the measure $\mu$ implies that:

**Theorem (Birkhoff Ergodic Theorem)**

For any $L^1(\Sigma, \mu)$ function $f : \Sigma \to \mathbb{R}$ we have that

$$\frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n x) \to \int f d\mu \text{ as } N \to +\infty,$$

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The central limit theorem gives stronger results where $1/N$ is replaced by $1/\sqrt{N}$.

**Theorem (Central Limit Theorem)**

Assume $f : \Sigma \rightarrow \mathbb{R}$ is a Lipschitz function not cohomologous to a constant (i.e., $f - \int fd\mu = u \circ \sigma - u$ where $u \in B$). Then there exists $\sigma^2 > 0$ such that we have that for any $\alpha < \beta$ we have

$$
\mu \left( \left\{ x \in \Sigma : \alpha \leq \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(\sigma^n x) - \int f d\mu \leq \beta \right\} \right) \rightarrow \frac{1}{\sqrt{2\pi \sigma}} \int_{\alpha}^{\beta} e^{-\frac{u^2}{2\sigma^2}} du
$$

as $N \rightarrow +\infty$. 

2. Large Deviation results

Recall that:

**Theorem (Birkhoff Ergodic Theorem)**

*For any $L^1(\Sigma, \mu)$ function $f : \Sigma \rightarrow \mathbb{R}$ we have that*

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Another form of generalisation of the Birkhoff theorem is the following.

**Theorem (Large Deviation Theorem)**

Let $f : \Sigma \to \mathbb{R}$ be Lipschitz. For each $\epsilon > 0$ there exists $C > 0, 0 < \rho < 1$ such that

$$\mu \left( \left\{ x \in \Sigma : \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n x) - \int f d\mu \right| > \epsilon \right\} \right) \leq C \rho^n$$

as $N \to +\infty$.

There is also be a corresponding version for measures $\frac{1}{N} \sum_{n=0}^{N-1} \delta_{\sigma^n x}$.
3. Error terms for Birkhoff Ergodic Theorem

Recall yet again that:

**Theorem (Birkhoff Ergodic Theorem)**

*For any $L^1(\Sigma, \mu)$ function $f : \Sigma \to \mathbb{R}$ we have that*

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The following is a simple consequence of the mixing.

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Let $f : \Sigma \to \mathbb{R}$ be Lipschitz. We can deduce that, for any $\delta > 0$ can write

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In particular, for any $\epsilon > 0$ can write

$$
\frac{1}{N} \sum_{n=0}^{N-1} f(\sigma^n x) = \int f d\mu + O \left( \frac{1}{N^{1/2-\epsilon}} \right).
$$
The strategy of the proof

Let us define a function $g : X \to \mathbb{R}$ by

$$
\psi(x) = \lim_{n \to +\infty} \frac{\mu[x_0, \cdots, x_{n-1}]}{\mu[x_1, \cdots, x_{n-1}]} \text{ for } a.e. (\mu)x = (x_n)_{n=0}^\infty \in \Sigma.
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Thus we need

- to define a suitable \( B \); and
- to prove there is a spectral gap for \( L \).
The space of functions $B$

Let $\mathcal{A}_n$ ($n \geq 0$) be the finite sigma algebra consisting of all cylinders $[i_0, \cdots, i_{n-1}]$ of length $n$ (N.B. those traditionally used in the definition of entropy).

Given $0 < \theta < 1$, let $B = B_\theta := \{ f : \| f \|_2^\theta := \infty \sum_{n=1}^{\infty} \| E(f | A_n) - E(f | A_{n-1}) \|_2^2 \theta^n < +\infty \}$ with norm $\| f \| = \| f \|_2 + \| f \|_\theta$.

Providing $\theta > \frac{1}{2}$ we have that $B$ contains the Lipschitz functions.
Let $\mathcal{A}_n$ ($n \geq 0$) be the finite sigma algebra consisting of all cylinders $[i_0, \ldots, i_{n-1}]$ of length $n$ (N.B. those traditionally used in the definition of entropy).

Let $\mathbb{E}(\cdot | \mathcal{A}_n) : L^2(X, \mathcal{B}, \mu) \to L^2(X, \mathcal{B}, \mu)$ be the usual expectation projection, i.e.,

$$\mathbb{E}(f | \mathcal{A}_n)(x) = \frac{\int_{[x_0, \ldots, x_{n-1}]} f d\mu}{\mu[x_0, \ldots, x_{n-1}]}$$

where $x = (x_n)_{n=0}^\infty$,

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The spectrum and an indirect approach

There is an operator theorem has a reassuringly familiar statement

**Theorem**

For $0 < \theta < 1$ sufficiently large, $L : B \to B$ defined by $Lf(x) = \sum_{\sigma y = x} g(y)f(y)$ is well defined. Moreover,

- $L(1) = 1$ (i.e., preserves the constant functions $\mathbb{C}$)
- The spectral radius of $L : B/\mathbb{C} \to B/\mathbb{C}$ is strictly smaller than 1

In particular, there is a spectral gap, as required.
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One might hope that a “traditional approach” would lead to the results on the spectrum of $L$ (e.g., Lasota-Yorke inequality, etc.).

Unfortunately, we couldn’t get that to work - so there is a more indirect approach working with Banach spaces of matrix valued functions, and operators on these (which then project down to operators on functions of the above form).
Generalizations

The Kusuoka measure is a special case of a more general class of invariant measures on $\Sigma$. 

Given any full shift $\Sigma = \{1, \cdots, k\}^\mathbb{Z}$, assume that we have:

- $d \times d$ matrices $A_1, \cdots, A_k$;
- and a positive definite $d \times d$ matrix $E$,

(for some $d \geq 1$) which satisfy

$$
\begin{align*}
  k \sum_{i=1}^{k} A_i A_i^T &= I \\
  k \sum_{i=1}^{k} A_i^T E A_i &= I
\end{align*}
$$

and a strong irreducibility condition.

We can define a (generalized) Kusuoka measure on $\Sigma$ using

$$
\mu(\lfloor i_0, \cdots, i_{n-1} \rfloor) = \text{trace} \left( A_{i_0} \cdots A_{i_{n-1}} \right)^T E A_{i_0} \cdots A_{i_{n-1}}.
$$

This is well defined and invariant under the shift $\sigma: \Sigma \rightarrow \Sigma$.

Theorem (Johansson-¨Oberg-P.)

The measure $\mu$ mixing exponentially fast, i.e., there exists $\alpha > 0$ such that for Lipschitz $f_1, f_2: \Sigma \rightarrow \mathbb{R}$ we can find $C > 0$ with

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\begin{align*}
  \int f_1 \circ \sigma^n d\mu - \int f_1 d\mu &\leq C \alpha^n \\
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$$\mu([i_0, \cdots, i_{n-1}]) = \text{trace} \left( (A_{i_0} \cdots A_{i_{n-1}})^T E (A_{i_0} \cdots A_{i_{n-1}}) \right).$$

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Aside: The origins of the Kusuoka measure

Question

Why was the Kusuoka measure introduced?

The Kusuoka measure arises naturally in the construction of a "laplacian" on function on $X$, via energy and harmonic functions.

Definition

Given $u \in C_0(X, \mathbb{R})$ we define the energy in terms of the values on smaller triangles (with vertices $v(i)_1$, $v(i)_2$, $v(i)_3$ corresponding to cylinders $i = [i_0, \ldots, i_{n-1}]$ in graphs approximating the fractal $X$.

$$ E(u) := \lim_{n \to +\infty} \sum_{i \in \{1, 2, 3\}} n \leq r < s \leq 3 (u_{v(i)_r} - u_{v(i)_s})^2 \in [0, +\infty] $$

and

$$ E(u, v) = \frac{1}{4} (E(u + v) - E(u - v)) $$

for $u, v \in C_0(X, \mathbb{R})$.
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\[
\mathcal{E}(u) := \lim_{n \to +\infty} \sum_{i \in \{1, 2, 3\}^n} \sum_{1 \leq r < s \leq 3} \left( u \left( v_r^{(i)} \right) - u \left( v_s^{(i)} \right) \right)^2 \in [0, +\infty]
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and $\mathcal{E}(u, v) = \frac{1}{4} \left( \mathcal{E}(u + v) - \mathcal{E}(u - v) \right)$ for $u, v \in C^0(X, \mathbb{R})$
Harmonic functions and Harmonic measures

We need to define analogues of harmonic functions and measures on the Sierpiński triangle $X$. 

Definition
Specifying the three values $u(0,1), u(1,0), u_1 \frac{\sqrt{3}}{2} \in \mathbb{R}$ there is a unique function $u \in C(X, \mathbb{R})$ achieving these values and minimizing $E(u)$. This is called a harmonic function.

If we quotient out by the constants, the space of harmonic functions is two dimensional.

Definition
We can associated to a harmonic function $u \in C(X, \mathbb{R})$ a harmonic measure $\nu_u$ on $X$ by

$$
\nu_u(\pi([i_0, \ldots, i_{n-1}])) = \frac{5}{3} \lim_{n \to \infty} E(u \circ \pi(i_0, \ldots, i_{n-1}, x_0, x_1, \ldots))
$$

where $i_0, \ldots, i_{n-1} \in \{1, 2, 3\}$. 

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**Definition**

Specifying the three values

$$u(0, 1), u(1, 0), u \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \in \mathbb{R}$$

there is a unique function $u \in C(X, \mathbb{R})$ achieving these three values and minimizing $E(u)$. This is called a **harmonic function**.

If we quotient out by the constants, the space of harmonic functions is two dimensional.
Harmonic functions and Harmonic measures

We need to define analogues of harmonic functions and measures on the Sierpiński triangle $X$.

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there is a unique function $u \in C(X, \mathbb{R})$ achieving these three values and minimizing $\mathcal{E}(u)$. This is called a *harmonic function*.

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**Definition**

We can associated to a harmonic function $u \in C(X, \mathbb{R})/\mathbb{R}$ a *harmonic measure* $\nu_u$ on $X$ by

$$\nu_u(\pi([i_0, \cdots, i_{n-1}]))) = \left(\frac{5}{3}\right)^n \mathcal{E}(u \circ \pi(i_0, \cdots, i_{n-1}, x_0, x_1, \cdots))$$

where $i_0, \cdots, i_{n-1} \in \{1, 2, 3\}^n$. 
The Kusuoka measure and the laplacian

Finally, fix a basis \( u_1, u_2 \) for harmonic functions satisfying \( E(u_1, u_2) = 0 \) and \( E(u_1) = E(u_2) = 1 \).
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The measure $\mu = \nu u_1 + \nu u_2$.

This is then used to define a laplacian.

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One defines a *Laplacian* $\Delta$ on suitable functions $f_1 \in C(X, \mathbb{R})$ by

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In particular, the Kusuoka measure gives the Laplacian desirable properties (that wouldn’t happen with the Bernoulli measure, say). For example,

**Lemma**

If $\Delta f \in L^2(\mu)$ then $\Delta(f^2) \in L^2(\mu)$.
Thank you for your time and attention