CONTINUITY OF ENTROPY MAP FOR NONUNIFORMLY HYPERBOLIC SYSTEMS

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Abstract. We prove that entropy map is upper semi-continuous for C^1 nonuniformly hyperbolic systems with domination, while it is not true for $C^{1+\alpha}$ nonuniformly hyperbolic systems in general. This goes a little against a common intuition that conclusions are parallel between $C^{1+\text{domination}}$ systems and $C^{1+\alpha}$ systems.

1. Introduction

The entropy map of a continuous transformation f on a metric space M is defined by $\mu \to h_\mu(f)$ on the set $\mathcal{M}_{inv}(M)$ of all f-invariant measures and it is generally not continuous (see [9]). However, it's still worth our effort to investigate the upper semi-continuity of entropy map since, for instance, it implies the existence of invariant measures of maximal entropy. It has been shown that entropy map is upper semi-continuous for expansive homeomorphisms of compact metric spaces [19], and then it's generalized to entropy expansive maps [3] as well as asymptotic entropy expansive maps [9]. In 1989 Newhouse [11] proved: (i) for any C^∞ maps the entropy map is upper semi-continuous; (ii) for $C^{1+\alpha}$ nonuniformly hyperbolic diffeomorphisms the entropy map, when restricted on the set of hyperbolic measures with the same "hyperbolic rate", is also upper semi-continuous. In the present paper, we remove the assumption on "hyperbolic rates" in [11] to show that for C^1 nonuniformly hyperbolic systems with domination property, the entropy map is upper semi-continuous.

Definition 1.1. Let M be a compact Riemannian manifold and $f : M \to M$ be a C^1 diffeomorphism. Given $\lambda_s, \lambda_u \gg \varepsilon > 0$, and for all $k \in \mathbb{N}$, we define $\Lambda_k = \Lambda_k(\lambda_s, \lambda_u; \varepsilon), k \geq 1$, to be all points $x \in M$ for which there is a splitting $T_xM = E^s_x \oplus E^u_x$ with the invariance property $(D_xf^m)E^s_x = E^s_{f^m x}$ and $(D_xf^m)E^u_x = E^u_{f^m x}$ and satisfying:

(a) $\|Df^n|_{E^s_{f^m x}}\| \leq e^{\varepsilon k} e^{-(\lambda_s - \varepsilon)n} e^{\varepsilon|m|}$, $\forall\, m \in \mathbb{Z}$, $n \geq 1$;

(b) $\|Df^{-n}|_{E^u_{f^m x}}\| \leq e^{\varepsilon k} e^{-(\lambda_u - \varepsilon)n} e^{\varepsilon|m|}$, $\forall\, m \in \mathbb{Z}$, $n \geq 1$;

(c) $tan(ang(E^s_{f^m x}, E^u_{f^m x})) \geq e^{-\varepsilon k} e^{-\varepsilon|m|}$.

$\Lambda = \Lambda(\lambda_s, \lambda_u; \varepsilon) = \bigcup_{k=1}^{+\infty} \Lambda_k$ is called a Pesin set.

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Denote by $\mathcal{M}_{inv}(\Lambda)$ the set of all invariant measures supported on Λ, i.e., $\mu \in \mathcal{M}_{inv}(\Lambda) \Leftrightarrow \mu(\Lambda) = 1$. For an ergodic $f-$invariant measure ν with non-zero Lyapunov exponents, we could define a Pesin set associated to it in the following way. Let Ω be the Oseledec basin of ν where all Lyapunov exponents exist, by Oseledec Theorem [14] $\nu(\Omega) = 1$. Denote by E^s and E^u the direct sum of the Oseledec splittings with respect to negative and positive Lyapunov exponents. Let λ_s be the norm of the largest Lyapunov exponent of vectors in E^s and λ_u be the smallest one in E^u and choose $0 < \varepsilon \ll \min\{\lambda_s, \lambda_u\}$. Then Ω is contained in the Pesin set $\Lambda = \Lambda(\lambda_s, \lambda_u; \varepsilon)$ in Definition 1.1, i.e., $\nu \in \mathcal{M}_{inv}(\Lambda)$. Observe that the splitting $T_xM = E^s(x) \oplus E^u(x)$ in Definition 1.1 is not necessary to be continuous with $x \in \Lambda$, and the angle between E^s and E^u may approach to zero along orbits in Λ. This discontinuity leads to an obstacle for the continuity property of entropy map on $\mathcal{M}_{inv}(\Lambda)$. In the present paper, we make an assumption that there is a domination between E^s and E^u, which ensures both continuity of the splittings and the uniformly bounded angles below between them. To be precise, a splitting $T_xM = E^s(x) \oplus E^u(x),\ x \in \Lambda$ is dominated if $\dfrac{\|D_xfv\|}{\|D_xfu\|} \leq \dfrac{1}{2}$ for any $v \in E^s(x)$ and $u \in E^u(x)$ with $\|v\| = 1, \|u\| = 1$.

Here is our main theorem in the paper.

Theorem 1.2. *Let f be a C^1 diffeomorphism of a compact manifold M. Let $\Lambda = \Lambda(\lambda_s, \lambda_u; \varepsilon)$ be a Pesin set with a dominated splitting $T_xM = E^s(x) \oplus E^u(x)$, $x \in \Lambda$. Then the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous on $\mathcal{M}_{inv}(\Lambda)$.*

Lack of domination may cause no upper semi-continuity of entropy map for C^r diffeomorphism for any $2 \leq r < \infty$ by a version of Downarowicz-Newhouse example [5]. This a little goes against a common intuition that conclusions are usually parallel between $C^{1+\text{domination}}$ systems and $C^{1+\alpha}$ systems (see for instance, [1][18]). Moreover, recall that the upper semi-continuity of entropy map is obtained for C^1 diffeomorphisms away from tangencies in [8]. However, due to the nonuniformity of hyperbolicity of (f, Λ) in Theorem 1.2, the system $(f, \overline{\Lambda})$ may be approximated by ones having homoclinic tangencies of some periodic points whose index different from $\dim E^s$, see for example in section 6.4 of [2], where the closure of the Pesin set $\overline{\Lambda} = M$ and Λ supports a hyperbolic SRB measure.

In section 2, we recall some definitions and basic facts about entropy, and give two lemmas needed to prove Theorem 1.2. In section 3, we will prove Theorem 1.2. By using a class of C^r $(2 \leq r < \infty)$ diffeomorphisms studied in [5] we illustrate in section 4 that entropy map could be not upper semi-continuous for nonuniformly hyperbolic systems without domination.

2. Preliminaries

Let M be a compact metric space and f a continuous map on M. Let μ be an f-invariant probability measure and $\xi = \{B_1, \cdots B_k\}$ a finite partition of M into measurable sets. The entropy of ξ with respect to μ is

$$H_\mu(\xi) = -\sum_{i=1}^{k} \mu(B_i) \log \mu(B_i).$$

The entropy of f with respect to μ and ξ is given by

$$h_\mu(f,\xi) = \lim_{n\to\infty} \frac{1}{n} H_\mu(f,\xi^n) = \inf_n \frac{1}{n} H_\mu(f,\xi^n)$$

where $\xi^n = \bigvee_{i=0}^{n-1} f^{-i}\xi$. The entropy of f with respect to μ is given by

$$h_\mu(f) = \sup_\xi h_\mu(f,\xi)$$

where ξ is taken over all finite measurable partitions of M. A partition $\alpha = \{A_0, A_1, \cdots, A_k\}$ is called a compact partition if $A_1, \cdots, A_k$ are disjoint compact subsets and $A_0 = M \setminus \bigcup_{1\le i\le k} A_i$. It is clear that $h_\mu(f) = \sup_\alpha h_\mu(f,\alpha)$, where α is taken over all finite compact measurable partitions of M.

Let F be a subset of M. A set $E \subseteq M$ is called a (n,δ)-spanning set of $F \subseteq M$ if $\forall\ x \in F,\ \exists\ y \in E$ such that $d(f^i(x), f^i(y)) \le \delta,\ 0 \le i < n$. Denote $r_n(F,\delta,f)$ the minimal cardinality of sets which (n,δ)-spans F with respect to f. Denote $r(F,\delta,f) = \limsup_{n\to+\infty} \frac{1}{n} \log r_n(F,\delta,f)$ and the topological entropy of f on F is defined by

$$h_{top}(f,F) = \lim_{\delta\to 0} r(F,\delta,f).$$

In particular, the topological entropy of f on M, $h_{top}(f,M)$, is denoted by $h_{top}(f)$.

For each $x \in M$, $n \in \mathbb{N}$, $r \in \mathbb{R}^+$, denote $B_n(x,r) = \{y \in M : d(f^i(x), f^i(y)) \le r, 0 \le i < n\}$, and $B_{+\infty}(x,r) = \{y \in M : d(f^n(x), f^n(y)) \le r, n \ge 0\}$. When f is a homeomorphism, one may also define $B_{\pm n}(x,r) = \{y \in M : d(f^i(x), f^i(y)) \le r, -n < i < n\}$ and $B_{\pm\infty}(x,r) = \{y \in M : d(f^i(x), f^i(y)) \le r, i \in \mathbb{Z}\}$. Denote

$$h^*_{loc}(x,r,f) = h(f, B_{\pm\infty}(x,r,f)).$$

Further let

$$h_{loc}(x,r,f) = \lim_{\delta\to 0} \limsup_{n\to+\infty} \frac{1}{n} \log r_n(B_n(x,r,f),\delta,f).$$

It's obvious that $h^*_{loc}(x,r,f) \le h_{loc}(x,r,f)$.

Lemma 2.1. *Let M be a compact Riemannian manifold and $f : M \to M$ be a diffeomorphism preserving a measure $\mu \in \mathcal{M}_{inv}(M)$. Then*

$$h_\mu(f) - h_\mu(f,\xi) \le \int h_{loc}(x,r,f)d\mu(x)$$

for any measurable partition ξ with $diam(\xi) \le r$.

Proof. For any given compact partition $\alpha = \{A_0, A_1, \cdots, A_k\}$ let

$$\delta_0 = \frac{1}{2} \min \Big\{ d(A_i, A_j), 1 \le i,j \le k, i \ne j \Big\}.$$

For any given $m \in \mathbf{N}$, take $\delta_1 \in (0, \delta_0)$ such that $d(x, y) < \delta_1$ implies that $d(f^i(x), f^i(y)) < \delta_0,\ 0 \le i < m$. Denote $\alpha_{f^m}^n = \bigvee_{i=0}^{n-1} f^{-mi}\alpha$. Then

$$\begin{aligned}
& \frac{1}{n}H_\mu(\alpha_{f^m}^n) - \frac{1}{n}H_\mu((\xi^m)_{f^m}^n) \qquad (\text{where } \ \xi^m = \bigvee_{i=0}^{m-1} f^{-i}\xi) \\
\le\ & \frac{1}{n}H_\mu(\alpha_{f^m}^n | (\xi^m)_{f^m}^n) \\
=\ & -\frac{1}{n}\sum_{B \in (\xi^m)_{f^m}^n} \mu(B) \sum_{A \in \alpha_{f^m}^n} \mu_B(A)\log\mu_B(A) \\
\le\ & \frac{1}{n}\sum_{B \in (\xi^m)_{f^m}^n} \mu(B)\log N_B(\alpha_{f^m}^n) \qquad\qquad (*)
\end{aligned}$$

where $N_B(\alpha_{f^m}^n) = \sharp\{A \in \alpha_{f^m}^n : A \cap B \ne \varnothing\}$.

Note that

$$N_B(\alpha_{f^m}^n) \le r_n(B_{mn}(x, r), \delta_0, f^m) \cdot 2^n \le r_{mn}(B_{mn}(x, r), \delta_1, f) \cdot 2^n$$

for any point $x \in B$, thus by $(*)$

$$\begin{aligned}
& \frac{1}{n}H_\mu(\alpha_{f^m}^n) - \frac{1}{n}H_\mu((\xi^m)_{f^m}^n) \\
\le\ & \frac{m}{mn}\sum_{B \in (\xi^m)_{f^m}^n} \int_B \log r_{mn}(B_{mn}(x, r), \delta_1, f)d\mu(x) + \log 2 \\
=\ & m\int \frac{1}{mn}\log r_{mn}(B_{mn}(x, r), \delta_1, f)d\mu(x) + \log 2.
\end{aligned}$$

When n is large enough, $\frac{1}{mn}\log r_{mn}(B_{mn}(x, r), \delta_1, f)$ is less than or equal to $h_{top}(f)$, which is a finite number for a diffeomorphism on a compact manifold. Applying Fatou Lemma we have

$$\begin{aligned}
& h_\mu(f^m, \alpha) - h_\mu(f^m, \xi^m) \\
=\ & \lim_{n\to+\infty}\Big(\frac{1}{n}H_\mu(\alpha_{f^m}^n) - \frac{1}{n}H_\mu((\xi^m)_{f^m}^n)\Big) \\
\le\ & m \cdot \limsup_{n\to+\infty}\int \frac{1}{mn}\log r_{mn}(B_{mn}(x, r), \delta_1, f)d\mu(x) + \log 2 \\
\le\ & m\int \limsup_{n\to+\infty}\frac{1}{mn}\log r_{mn}(B_{mn}(x, r), \delta_1, f)d\mu(x) + \log 2 \\
\le\ & m\int h_{loc}(x, r, f)d\mu(x) + \log 2
\end{aligned}$$

for any compact partition α and any $m \in \mathbb{N}$. Note that $h_\mu(f^m, \xi^m) = mh_\mu(f, \xi)$, $\forall m \in \mathbb{N}$. It follows that

$$h_\mu(f) - h_\mu(f, \xi) \le \int h_{loc}(x, r, f)d\mu(x).$$

$\square$

Remark 2.2. The local entropy originates from Bowen [3], where it is used to bound the difference between metric entropy and the metric entropy with respect to a partition with small diameter. In [3], the right-hand side of the inequality is

$$\lim_{\delta\to 0}\limsup_{n\to+\infty}\frac{1}{n}\sup_{x\in M}\log r_n(B_n(x,r,f),\delta,f),$$

which is called the local entropy of f. It's obvious that this quantity is not smaller than $\sup_{x\in M} h_{loc}(x,r,f)$ and thus $\int h_{loc}(x,r,f)d\mu(x)$. Here $\int h_{loc}(x,r,f)d\mu(x)$, which could be called local entropy of (f,μ), is slightly different from the standard local entropy of f in [3]. This quantity enables us to deal with local entropy in an open set which has large measure for any invariant measure ν near μ. The hyperbolicity assumption of measures ν guarantees "uniform hyperbolicity"(average along the orbit) in large measure sets (see Proposition 3.1) and thus small local entropy of (f,ν). In this way we can control the difference between metric entropy and the metric entropy with respect to a partition with small diameter for all nearby ν in Proposition 3.5, which is a necessary step to prove Theorem 1.2.

For a continuous map f on a compact metric space M, given $\nu\in\mathcal{M}_{inv}(M)$ and a Borel set A, by Birkhoff Ergodic Theorem, the set of points for which the limit of $\frac{1}{n}\sum_{i=0}^{n-1}\chi_A(f^i(x))$ exists is measurable and has measure 1.

Lemma 2.3. *For any given* $0<\gamma<1,\ 0<\eta<1$, *there exists* $\sigma=\frac{1}{2}\gamma\eta$ *such that for any* $\nu\in\mathcal{M}_{inv}(M)$, *any Borel set* A *with* $\nu(A)>1-\sigma$, *we have*

$$\nu\{x:\bar f_A(x)>1-\gamma\}>1-\frac{1}{2}\eta$$

where $\bar f_A(x)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\chi_A(f^i(x))$ *whenever exists.*

Proof. By Birkhoff Ergodic Theorem, $\int\chi_A d\nu(x)=\int\bar f_A(x)d\nu(x)$. Let $E=\{x:\bar f_A(x)>1-\gamma\}$, then

$$\nu(A)=\int_E\bar f_A(x)d\nu(x)+\int_{M\setminus E}\bar f_A(x)d\nu(x)\le\nu(E)+(1-\gamma)(1-\nu(E)).$$

Choose $\sigma=\frac{1}{2}\gamma\eta$, then $\nu(A)>1-\sigma$ implies that $\nu(E)>1-\frac{1}{2}\eta$. □

Remark 2.4. Lemma 2.3 is also true for f^{-1}.

3. Proof of Theorem 1.2

In this section, we prepare several lemmas and propositions and then prove Theorem 1.2.

Recall that $f:M\to M$ is a C^1 diffeomorphism, which has a Pesin set $\Lambda=\Lambda(\lambda_s,\lambda_u;\varepsilon)$ with a dominated splitting $E^s(x)\oplus E^u(x)$, $x\in\Lambda$.

Proposition 3.1. *Given* $\mu\in\mathcal{M}_{inv}(\Lambda)$ *and* $0<\eta<1$ *there exist* $\rho>0$ *and* $L>0$ *with the following property. For any* $\nu\in B_\rho(\mu)\cap\mathcal{M}_{inv}(\Lambda)$ *there exists a measurable set* T *with* $\nu(T)>1-\eta$ *such that*

$$\lim_{k\to+\infty}\frac{1}{k}\sum_{i=1}^{k}\frac{1}{L}\log\left\|Df^{-L}|_{E^u(f^{iL}(x))}\right\|<-\lambda_u+2\varepsilon, \tag{1}$$

$$\lim_{k\to+\infty}\frac{1}{k}\sum_{i=1}^{k}\frac{1}{L}\log\left\|Df^{L}|_{E^s(f^{-iL}(x))}\right\|<-\lambda_s+2\varepsilon \tag{2}$$

for any $x\in T$, where $B_\rho(\mu)$ denotes the set of all f-invariant measures with distance from μ less than ρ.

Proof. For an integer $n\geq 1$ set

$$A_n^\varepsilon=\Big\{x\in\Lambda:\ \frac{1}{m}\log\left\|Df^{-m}|_{E^u(x)}\right\|<-\lambda_u+\varepsilon,\ \forall m\geq n\Big\}.$$

Then $A_1^\varepsilon\subset\cdots\subset A_n^\varepsilon\subset A_{n+1}^\varepsilon\subset\cdots$, and $\mu(\bigcup_{n=1}^{\infty}A_n^\varepsilon)=1$. Let

$$c=\max\Big\{\sup_{x\in\Lambda}\left\|Df^{-1}|_{E^u(x)}\right\|,\ \sup_{x\in\Lambda}\left\|Df|_{E^s(x)}\right\|,\ 2\Big\}$$

and let $0<\eta<1$ be given in the condition of the proposition. Take $\gamma<\dfrac{\varepsilon}{2\log c}$ with $0<\gamma<1$ and take $\sigma=\dfrac{1}{2}\gamma\eta$ as in Lemma 2.3. Clearly $0<\sigma<1$. Take N such that $\mu(\bigcup_{n=1}^{N}A_n^\varepsilon)>1-\sigma$. Let

$$U_N^\varepsilon=\Big\{x\in\Lambda:\ \frac{1}{N}\log\left\|Df^{-N}|_{E^u(x)}\right\|<-\lambda_u+\varepsilon\Big\},$$

then $\bigcup_{n=1}^{N}A_n^\varepsilon\subseteq U_N^\varepsilon$. Since U_N^ε is open, $\nu(U_N^\varepsilon)>1-\sigma$ for any $\nu\in\mathcal{M}_{inv}(\Lambda)$ close enough to μ. Denote $\bar f_{U_N^\varepsilon}(x)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\chi_{U_N^\varepsilon}(f^i(x))$ whenever exists and $\widetilde{U_N^\varepsilon}=\Big\{x\in\Lambda:\bar f_{U_N^\varepsilon}(x)>1-\gamma\Big\}$. By Lemma 2.3, we get that $\nu(\widetilde{U_N^\varepsilon})>1-\dfrac{1}{2}\eta$.

Following the same procedure for f^{-1} and $\dfrac{1}{n}\log\left\|Df^n|_{E^s(x)}\right\|$, we get N' and thus $\widetilde{(U_{N'}^\varepsilon)}'$ with $\nu(\widetilde{(U_{N'}^\varepsilon)}')>1-\dfrac{1}{2}\eta$ for $\nu\in\mathcal{M}_{inv}(\Lambda)$ close to μ. Then we get a set $T=\widetilde{U_N^\varepsilon}\bigcap\widetilde{(U_{N'}^\varepsilon)}'$ and a constant $\rho>0$ such that $\nu(T)>1-\eta$ for any $\nu\in B_\rho(\mu)\cap\mathcal{M}_{inv}(\Lambda)$.

Let $L=2\max\Big\{[\dfrac{2N\log c}{\varepsilon}],[\dfrac{2N'\log c}{\varepsilon}]\Big\}$.

For $x\in T$ and $i\in\mathbf{Z}^+$, choose a sequence of integers $\{n_j^i\}_{j=1}^{\ell+1}$,

$$(i-1)L=n_{\ell+1}^i<n_\ell^i<n_{\ell-1}^i<\cdots<n_1^i=iL$$

by the following procedure

$$n_{j+1}^i=\begin{cases}n_j^i-N, & n_j^i\geq(i-1)L+N \text{ and } f^{n_j^i}(x)\in U_N^\varepsilon\\ n_j^i-1, & \text{otherwise.}\end{cases}$$

where $1\leq j\leq\ell$. Write $\{n_1^i,\cdots,n_\ell^i\}$ as a disjoint union $A_i\bigcup B_i\bigcup C_i$, where

$$A_i=\Big\{n_j^i\geq(i-1)L+N,\ \text{ and } f^{n_j^i}(x)\in U_N^\varepsilon\Big\},$$

$$B_i = \{n_j^i \geq (i-1)L + N \text{ and } f^{n_j^i}(x) \notin U_N^\varepsilon\},$$
$$C_i = \{(i-1)L < n_j^i < (i-1)L + N\}.$$

It's obvious that $0 \leq \sharp C_i < N$, $0 \leq \sharp A_i \leq [\frac{L}{N}]$. Thus,

$$\begin{aligned}
&\log \left\|Df^{-L}|_{E^u(f^{iL}(x))}\right\| \\
\leq\ & \sum_{j=1}^{\ell} \log \left\|Df^{-(n_j^i - n_{j+1}^i)}|_{E^u(f^{n_j^i}(x))}\right\| \\
\leq\ & N(-\lambda_u + \varepsilon) \cdot \sharp A_i + \log c \cdot \sharp B_i + \log c \cdot \sharp C_i \\
<\ & (-\lambda_u + \varepsilon)L + \log c \cdot (N + \sharp B_i).
\end{aligned}$$

By the definition of $\widetilde{U_N^\varepsilon}$, for any k large enough, $\sum\limits_{i=1}^{k} \sharp B_i \leq kL\gamma$. Therefore,

$$\begin{aligned}
&\frac{1}{k}\sum_{i=1}^{k} \frac{1}{L} \log \left\|Df^{-L}|_{E^u(f^{iL}(x))}\right\| \\
\leq\ & \frac{1}{kL}(kL \cdot (-\lambda_u + \varepsilon) + \log c \cdot Nk + \log c \cdot kL\gamma) \\
<\ & (-\lambda_u + \varepsilon) + \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon < -\lambda_u + 2\varepsilon.
\end{aligned}$$

Hence,

$$\lim_{k\to+\infty} \frac{1}{k}\sum_{i=1}^{k} \frac{1}{L} \log \left\|Df^{-L}|_{E^u(f^{iL}(x))}\right\| < -\lambda_u + 2\varepsilon, \quad \forall x \in T$$

and we get (1).

Replace f and $E^u(x)$ by f^{-1} and $E^s(x)$ respectively, we get (2) analogously. □

The following lemma comes from Burns and Wilkinson [4], which uses locally invariant fake foliations to avoid the assumption of dynamical coherence, a construction that goes back to Hirch, Pugh, and Shub [6]. Given a foliations $\mathcal{F}$ and a point y in domain, we denote $\mathcal{F}(y)$ the leaf through y and by $\mathcal{F}(y, \rho)$ we mean the neighborhood of radius $\rho > 0$ around y inside the leaf.

Lemma 3.2. *Let $f : M \to M$ be a C^1 diffeomorphism. Assume that Δ is an f-invariant compact set and the tangent space of which admits f-dominated splitting: $T_x(M) = E^s(x) \oplus E^u(x), \forall x \in \Delta$. Let angles between f-invariant subbundles E^s and E^u are bounded from zero by θ for every $x \in \Delta$. Then for any $0 < \zeta < \dfrac{\theta}{4}$, $\exists$ $\rho > r_0 > 0$, for any $x \in \Delta$, the neighborhood $B(x, \rho)$ admits foliations $\mathcal{F}_x^s$ and $\mathcal{F}_x^u$, such that for any $y \in B(x, r_0)$ and $* \in \{s, u\}$,*

(1) almost tangency: leaf $\mathcal{F}_x^*(y)$ is C^1 and $T_y\mathcal{F}_x^*(y)$ lies in a cone of width ζ of $E^*(x)$,

(2) local invariance: $f\mathcal{F}_x^*(y, r_0) \subseteq \mathcal{F}_{fx}^*(fy)$, $f^{-1}\mathcal{F}_x^*(y, r_0) \subseteq \mathcal{F}_{f^{-1}x}^*(f^{-1}y)$.

By Lemma 3.2 (1) one can define local product structure on the r-neighborhoods of every $x \in \Delta$, for a small $r > 0$, as used in [8].

For $y, z \in B(x, \rho)$, write $[y, z]_{s,u} = a$ if $\mathcal{F}^s_x(y)$ intersects $\mathcal{F}^u_x(z)$ at $a \in B(x, \rho)$. By transversality of (1), the intersection point a is unique whenever it exists. We could find $r_1 \in [0, r_0]$ independent of x such that $[y, z]_{s,u}$ is well defined whenever $y, z \in B(x, r_1)$, and for any $y \in B(x, r_1)$ there exists $y_* \in \mathcal{F}^*_x(x)$ such that $[y_s, y_u]_{u,s} = y$. Lemma 3.2 implies that the locally invariant foliations $\mathcal{F}^*_x$ are transverse with angles uniformly bounded from below. Therefore, $\exists\ \ell > 0$ independent of x such that for any $y \in B(x, r')$ we have $y_* \in \mathcal{F}^*_x(x, \ell r')$ for $\ell r' < r_1$. Furthermore, by locally invariance of foliations we get that $y \in B_{\pm 2}(x, r')$ implies that $f^{\pm 1}(y_*) = (f^{\pm 1} y)_*$, where recall that $B_{\pm 2}(x, r') = \{y \in M : d(f^i y, f^i x) \le r', -2 < i < 2\}$.

Also note that $y_{s\backslash u} = x$ for $y \in B(x, r')$ implies that $y \in \mathcal{F}^{u\backslash s}_x(x, \ell r')$, therefore $y_s = y_u = x$ implies that $y = x$ for $y \in B(x, r')$.

Since there exists domination on the Pesin set $\Lambda = \Lambda(\lambda_s, \lambda_u; \varepsilon)$, we could extend it to the closure of Λ, then the process above could be done in $\overline{\Lambda}$. Therefore, we get r' independent of x in Λ such that $y \in B(x, r')$ implies that $y = x$.

Lemma 3.3 (Pliss[16])**.** *Let $a_* \le c_2 < c_1$ and $\theta = \dfrac{c_1 - c_2}{c_1 - a_*}$. For given real numbers $a_1, \cdots, a_N$ with $\sum_{i=1}^{N} a_i \le c_2 N$ and $a_i \ge a_*$ for every i, there exists $\ell \ge N\theta$ and $1 \le n_1 < n_2 < \cdots < n_\ell \le N$, such that*

$$\sum_{i=k+1}^{n_j} a_i \le c_1(n_j - k), \qquad 0 \le k < n_j, \quad 1 \le j \le \ell.$$

By (1) of Proposition 3.1 , for every $x \in T$ and every k large enough,

$$\sum_{i=1}^{k} \log \left\| Df^{-L}\big|_{E^u(f^{iL}(x))} \right\| \le (-\lambda_u + 2\varepsilon) L k.$$

Take $a_* = \inf_{x \in \Lambda} \{\log \|Df^{-L}\ |_{E^u(x)}\ \|\}$, $c_1 = (-\lambda_u + 2\varepsilon)L$, $c_2 = (-\lambda_u + 3\varepsilon)L$. Applying Lemma 3.3 it is easy to find an infinite sequence

$$1 \le n_1 < n_2 < \cdots < n_j < \cdots$$

such that

$$\sum_{i=k+1}^{n_j} \log \left\| Df^{-L}\big|_{E^u(f^{iL}(x))} \right\| \le (-\lambda_u + 3\varepsilon)(n_j - k) L,$$

$0 \le k < n_j,\ j = 1, 2 \cdots$.

Choose $r'' > 0$ and $\zeta > 0$ such that $\dfrac{\|D_y f^{-L} v\|}{\|D_x f^{-L} u\|} < e^{\varepsilon L}$ and $\dfrac{\|D_y f^{L} v\|}{\|D_x f^{L} u\|} < e^{\varepsilon L}$ for $d(x, y) < r''$, $\angle(u, v) \le \dfrac{\zeta}{2}$, $\|u\| = \|v\| = 1$.

Take $r = \min\left\{ r', r'' \right\}$, then

$$f^{-(n_j - k)L} \mathcal{F}^u_{f^{n_j L} x}(f^{n_j L} x, \ell r) \subseteq \mathcal{F}^u_{f^{kL} x}(f^{kL} x, e^{(-\lambda_u + 4\varepsilon)(n_j - k)L} \ell r)$$

for $0 \le k < n_j, j = 1, 2\cdots$. In particular,

$$f^{-n_jL}\mathcal{F}^u_{f^{n_jL}x}(f^{n_jL}x, \ell r) \subseteq \mathcal{F}^u_x(x, e^{(-\lambda_u+4\varepsilon)Ln_j}\ell r), \quad j = 1, 2\cdots.$$

For $y \in B_{+\infty}(x, r)$, $f^i(y_u) = (f^iy)_u$, $\forall i \in \mathbb{N}$, thus $y_u = f^{-n_jL}(f^{n_jL}y)_u \in \mathcal{F}^u_x(x, e^{(-\lambda_u+4\varepsilon)Ln_j}\ell r)$, $\forall j \in \mathbb{N}$. Therefore y_u belongs to the intersection of all $\mathcal{F}^u_x(x, e^{(-\lambda_u+4\varepsilon)Ln_j}\ell r)$ over all j which reduces to $\{x\}$. Analogously, for $y \in B_{-\infty}(x, r)$, we get that $y_s = x$. Thus $y \in B_{\pm\infty}(x, r)$ implies that $y = x$.

To conclude, we have obtained the following :

Claim 3.4. *For any $\sigma > 0$ there exist $r > 0$ and $\rho > 0$ satisfying the following property. For any $\nu \in B_\rho(\mu) \cap \mathcal{M}_{inv}(\Lambda)$, there exists a Borel set T with $\nu(T) > 1-\sigma$ such that:*

$$B_{\pm\infty}(x, r) = \{x\}, \quad \forall x \in T.$$

Claim 3.4 says that, fixing a small $r > 0$, for ν close to μ, $h^*_{loc}(x, r, f) = 0$ on a set with large ν-measure. To estimate the difference between $h_\mu(f)$ and $h_\mu(f, \xi)$, by Lemma 2.1 we need to deal with $h_{loc}(x, r, f)$. One always has that $h^*_{loc}(x, r, f) \le h_{loc}(x, r, f)$ but the inverse inequality is generally not true. However, we are going to show that, $h_{loc}(x, r, f)$ is still small on a set with large measure. Combining Claim 3.4 with Lemma 2.1 we aim to deduce the following proposition.

Proposition 3.5. *Let $\mu \in \mathcal{M}_{inv}(\Lambda)$ and $\tau > 0$. There exist $r > 0$ and $\rho > 0$ such that*

$$h_\nu(f) - h_\nu(f, \xi) \le \tau$$

for any $\nu \in B(\mu, \rho) \cap \mathcal{M}_{inv}(\Lambda)$ and any measurable partition ξ with diam$(\xi) \le r$.

Proof. Let $C_0 = \sup_{x\in\Lambda}\{\|D_xf\| + 1\}$ and $C = h_{top}(f, \Lambda)$. It's seen that both of them are finite. We assume that $C > 0$, otherwise the entropy map for f is upper semi-continuous and we complete the proof. Take $\eta = \dfrac{\tau}{2C}$, $\gamma = \dfrac{\tau}{2\log C_0}$ in Lemma 2.3, we get $\sigma = \dfrac{\eta\gamma}{2} \le \dfrac{\eta}{2}$. By Claim 3.4 we get $r > 0$ and $\rho > 0$ with the property that for any $\nu \in B(\mu, \rho)$ there exists a Borel set $T = T(\nu)$ with $\nu(T) > 1 - \sigma$ such that

$$h^*_{loc}(x, r, f) = 0, \quad x \in T. \tag{3.1}$$

We could assume that T is compact by the regularity of measure.

For any $\delta > 0$, $\beta > 0$ and $x \in T$, by (3.1) we get $m(x) > 0$, $N(x) > 0$ as well as an open neighborhood $V(x)$ of x such that $\forall y \in V(x)$,

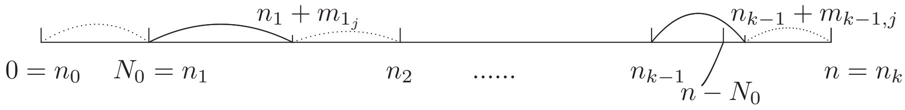
$$r_{m(x)}(B_{\pm N(x)}(y, r), \frac{\delta}{4}) \le e^{\beta m(x)}.$$

By compactness of T, $\exists\, x_1, \cdots, x_s$ such that $T \subseteq \bigcup_{i=1,\cdots,s} V(x_i) := W$. Then $\nu(W) > 1 - \sigma$. By Lemma 2.3, $\nu(\widetilde{W}) > 1 - \frac{\eta}{2}$, where $\widetilde{W} = \{x : \bar{f}_W(x) > 1 - \gamma\}$. Denote $m_i = m(x_i)$, $N_i = N(x_i)$, $N_0 = \max_{1\le i\le s}\{m_i, N_i\}$ and $W' = f^{-N_0}(W) \cap \widetilde{W}$. Since $\sigma < \dfrac{\eta}{2}$, we get that $\nu(W') > 1 - \eta$. For $x \in W'$ take $n > 2N_0$ large enough such that $\sharp\{0 \le i < n : f^i(x) \in W\} > (1 - \gamma)n$. We define a sequence $0 = n_0 < n_1 < \cdots < n_{k-1} < n_k = n$ of integers as follows. Let $n_1 = N_0$ then $f^{n_1}(x) \in W$. Assume that $N_0 \le n_i < n$ has been defined. Now suppose $n_i \le n - N_0$. Since $f^{n_i}(x) \in W$, there exists x_{i_j} such that $f^{n_i}(x) \in V(x_{i_j})$, and then we take

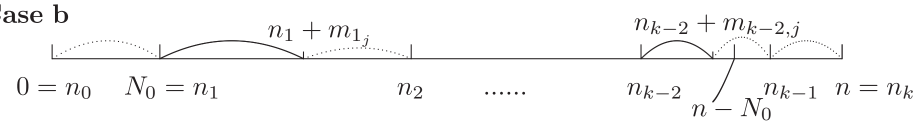
$$n_{i+1} = \min\{\min\{k : k \ge n_i + m_{i_j}, f^k(x) \in W\}, n\}.$$

Suppose $n - N_0 < n_i < n$, we take $n_{i+1} = n$. For different relations between n_{k-1} and $n - N_0$, the sequence $\{n_i\}_{i=1}^{k}$ has two following cases:

Case a



Case b



Note that for any $\delta > 0$ and any integer $\ell \geq 1$, any ball with radius δ could be (ℓ, δ)-spanned by $C_0^{\ell-1}$ points, where $C_0 = \sup_{x \in \Lambda}\{[\|D_x f\|] + 1\}$. When $i = 0$, $f^{n_0}(B_n(x,r)) = B_n(x,r)$ could be $(N_0, \frac{\delta}{2})$-spanned by $\kappa C_0^{N_0-1}$ points, where κ is the minimal cardinality of sets which $(1, \frac{\delta}{2})$-span M. When $1 \leq i \leq k-2$, we have $N_0 \leq n_i \leq n - N_0$, it holds that

$$f^{n_i}(x) \in V(x_{i_j}), \quad f^{n_i}(B_n(x,r)) \subset B_{\pm N_{i_j}}(f^{n_i}(x), r),$$

so $f^{n_i}(B_n(x,r))$ could be $(m_{i_j}, \frac{\delta}{4})$-spanned by $e^{m_{i_j}\beta}$ points. Since any ball with radius $\frac{\delta}{4}$ could be $(m_{i_j}, \frac{\delta}{4})$-spanned by $C_0^{m_{i_j}-1}$ points, we get that $f^{n_i+m_{i_j}}(B_n(x,r))$ could be spanned by $e^{m_{i_j}\beta}C_0^{n_{i+1}-n_i-m_{i_j}-1}$ points. Thus $f^{n_i}(B_n(x,r))$ could be $(n_{i+1} - n_i, \frac{\delta}{2})$-spanned by $e^{2m_{i_j}\beta}C_0^{n_{i+1}-n_i-m_{i_j}-1}$ points.

When $i = k-1$ and for Case a, since $n_{k-1} \leq n - N_0$, just as the discuss above for $1 \leq i \leq k-2$, $f^{n_{k-1}}(B_n(x,r))$ could be $(n - n_{k-1}, \frac{\delta}{2})$-spanned by $e^{2m_{k-1,j}\beta}C_0^{n-n_{k-1}-m_{k-1_j}-1}$ points. For Case b, since $(n - n_{k-1}) \leq N_0 - 1$, $f^{n_{k-1}}(B_n(x,r))$ could be $(\frac{\delta}{2}, n - n_{k-1})$-spanned by $\kappa C_0^{N_0-2}$ points.

By Lemma 2.1 of [3], which says that $r_n(B_n(x,r),\delta) \leq \prod_{i=0}^{k-1} r_{n_{i+1}-n_i}(f^{n_i}B_n(x,r), \frac{\delta}{2})$, we get that

$$r_n(B_n(x,r),\delta) \leq \begin{cases} \kappa C_0^{N_0+\gamma n}e^{2\beta n}, & \text{when } \textbf{Case a}, \\ \kappa^2 C_0^{2N_0+\gamma n}e^{2\beta n}, & \text{when } \textbf{Case b}. \end{cases}$$

Therefore, for both cases, $\forall x \in W'$, $\forall \delta > 0$, $r_n(B_n(x,r),\delta) \leq \kappa^2 C_0^{2N_0+\gamma n}e^{2\beta n}$ for any n large enough. Thus,

$$\begin{aligned} & \limsup_{n\to+\infty} \frac{1}{n}\log r_n(B_n(x,r),\delta) \\ \leq\ & \lim_{n\to+\infty}\left(\frac{1}{n}\log\kappa^2 + 2\beta + \frac{2N_0}{n}\log C_0 + \gamma\log C_0\right) \\ =\ & 2\beta + \gamma\log C_0. \end{aligned}$$

By the choice of γ and the arbitrariness of β, we get that

$$h_{loc}(x,r,f) = \lim_{\delta\to 0}\limsup_{n\to+\infty}\frac{1}{n}\log r_n(B_n(x,r),\delta) \le \frac{\tau}{2}, \qquad \forall x\in W'.$$

For any measurable partition ξ with $diam(\xi)\le r$ by Lemma 2.1 it holds that

$$h_\nu(f) - h_\nu(f,\xi) \le \int h_{loc}(x,r,f)d\nu(x)$$

and thus

$$\begin{aligned} h_\nu(f) - h_\nu(f,\xi) &\le \int_{W'} h_{loc}(x,r,f)d\nu(x) + \int_{M\setminus W'} h_{loc}(x,r,f)d\nu(x) \\ &\le \frac{\tau}{2} + \eta\cdot C \le \frac{\tau}{2}+\frac{\tau}{2} \le \tau. \end{aligned}$$

This completes the proof of Proposition 3.5. □

We are now turning to the proof of Theorem 1.2.

Proof of Theorem 1.2. For any given $\mu\in\mathcal{M}_{inv}(\Lambda)$ we will show that the entropy map is upper semi-continuous at μ. For μ and a real $\tau>0$, we can choose two constants $r>0,\ \rho>0$ as in Proposition 3.5 and a partition ξ with $diam(\xi)<r$ and $\mu(\partial\xi)=0$. From Proposition 3.5, we know that

$$h_\nu(f) - h_\nu(f,\xi) \le \tau, \quad \forall \nu\in B(\mu,\rho)\cap\mathcal{M}_{inv}(\Lambda).$$

Since $h_\mu(f) = \sup_\xi h_\mu(f,\xi)$, we can shrink $diam(\xi)$ if necessary such that

$$h_\mu(f,\xi) - h_\mu(f) \le \tau.$$

Note for a fixed n and a partition ξ with $\mu(\partial\xi)=0$, $\frac{1}{n}H_\nu(f,\xi^n)$ is continuous at μ. Thus $h_\nu(f,\xi) = \inf_n \frac{1}{n}H_\nu(f,\xi^n)$ is upper semi-continuous at μ. Shrink $\rho>0$ if necessary, we get

$$h_\nu(f,\xi) - h_\mu(f,\xi) \le \tau.$$

Therefore, for $\nu\in\mathcal{M}_{inv}(\Lambda)\cap B(\mu,\rho)$,

$$\begin{aligned} h_\nu(f) - h_\mu(f) &= (h_\nu(f)-h_\nu(f,\xi)) + (h_\nu(f,\xi)-h_\mu(f,\xi)) + (h_\mu(f,\xi)-h_\mu(f)) \\ &\le \tau+\tau+\tau \\ &\le 3\tau, \end{aligned}$$

which shows that the entropy map is upper semi-continuous at μ. □

4. C^r $(2\le r<\infty)$ DIFFEOMORPHISMS WITHOUT DOMINATION

In this section, by some brief analysis of the techniques in [5] we illustrate the examples of C^r $(2\le r<\infty)$ nonuniformly hyperbolic system without domination for which the entropy map is not upper semi-continuous. For a detail proof, readers may refer to [5].

We denote $C^r(M)(2\le r<+\infty)$ as the set of C^r diffeomorphisms on a smooth surface M. We can choose an open subset $\mathcal{U}\subset C^r(M)$ such that each f in it has a hyperbolic basic set $\Delta(f)$ with the same adapted neighborhood $U\subset M$ which has persistent homoclinic tangencies, i.e. there exist $x,y\in\Delta(f)$ such that

$W^s(x)$ and $W^u(y)$ have tangencies. This is possible according to Proposition 1 in Chapter 6 of [15]. Let $\widetilde{H}_n(f)$ be the set of periodic hyperbolic points p which are homoclinic related to $\Delta(f)$ (i.e. $W^s(\Delta(f))\backslash\Delta(f)$ and $W^u(O(p))\backslash O(p)$ have nonempty transverse intersections and vice versa) with least period less than or equal to n, and let $\widetilde{H}(f) = \bigcup_{n=1}^{+\infty} \widetilde{H}_n(f)$. Let $\widetilde{\tau}(f)$ be the least integer n such that $\widetilde{H}_n(f) \neq \emptyset$ and let $\mathcal{D}_m$ be the subset of $\mathcal{U}$ such that $\widetilde{\tau}(f) = m$. For $p \in \widetilde{H}(f)$, denote $\chi(p) = \frac{1}{\pi(p)} \min\{\log|\lambda_s^{-1}(p)|, \log|\lambda_u(p)|\}$, where $|\lambda_s(p)| < 1$ and $|\lambda_u(p)| > 1$ are the norms of the two eigenvalues of $D_p f^{\pi(p)}$ respectively, and $\pi(p)$ the least period of p. Let μ_p be the periodic measure supported on p, i.e. $\mu(p) = \dfrac{1}{\pi(p)} \sum_{i=1}^{\pi(p)-1} \delta_{f^i(p)}$. For an ergodic hyperbolic measure μ on M, let $\chi(\mu) = \min\{|\chi_s(\mu)|, |\chi_u(\mu)|\}$, where $\chi_s(\mu), \chi_u(\mu)$ are the two Lyapunov exponents of μ. In the sequel, by (C^r, ϵ)-perturbation we mean that the perturbation is done in the ϵ-neighborhood in C^r topology. By C^r perturbation, we mean (C^r, ϵ)-perturbation for any sufficiently small ϵ.

Let $f \in \mathcal{D}_m$, $n \geq m$, for any $p \in \widetilde{H}_n(f)$, we first C^r-perturb f to get a homoclinic tangency for $O(p)$ (see Lemma 8.4 in [10]). Then according to Proposition 5 and Lemma 3 in [7] by a further C^r small perturbation, one can get an interval I of tangencies between $W^u(p)$ and $W^s(p)$. Near this interval we take one more C^r small perturbation g to create a curve $J \subset W^u(p, g)$ with N bumps as in Figure 1.

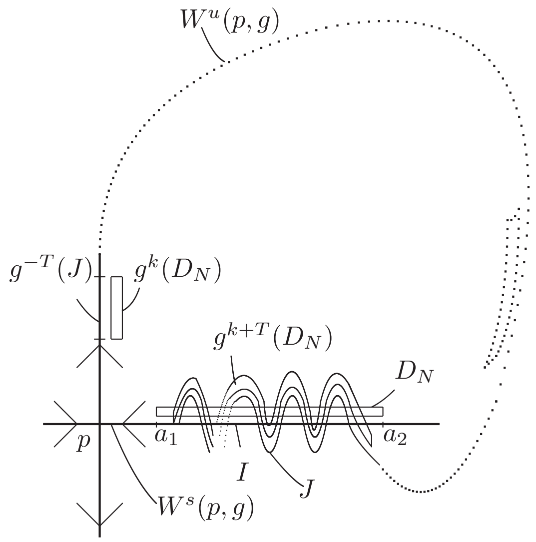


FIGURE 1. Creation of small basic sets

This perturbation can be done as follows. Denote $I = \{a_1 \leq x \leq a_2, y = 0\}$, $J = \{a_1 \leq x \leq a_2 : y = A\cos\omega(x-c)\}$. To keep the perturbation to be C^r-small, we only need require that $A \cdot \omega^r \leq \epsilon$, where ϵ could be arbitrarily small. For any fixed small $\epsilon > 0$, let $A = \epsilon(\dfrac{a_2 - a_1}{\pi N})^r$, $\omega = \dfrac{\pi\, N}{a_2 - a_1}$, $c = \dfrac{a_1 + a_2}{2}$.

To create small hyperbolic basic sets, consider a small rectangle D_N close to I with distance less than $\frac{A}{4}$ and consider the iterations of g^{k+T} (where k denote the first k iterations near p). In this way, a periodic hyperbolic basic set $\Delta(p,N)$ is obtained by transversal intersections. It's an N-horseshoe with topological entropy $\dfrac{\log N}{k+T}$. Note that k increases with the increase of N:

$$A \cdot |\lambda_u|^k \gtrsim 1, \quad |\lambda_s|^k \lesssim A,$$

where by $a \gtrsim b$ we mean that $a \geq const \cdot b$, and the $const$ is independent of N and $k(N)$ ($a \lesssim b$ is defined similarly). So we get $k = \dfrac{r \log N}{\chi(p)} + const$ and thus

$$h_{top}(\Delta(p,N),g) = \frac{\log N}{\dfrac{r \log N}{\chi(p)} + const + T}.$$

For $n \in \mathbf{N}$, choose $N(n)$ large enough such that

$$h_{top}(\Delta(p,N(n)),g) > \frac{\chi(p)}{r} - \frac{1}{n}.$$

By Variational Principle, there exists an ergodic measure ν_n supported on $\Delta(p,N(n))$ such that $h_{\nu_n}(g) > \dfrac{\chi(p)}{r} - \dfrac{1}{n}$. By estimating one sees that Dg^{k+T} expands unstable direction in $\Delta(p,N)$ about N times and contracts the stable direction about $1/N$ times, so $\chi(\nu_n)$ of any ergodic measure ν_n on $\Delta(p,N(n))$ will be close to $\frac{\log N}{k+T} \simeq \dfrac{\chi(p)}{r}$. Moreover, since by iterations of g, $\Delta(p,N)$ spends most of time near p, ν_n is close to the periodic measure μ_p. Let $N(n)$ be larger if necessary such that

$$\chi(\nu_n) > \frac{\chi(p)}{r} - \frac{1}{n} \quad \text{and} \quad d(\nu_n,\mu_p) < \frac{1}{n}.$$

Denote $\widetilde{\Delta}(p,n) = \Delta(p,N(n))$.

To conclude, for any diffeomorphism $f \in \mathcal{D}_m$, by any C^r small perturbation and for any positive integer n, we get a diffeomorphism g_n satisfying property $\mathcal{S}_n$:

(1) there exists a hyperbolic basic set $\widetilde{\Delta}(p,n)$ and an ergodic measure ν_n on $\widetilde{\Delta}(p,n)$ such that

$$h_{\nu_n}(g_n) > \frac{\chi(p)}{r} - \frac{1}{n},$$

(2) for any ergodic measure μ_n on $\widetilde{\Delta}(p,n)$, we have

$$\chi(\mu_n) > \frac{\chi(p)}{r} - \frac{1}{n} \text{ and } (\mu_n,\mu_p) < \frac{1}{n}.$$

Denote $\mathcal{D}_{m,n} (n \geq m)$ as the subset of $\mathcal{D}_m$ satisfying property $\mathcal{S}_n$. It's obvious that property $\mathcal{S}_n$ is an open property. From above discussion, we see that $\mathcal{D}_{m,n}$ is an open dense subset of $\mathcal{D}_m$. Let

$$\mathcal{R} = \bigcup_{m=1}^{+\infty} \bigcap_{n=m}^{+\infty} \mathcal{D}_{m,n},$$

then $\mathcal{R}$ is a residual subset of $\mathcal{U}$. For any $f \in \mathcal{R}$, any $p \in \widetilde{H}_n(f)$, there exists a sequence of ergodic measures $\{\nu_n\}$ such that $\nu_n \to \mu_p$ and $\chi(\nu_n) > \dfrac{1}{2r}\chi(p)$. By

Definition 1.1, $\{\nu_n\}$ and μ_p are supported on a Pesin set $\Lambda(\frac{1}{2r}\chi(p), \frac{1}{2r}\chi(p); \varepsilon)$. But at the same time, $h_{\nu_n}(f) \to \frac{1}{r}\chi(p)$ while $h_{\mu_p}(f) = 0$, which implies that the entropy map of $f \in \mathcal{R}$ is not upper semi-continuous at μ_p on the Pesin set $\Lambda(\frac{1}{2r}\chi(p), \frac{1}{2r}\chi(p); \varepsilon)$.

Note that although for each fixed n, ν_n is supported on $\widetilde{\Delta}(p, n)$ which is uniformly hyperbolic and the angles between E^s and E^u is uniformly bounded below by about $const \cdot \frac{1}{(N(n))^{r-1}}$, the angles of the Oseledec splittings for the sequence ν_n, $n \geq 1$, may be arbitrary small as n goes to infinity. Therefore there is no domination between E^s and E^u over $\Lambda(\frac{1}{2r}\chi(p), \frac{1}{2r}\chi(p); \varepsilon)$.

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