Error term of the prime periodic orbit theorem for expanding semiflows

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Counting periodic orbits for hyperbolic flows (1)

Theorem (*Prime Periodic Orbit Theorem*, **Margulis, Parry-Pollicott)**

For a topologically mixing Anosov flow $F^t : N \to N$,

$$\pi(T) := \#\{\gamma \in \mathrm{P.P.O} \mid |\gamma| \le T\} = (1 + o(1)) \cdot \int_1^T \frac{e^{ht}}{t} dt$$

as $T \to \infty$. (h: topological entropy)

More recently, it is obtained that

Theorem (Pollicott-Sharp, Stoyanov)

For a contact Anosov flow $F^t : N \to N$ (especially, for the geodesic flow on a negatively curved manifold), the (relative) error term $o(1)$ above is actually exponentially small as $T \to \infty$, that is, $\mathcal{O}(e^{-\varepsilon T})$ for some $\varepsilon > 0$.

Counting periodic orbits for hyperbolic flows (2)

For the geodesic flows on closed hyperbolic surfaces S, the following precise asymptotic formula admitting a few "resonance terms" is known (by using Selberg trace formula).

Theorem (Huber, 1961)

For the geodesic flow $F^t : T_1 S \to T_1 S$, we have

$$(\star) \qquad \pi(T) = \int_1^T \frac{e^{ht}}{t} dt + \sum_{j=1}^{k} \int_1^T \frac{e^{\chi_j t}}{t} dt + \mathcal{O}(e^{\rho T})$$

as $T \to \infty$, where $\rho = (3/4)h$, $\rho < \chi_j < h$, $1 \le j \le k$, are real constants. (Actually $h = 1$ and $\chi_j = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_j}$, $\lambda_j \in$ e.v. of Δ_M.)

Question: How general does this kind of asymptotic formula holds true?

"Theorem"([T]) For a contact Anosov flow $F^t : N^3 \to N^3$ in 3-dim, we have the formula $(\star)$ with error term

$$\mathcal{O}\left(\exp\left(\left(\frac{h + (\lambda_{\max}/2)}{2}\right)\right)\right)$$

where

$$\lambda_{\max} := \lim_{t\to\infty} \frac{1}{t} \log \max |\det(DF^t|_{E_u})| \geq h$$

and χ_j (now) are complex numbers s.t. $\rho < \mathrm{Re}\chi_j < h$.

- This is very good when $\lambda_{\max} \sim h$ ($\Rightarrow \rho \approx (3/4)h$).
- But vacuous when $\lambda_{\max} \geq 2h$ ($\Rightarrow \rho \geq h$).

Question: Why does this happen? How we can modify the arguement?

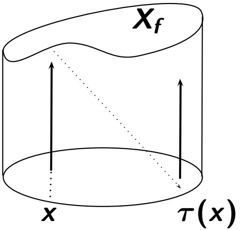
Simplified model

As a simplified model of Anosov flows, we henceforth consider suspension flow of expanding map on S^1:

Let $\tau : S^1 \to S^1$ be the map $x \mapsto \ell x$ with $\ell \geq 2$. Let $T^t : X_f \to X_f$ be the suspension semi-flow of τ with C^∞ ceiling function $f : S^1 \to \mathbb{R}$.

$$X_f = \{(x, s) \in S^1 \times \mathbb{R} \mid 0 \leq s < f(x)\}$$

This is an expanding semiflow



Transfer operator

We consider the one parameter (semi)group of transfer operators

$$\mathcal{L}^t : C^\infty(X_f) \to C^\infty(X_f), \qquad \mathcal{L}^t u(z) = \sum_{T^t(w)=z} u(w).$$

Its Atiyah-Bott trace is

$$\mathrm{Tr}^\flat \mathcal{L}^t = \sum_{n\geq 1} \sum_{\gamma \in \mathrm{P.P.O.}} \frac{|\gamma|}{1 - D_\gamma^{-n}} \cdot \delta(t - n|\gamma|)$$

where $D_\gamma > 1$ is the expansion rate along a prime periodic orbit γ.
Then we obtain

$$\int_1^T \frac{1}{t} \mathrm{Tr}^\flat \mathcal{L}^t dt = \pi(T) + \mathcal{O}(e^{(h/2)T}).$$

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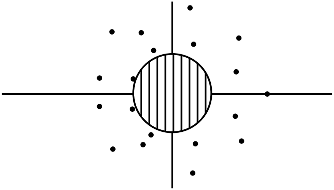
Essential spectral radius

If a bounded operator $\boldsymbol{L} : \boldsymbol{B} \to \boldsymbol{B}$ on a B-space $\boldsymbol{B}$ is written

$$\boldsymbol{L} = \boldsymbol{L_0} + \boldsymbol{K}, \quad \boldsymbol{K} : \text{compact}, \ \|\boldsymbol{L_0}\| < \boldsymbol{\lambda},$$

the spectral set of $\boldsymbol{L}$ in $\{|\boldsymbol{z}| > \boldsymbol{\lambda}\}$ consists of discrete eigenvalues. The infimum of such $\boldsymbol{\lambda} > \mathbf{0}$ is called the essential spectral radius.

Question: How small can we make the essential spectral radius of $\mathcal{L}^t : \boldsymbol{B} \to \boldsymbol{B}$ by choosing appropriate function spaces $\boldsymbol{B}$?



Remark: $\rho_{ess}(\mathcal{L}^t) < \exp(\rho t)$
$\Rightarrow$ error term in $(\star) \leq \exp((\rho + h)t/2)$
(by a technical reason).

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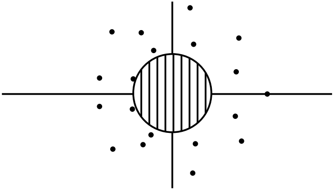

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Essential spectral radius

If a bounded operator $L : B \to B$ on a B-space B is written

$$L = L_0 + K, \quad K : \text{compact}, \ \|L_0\| < \lambda,$$

the spectral set of L in $\{|z| > \lambda\}$ consists of discrete eigenvalues. The infimum of such $\lambda > 0$ is called the essential spectral radius.

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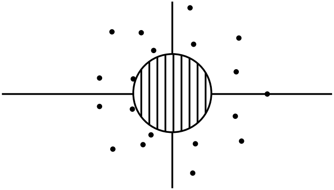

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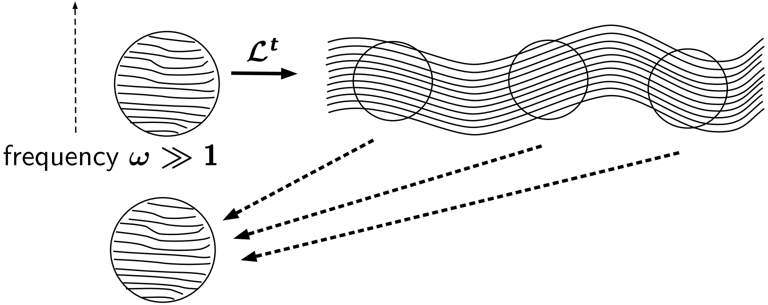
Decomposition w.r.t. frequency in the flow direction

The semiflow $\boldsymbol{F^t}$ is locally just a translation along the flow line.

If we decompose functions $\boldsymbol{X_f}$ with respect to frequency $\boldsymbol{\omega}$ in the flow direction, the decomposition is preserved, so that

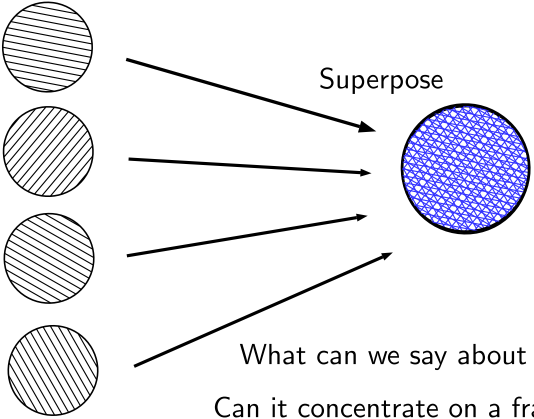
$$\mathcal{L}^t = \bigoplus_{\omega} \mathcal{L}^t_{\omega} \qquad \text{(almost)}$$

We estimate the operator norm of $\mathcal{L}^t_{\omega}$ in the limit $|\omega| \to \infty$.



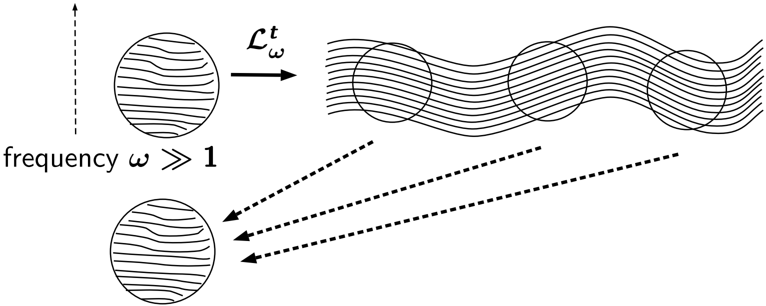
The essence of the problem

Consider superposition of some plane waves $\varphi_k(z) = A_k \exp(i\xi_k \cdot z)$ with $|\xi_k| \sim \omega \gg 1$, $|A_k| = a$



What can we say about the result?

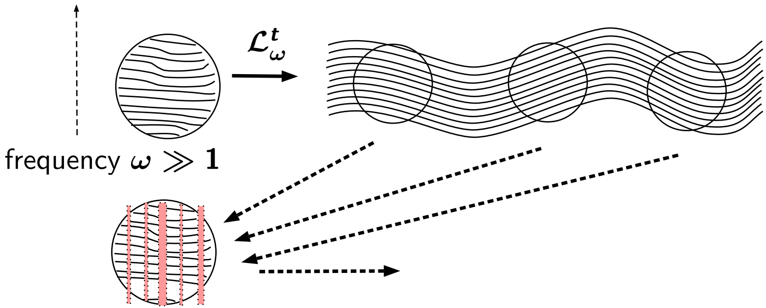
Can it concentrate on a fractal like subset?



- L^∞ estimate gives the trivial bound $\|\mathcal{L}^t_\omega\|_{L^\infty} \lesssim \exp(ht)$.
- L^2 estimate + "transversality condition" give the bound $\|\mathcal{L}^t_\omega\|_{L^2} \lesssim \exp((\chi_{\max}/2)t)$
  This corresponds to the estimate in "Theorem".

The essence of the problem (again)

The problem in the L^2 estimate is that we do not how the result of superposition can concentrate on a small fractal like subset on which the expansion of F^t is relatively large.



Idea: Let us consider the L^{2p} norm!

Main result

Theorem (Spectrum of $\mathcal{L}^t$)

There is a Banach space $C^\infty(X_f) \subset \mathcal{B}^{2p}(X_f) \subset L^{2p}(X_f)$ *for* $p \in \mathbb{N}$ *such that* $\mathcal{L}^t : \mathcal{B}^{2p}(X_f) \to \mathcal{B}^{2p}(X_f)$ *for sufficiently large* t *is bounded and the spectral radius of* $\mathcal{L}^t$ *is* e^{ht}.

For any $\varepsilon > 0$, *there exists an open and dense subset* $\mathcal{U}_p(\varepsilon) \subset C^\infty_+(S^1)$ *such that, if* $f \in \mathcal{U}_{2p}(\varepsilon)$, *the essential spectral radius of* $\mathcal{L}^t$ *for large* t *is bounded by* $\exp((\rho_{2p}(f) + \varepsilon)t)$ *where*

$$\rho_{2p}(f) := \frac{1}{2}\left(1 + \frac{\max\{p, \chi_{\max}(f)/h(f)\} - 1}{p}\right) \cdot h(f)$$

Theorem (restatement of the last part)

...., the essential spectral radius of $\mathcal{L}^t$ *for large* t *is bounded by* $\exp((\rho_{2p}(f) + \varepsilon)t)$ *where*

$$\rho_{2p}(f) := \frac{1}{2}\left(1 + \frac{\max\{p, \chi_{\max}(f)/h(f)\} - 1}{p}\right) \cdot h(f)$$

For the case $p = 1$, since $\chi_{\max}(f)/h(f) \geq 1$, we have

$$\rho_2(f) = \chi_{\max}(f)/2.$$

If we set $p_0 = \lceil \chi_{\max}(f)/h(f) \rceil$, we have

$$\min_{p \geq 1} \rho_{2p(f)} \leq \rho_{2p_0}(f) \leq \left(1 - \frac{1}{2p_0}\right) \cdot h(f)$$

This is always smaller than $h(f)$!

A consequence of the main theorem

Theorem (An asymptotic formula for $\pi(T)$)

For the expanding semi-flow $F^t : X_f \to X_f$ for C^∞ generic ceiling function f, we have, for any $\varepsilon > 0$, that

$$\pi(T) = \int_1^T \frac{e^{ht}}{t} dt + \sum_{j=1}^{k} \int_1^T \frac{e^{\chi_j t}}{t} dt + \mathcal{O}(e^{(\bar{\rho}+\varepsilon)T})$$

as $T \to \infty$, where

$$\bar{\rho} = \frac{h(f) + \min_{p \geq 1} \rho_p(f)}{2} \leq \left(1 - \frac{1}{4\lceil \chi_{\max}(f)/h(f) \rceil}\right) \cdot h(f).$$

and $\chi_j \in \mathbb{C}$, $1 \leq j \leq k$.

Idea of the proof

Theorem

For any $\varepsilon > 0$, there exists an open and dense subset $\mathcal{U}_p(\varepsilon) \subset C^\infty_+(S^1)$ such that, if $f \in \mathcal{U}_{2p}(\varepsilon)$, the essential spectral radius of $\mathcal{L}^t$ for large t is bounded by $\exp((\rho_{2p}(f) + \varepsilon)t)$ where

$$\rho_{2p}(f) := \frac{1}{2}\left(1 + \frac{\max\{p, \chi_{\max}(f)/h(f)\} - 1}{p}\right) \cdot h(f)$$

Idea of the proof:

$$\exp((\chi_{\max} + \max\{0, hp - \chi_{\max}\} + (p-1)h)/2p).$$

Question: How about non-integer p?

Thank you for your attention!