The Lagrange Spectrum of some square-tiled surface

(joint with P. Hubert, S. Lelièvre, C. Ulcigrai)

Luca Marchese

LAGA, Université Paris 13

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Classical Lagrange Spectrum

-Continued fraction expansion: $\alpha = [a_1, a_2, \dots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}$

-Define $L : \mathbf{R} \setminus \mathbf{Q} \to \mathbf{R}_+$ by $L(\alpha) := \limsup \; , \to\infty \; \frac{1}{\cdot | \; \alpha - \; |}$

-**Def/Prop** α is *badly approximable* iff $L(\alpha) < +\infty$ iff $\exists L > 0$ such that $\forall \epsilon > 0$ and all but finitely many p, q:

$$\left| \alpha - \frac{p}{q} \right| > \frac{1}{L + \epsilon} \cdot \frac{1}{q^2}.$$

In particular $L(\alpha)$ is the biggest L satisfying the condition.

-The set of such α is a totally disconnected, *thick* subset of $\mathbf{R}$ with measure zero.

-The classical *Lagrange Spectrum* is the set $\mathcal{L} \subset \mathbf{R}_+$ defined by

$$\mathcal{L} := \{L(\alpha); \alpha \text{ badly approximable}\}$$

Translation Surfaces

-Vectors $\zeta_1, \ldots, \zeta$ in $\mathbf{R}^2$ defining a polygon $P \subset \mathbf{R}^2$ with $2d$ sides, corresponding to two rearrangements of the ζ 's.

-*Translation surface*: $X = P/\partial P$, where any two sides in ∂P are identified iff they correspond to the same ζ .

-Genus g flat surface, with conical points $p_1, \ldots, p$, the conical angle at any p being $2(K + 1)\pi$ for any $i = 1, \ldots, r$. Constraint: $2g - 2 = k_1 + \cdots + k$. Normalization $\mathrm{Area}(X) = 1$.

-*Saddle connection:* geodesic segment γ in X connecting two conical points with no other p in its interior. Let $\mathrm{Hol}(\gamma)$ be its *planar development*. Point set $\mathrm{Hol}(X)$ whose elements are $\mathrm{Hol}(\gamma)$ for γ saddle connection.

-*Moduli space:* the orbifold $\mathcal{H}(k_1, \ldots, k\ )$ of all X whose conical points have prescribed angle. Local coordinates are the vectors $\zeta_1, \ldots, \zeta$.

-*Mahler criterion:* $\mathcal{K} \subset \mathcal{H}(k_1, \ldots, k\ )$ has compact closure iff $\exists c > 0$ such that for any $X \in \mathcal{K}$ we have

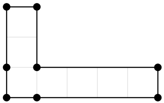
$$\mathrm{Sys}(X) := \min\{|v|, v \in \mathrm{Hol}(X)\} > c$$

Examples of translation surfaces

-*Standard torus* $X_1 := \mathbf{R}^2/\mathbf{Z}^2$. $\mathrm{Hol}(X_1) = \{(p,q); \gcd(p,q) = 1\}$.

-*Generic flat torus* $X_2 := \mathbf{R}^2/G \cdot \mathbf{Z}^2$, where $G \in \mathrm{SL}(2, \mathbf{Z})$. $\mathrm{Hol}(X_2) = G \cdot \mathrm{Hol}(X_1)$.

-*Square tiled surface.* Set $\zeta \; = (1,0)$, $\zeta \; = (0,-2)$, $\zeta \; = (4,0)$ and $\zeta \; = (0,-1)$. Let $E_3 = P/\partial P$, where sides in P appear in the two orderings (A, B, C, D) and (B, D, A, C).

$E_3 :=$ 

There exists a covering $\pi : E_3 \to X_1$ compatible with the two flat structures.

-*Generic surface in* $\mathcal{H}(2)$*:* $X_4 = P'/\partial P'$, where the identifications in P' are as in P and the sides ζ' of P' are generic perturbations of the sides $\zeta \;$ of P.

Action of $\mathrm{SL}(2, \mathbf{R})$

-For $X \in \mathcal{H}(k_1, \ldots, k\ )$ take polygon $P \subset \mathbf{R}^2$ with $X = P/\partial P$.
-For $G \in \mathrm{SL}(2, \mathbf{R})$ define $G \cdot X := GP/\partial GP$. It does not depend on P but just on X and G. Set

$$g\ := \begin{pmatrix} e & 0 \\ 0 & e^{-} \end{pmatrix} \ ; \ r_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

-*Stabilizer.* For any X set $\mathrm{Stab}(X) := \{G \in \mathrm{SL}(2, \mathbf{R}); G \cdot X = X\}$.

- We have $\mathrm{Stab}(X_1) = \mathrm{Stab}(X_2) = \mathrm{SL}(2, \mathbf{Z})$.
- $\mathrm{Stab}(E_3)$ is a finite index subgroup of $\mathrm{SL}(2, \mathbf{Z})$.
- $\mathrm{Stab}(X_4)$ for generic $X_4 \in \mathcal{H}(2)$.

-*Nice submanifolds* (Eskin-Mirzakhani and Es.-Mir.-Mohammadi). Any $\mathrm{SL}(2, \mathbf{R})$-orbit closure $\mathcal{M}$ is an affine submanifold of the moduli space carrying a nice $\mathrm{SL}(2, \mathbf{R})$-ergodic and $g\ $-ergodic probability measure.

Geometric interpretation and generalization

Consider $X_2 = \mathbf{R}^2/(\zeta\ \ \mathbf{Z} \oplus \zeta\ \ \mathbf{Z})$ and set $\alpha := \frac{|\mathrm{Re}(\zeta\ \ )|}{|\mathrm{Re}(\zeta\ \ +\zeta\ \ )|}$.

$$
\begin{aligned}
L(\alpha) &\overset{(\ )}{=} \limsup_{|\mathrm{Im}(\ \ )|\to\infty} \frac{1}{|\mathrm{Re}(v)|\cdot|\mathrm{Im}(v)|} \quad \text{where } v \in \mathrm{Hol}(X_2) \\
&\overset{(\ )}{=} \limsup_{\ \to+\infty} \frac{2}{\mathrm{Sys}^2(g\ \ \cdot X_2)} \\
&\overset{(\ )}{=} \limsup_{\ \to+\infty} [a\ _{-1}, a\ _{-2}, \dots a_1] + a\ \ + [a\ _{+1}, a\ _{+2}, \dots]
\end{aligned}
$$

-Equality (B) holds for any translation surface X. Generalize the classical function $\alpha \mapsto L(\alpha)$ by

$$
L(X) := \limsup_{|\mathrm{Im}(\ \ )|\to\infty} \frac{1}{|\mathrm{Re}(v)|\cdot|\mathrm{Im}(v)|} = \limsup_{\ \to+\infty} \frac{2}{\mathrm{Sys}^2(g\ \ \cdot X)}.
$$

-The *Lagrange Spectrum* of a nice manifold $\mathcal{M}$ is the set

$$
\mathcal{L}(\mathcal{M}) := \{L(X); X \in \mathcal{M}\} \subset \mathbf{R}_+.
$$

-Equality (A) can be generalized in terms of *interval exchange transformations*.

-Formula (C) generalizes to different formulas with renormalization

1. Hubert-M.-Ulcigrai: $\mathcal{L}(\mathcal{M})$ is closed for any $\mathcal{M}$, moreover closed Teichmuller geodesics in $\mathcal{M}$ provide a dense subset of values. Tool: *Rauzy-Veech* induction.
2. Artigiani-M.-Ulcigrai: If $\exists X \in \mathcal{M}$ such that $\mathrm{Stab}(X)$ is a lattice in $\mathrm{SL}(2, \mathbf{R})$ then $\mathcal{L}(\mathcal{M})$ has Hall's ray. Tool: $\mathrm{Stab}(X)$ acting by homographies.

Today's statement

-Consider the finite words $a := 1, 4, 2, 4$ and $b := 1, 3$. Let $\tilde{\sigma} : \Xi \to \Xi$ be the shift on $\Xi := \{a, b\}^{\mathbf{Z}}$.
-Consider the subset $\Xi_0 \subset \Xi$ of those $\bar{\xi} = (\xi \ )_{\ \in \mathbf{Z}}$ such that

$$\xi_0 = a$$
$$\xi \quad \text{for infinitely many} \, n \in \mathbf{Z}_+$$

then let $\sigma : \Xi \to \Xi$ be the first return of $\tilde{\sigma}$ to Ξ_0.
-Define a function $L^\sigma : \Xi \to \mathbf{R}_+$ by

$$L^\sigma(\bar{\xi}) := \limsup_{\ \to +\infty} [\sigma \ (\bar{\xi})]_{(-)} + [\sigma \ (\bar{\xi})]_{(+)} \text{ where}$$
$$[\bar{\xi}]_{(-)} := [1, 4, \xi_{-1}, \xi_{-2}, \dots] \text{ and}$$
$$[\bar{\xi}]_{(+)} := [1, 4, \xi_1, \xi_2, \dots].$$

$$\phi_1 := 7 + 14 \cdot [\overline{3,1}] = 10,696277\ldots$$
$$\phi_2 := 14 \cdot [1,4,\overline{1,3}] = 11,582576\ldots$$
$$\phi_\infty := 14 \cdot [1,4,\overline{1,4,2,4}] = 11,593101\ldots$$

An open interval (x,y) is a *gap* in a Lagrange spectrum $\mathcal{L}(\mathcal{M})$ if $(x,y) \cap \mathcal{L}(\mathcal{M}) = \emptyset$ and $x, y \in \mathcal{L}(\mathcal{M})$.

Theorem[Lelièvre-Hubert-M.-Ulcigrai] *Consider the closed orbit* $\mathcal{M} := \mathrm{SL}(2,\mathbf{R}) \cdot E_3$.

-ϕ_1 *is the minimum of* $\mathcal{L}(\mathcal{M})$*, moreover* (ϕ_1, ϕ_2) *is a gap in* $\mathcal{L}(\mathcal{M})$.

-$[\phi_2, \phi_\infty] \cap \mathcal{L}(\mathcal{M})$ *is the set* $\mathbf{K}$ *of values of* $L^\sigma : \Xi_0 \to \mathbf{R}_+$.

Corollary *Above its isolated minimum, the Lagrange spectrum* $\mathcal{L}(\mathcal{M})$ *has a rich structure.*

Question: is $\mathbf{K}$ a Cantor set? Do we have a constant $c > 0$ such that for any $\phi > \phi_2$ we have $HD\left((\phi_2, \phi)\right) > c$?

Elements of the proof

-Parametrize $\mathcal{L}(\mathcal{M})$ by $\alpha \mapsto L(E_3, \alpha) := L(r_{\arctan \alpha})$
-Any rational slope p/q induce a *cylinder decomposition* of E_3. Set

$$m(p/q, X) := \min \mathrm{Deg}(t \mapsto \pi \circ \gamma(t))$$

(minimum taken over all saddle connection $t \mapsto \gamma(t)$ in direction p/q on the square tiled surface X).

$$L(E_3, \alpha) = 7 \cdot \limsup_{\ ,\ \to +\infty} \frac{1}{q \cdot |q\alpha - p|} \cdot \frac{1}{m^2(p/q)}$$

$$\overset{(*)}{=} 7 \cdot \limsup_{\ \to \infty} \max_{1 \leq \ \leq \ } \frac{D(n, i, \alpha)}{m^2(\infty; R \cdot g(a_1, \dots, a_{\ -1}, i) \cdot E_3)}.$$

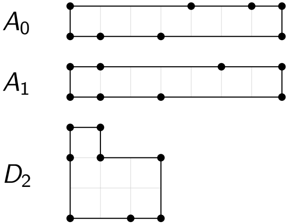
In $(*)$ for any n and any i with $1 \leq i \leq a$ we set

$$g(a_1, \dots, a_{\ -1}, i) := (T\ \ R)(T^{-\ \ -1}R) \dots (T^{-\ 2}R)(T^{\ 1}R)$$

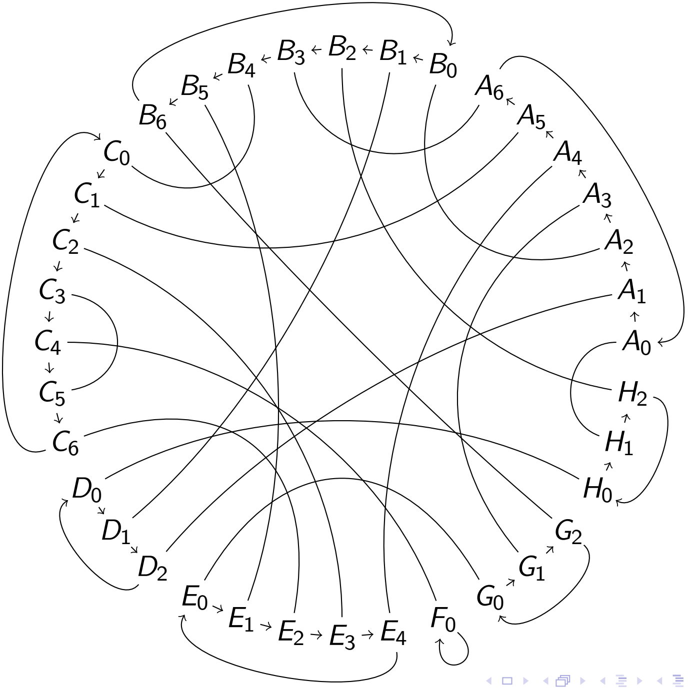
$$D(n, i, \alpha) := [a_{\ }, \dots, a_1] + a_{\ +1} + [a_{\ +2}, a_{\ +3}, \dots] \text{ if } i = a$$

$$D(n, i, \alpha) := [i, \dots, a_1] + [a_{\ } - i, a_{\ +1}, \dots] \text{ if } 1 \leq i < a\ .$$

The generators $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of $\mathrm{SL}(2, \mathbf{Z})$ are the arcs of a graph, whose vertices are the element of the $\mathrm{SL}(2, \mathbf{Z})$-orbit of E_3. For example we have $T \cdot A_0 = A_1$ and $S \cdot A_1 = D_2$, where



The global picture is...



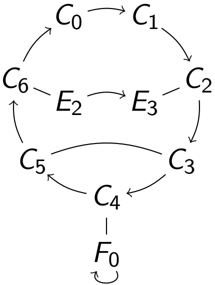
B_0
B_1
B_2
B_3
B_4
B_5
B_6
A_6
A_5
A_4
A_3
A_2
A_1
A_0
C_0
C_1
C_2
C_3
C_4
C_5
C_6
H_2
H_1
H_0
D_0
D_1
D_2
G_2
G_1
G_0
E_0
E_1
E_2
E_3
E_4
F_0

We follow methods from T. W. Cusick, M. E. Flahive: *The Markoff and Lagrange Spectra*. Mathematics Surveys and Monographs, 30, 1989.

Assume

$$L(E_3, \alpha) < 7 \cdot \frac{[1, \overline{1, 6}] + 6 + [\overline{6, 1}]}{4} = 11,688957\ldots$$

Then definitely $a \ \leq 5$ and $g(a_1, \ldots, a \ ) \cdot E_3$ belongs to the subgraph



If

$$L(E_3, \alpha) < 7 \cdot \frac{[1,4,2,\overline{1,5}] + 5 + [1,5,1,\overline{1,5}]}{4} = 11,655309\ldots$$

then the renormalization only uses the Gauss steps

$$\begin{tikzcd}
& C_6 \arrow[dr, bend right, "+1"'] & \\
C_5 \arrow[ur, "-4"] \arrow[loop left, "+2"] & & E_3 \arrow[ll, "-4"'] \arrow[ul, bend right, "-3"']
\end{tikzcd}$$

where for example $E_3 \overset{3-}{\longrightarrow} C_6$ encodes the operation $C_6 = T^{-3}S \cdot E_3$.

-Compare all these steps and see that the maximum is always taken by the Farey step at the middle of $C_5 \overset{2+}{\longrightarrow} C_5$.

Thank you!