Takens’ Last Problem and Existence of Non-trivial Wandering Domains

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What is a wandering domain?

\( M \): compact smooth manifold

**Definition**

\( D \) is a *wandering domain* for \( f \in \text{Diff}^r(M) \) if

- \( D \) : nonempty, connected, open set of \( M \)
- \( f^i(D) \cap f^j(D) = \emptyset \) for all \( i, j \in \mathbb{Z} \) with \( i \neq j \)
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Trivial example

\[ f(D) \]

sink

\[ f^2(D) \]

source

\[ D \]
Non-trivial example

[Bohl(1916), Denjoy(1932)]

\[ \exists f \in \text{Diff}^1(S^1), \exists \{ D_i \}_{i \in \mathbb{N}} \subset S^1 \text{ with } f(D_i) = D_{i+1} \text{ s.t.} \]

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- rotation number of \( f \) is irrational;

\[ \hat{\alpha} := S^1 \cap \bigcap_{n=1}^{\infty} \text{Int}(D^n) \] is a \( f \)-invariant Cantor set satisfying \( \hat{\alpha} = \emptyset \).
Non-trivial example

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- rotation number of \( f \) is irrational;
- \( D_i \cap D_j = \emptyset \) for \( \forall i, j \in \mathbb{N} \) with \( i \neq j \);
- \( \Lambda := S^1 \setminus \bigcup_{i \in \mathbb{N}} \text{Int}(D_n) \) is a \( f \)-invariant Cantor set satisfying \( \Lambda = \omega_D \).
Non-trivial example

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Note: \( \forall f \in \text{Diff}^2(S^2) \) has no wandering domain
In 1-dimension

Definition (de Melo & van Strien)

For a circle diffeomorphism $f$, an open interval in $S^1$ is a non-trivial wandering domain if

- $D, f(D), f^2(D), \ldots$ are pairwise disjoint;
- the $\omega$-limit set of $D$ is not equal to a single periodic orbit.
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In 2-dimensions

\[ B_f(\{p\}) \]

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In 2-dimensions

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sink

\[ B_f(\{q\}) \]

(dissipative) saddle-node
**Definition**

Let $M$: closed surface and $f \in \text{Diff}^r(M)$. An open set $D \subset M$ is a *non-trivial wandering domain* if

- $f^i(D) \cap f^j(D) \neq \emptyset$ if $i \neq j$;
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![Attracting irrational rotation](image)

$\omega_D = S^1$
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![attracting irrational rotation](attachment:image.png)

\[ \omega_D = S^1 \]
In 2-dimension

\[ f \in \text{Diff}^r(S^2) \text{ with } \]
- sink \( p_0 \);
- horseshoe \( \Lambda \);
- source \( p_\infty \)
In 2-dimension

$f \in \text{Diff}^r(S^2)$ with

- sink $p_0$;
- horseshoe $\Lambda$;
- source $p_\infty$

$D \subset \text{basin of } p_0 \implies f^i(D) \cap f^j(D) = \emptyset \text{ if } i \neq j$;

$\omega_D = \{p_0\}$;

$\lim_{n \to \infty} \text{diam}(f^n(D)) \to 0$
In 2-dimension

\[ D \subset R \setminus f(R) \implies f^i(D) \cap f^j(D) = \emptyset \text{ if } i \neq j; \]
\[ \omega_D = \{ p_0 \} \cup \Lambda; \]
\[ \lim_{n \to \infty} \text{diam}(f^n(D)) = c > 0 \]

\( f \in \text{Diff}^r(S^2) \) with
- sink \( p_0 \);
- horseshoe \( \Lambda \);
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Non-trivial wandering domains in 2-dimension

$M$: closed surface

**Definition**
A nonempty connected open set $D$ in $M$ is a *non-trivial wandering domain* for $f \in \text{Diff}^r(M)$ if

- $f^i(D) \cap f^j(D) = \emptyset$ for any $i, j \in \mathbb{Z}$ with $i \neq j$;
- there is a non-trivial basic set $\Lambda$ such that, for any $x \in D$, the $\omega$-limit set $\omega(x)$ contains $\Lambda$.

**Definition**
A non-trivial wandering domain $D$ is called *contracting* if the diameter of $f^n(D)$ converges to zero as $n \to \infty$. 
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Smale horseshoe map

$f \in \text{Diff}^r(S^2)$ with

- sink $p_0$;
- horseshoe $\Lambda$;
- source $p_\infty$

There are trivial wandering domains but no non-trivial ones.
Question

Does there exist a diffeomorphism having non-trivial wandering domains?
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Colli-Vargas’ example
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\[ \tau^S \tau^U > 1 \]
Colli-Vargas’ example

\[ D_0 \subset G_{i_0} \]
Colli-Vargas’ example

\[ \exists k_0 \in \mathbb{N} \text{ s.t. } f^{k_0}(D_0) \cap B_{i_0} \neq \emptyset \]
Colli-Vargas’ example

center of $f_{\delta_1}^{k_1}(D_0) = \text{center of } \mathcal{B}_{i_0}$
Colli-Vargas’ example

\[ D_1 \subset G_{i_1}, \quad D_1 \cap f_{\delta_0}^{k_0}(D_0) \neq \emptyset \]
Colli-Vargas’ example

\[ f_{\delta_1}^{k_1}(D_1) \cap B_{i_1} \neq \emptyset \]
Colli-Vargas’ example

\[ D_0 \xrightarrow{f_{\delta_0}^{k_0}} D_1 \xrightarrow{f_{\delta_1}^{k_1}} D_2 \]
Colli-Vargases' example

\[ D_0 \xrightarrow{f_{\delta_0}^k} D_1 \xrightarrow{f_{\delta_1}^k} D_2 \xrightarrow{f_{\delta_2}^k} D_3 \xrightarrow{\ldots} \]
There exists a 2-dimensional $C^r$, $r \geq 2$, diffeomorphism having a **contracting non-trivial wandering domain** whose $\omega$-limit set is contained in the **horseshoe $\Lambda$** with its **homoclinic tangencies**.

[Colli-Varg'as '01]
Colli-Vargas’ example

There exists a 2-dimensional $C^r$, $r \geq 2$, diffeomorphism having a contracting non-trivial wandering domain whose $\omega$-limit set is contained in the horseshoe $\Lambda$ with its homoclinic tangencies.

[Colli-Vargas ’01]
Main results

$M$: closed surface $f \in \text{Diff}^r(M)$, $r \geq 2$, with
- saddle fixed point $p$;
- homoclinic tangency for $p$

$\exists$ open set $\mathcal{N}_f \subset \text{Diff}^r(M)$
s.t. $f \in \text{Cl}(\mathcal{N}_f)$ and $\mathcal{N}_f$ has persistent tangencies.

$\mathcal{N} \overset{\text{def}}{=} \bigcup_f \mathcal{N}_f$: Newhouse open set
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**Theorem A (Colli-Vargas’ conjecture ’01)**

The Newhouse open set $\mathcal{N}$ is contained in the closure of a subset of $\text{Diff}^r(M)$, $2 \leq r < \infty$, whose any diffeomorphism has a contracting non-trivial wandering domains.
Main results

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\[ \mathcal{N} \overset{\text{def}}{=} \bigcup_f \mathcal{N}_f: \text{Newhouse open set} \]

**Theorem A (Colli-Vargas’ conjecture ’01)**

The Newhouse open set \( \mathcal{N} \) is contained in the closure of a subset of \( \text{Diff}^r(M), \ 2 \leq r < \infty, \) whose any diffeomorphism has a contracting non-trivial wandering domains.
**Takens’ Last Problem**


\(X\): compact state space  
\(\varphi : X \rightarrow X\): continuous map  

**Definition**

An orbit \(\{x, \varphi(x), \varphi^2(x), \ldots\}\) has **historic behavior** if the measure

\[
\mu_{x,n} := \frac{1}{n+1} \sum_{i=0}^{n} \delta_{\varphi^i(x)},
\]

where \(\delta_{\varphi^i(x)}\) is the Dirac measure on \(X\) supported at \(\varphi^i(x)\) does not converge in the weak topology as \(n \rightarrow \infty\).
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Takens’ last problem [Takens 2008]

Whether are there *persistent classes* of smooth dynamical systems such that the set of initial states which give rise to orbits with *historic behavior* has *positive Lebesgue measure*?

Theorem B (Answer to Takesn’s last problem)

The Newhouse open set \( \mathcal{N} \subset \text{Diff}^r(M) \) has a dense subset where any diffeomorphism \( f \) has a contracting non-trivial wandering domain \( D \) such that, for any \( x \in D \), the forward orbit of \( x \) under \( f \) has *historic behavior*. 
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Hénon family $f_{a,b} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$f_{a,b}(x, y) = (1 - ax^2 + y, bx)$

where $a, b$: real parameters
A flowchart showing the behavior of the Hénon family of maps. The text states:

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An application to Hénon maps

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Open problem [van Strien ’10], [Lyubich-Martens ’11]

Does the Hénon family have a non-trivial wandering domain?

Corollary C

There is an open set $\mathcal{O}$ of the parameter space of Hénon family with $\text{Cl}(\mathcal{O}) \ni (2, 0)$ such that for every $(a, b) \in \mathcal{O}$, $f_{a,b}$ is approximated by $C^r$ diffeomorphisms, $2 \leq r < \infty$, which have historic & non-trivial wandering domains.
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Thank you for your kind attention!

The paper can be downloaded from:

arXiv:1503.06258

or

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