Building Thermodynamics For Non-uniformly Hyperbolic Maps

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School and Conference on Dynamical Systems
ICTP, Trieste

August 7, 2015

The General Setting

X a compact metric space with metric d;
$f : X \to X$ a continuous map of finite topological entropy;
φ be a continuous function (potential) on X;
$\mathcal{M}(f)$ the space of all f-invariant Borel probability measures on X;
$\mu_\varphi \in \mathcal{M}(f)$ an equilibrium measure

$$P(\varphi) := -\inf_{\mu \in \mathcal{M}(f)} E_\mu(\varphi) = -E_{\mu_\varphi}(\varphi),$$

where $P(\varphi)$ is the topological pressure of φ and

$$E_\mu(\varphi) = -(h_\mu(f) + \int_X \varphi \, d\mu)$$

the free energy of the system w.r.t. μ.
It suffices to take the infimum over the space $\mathcal{M}^e(f) \subset \mathcal{M}(f)$ of ergodic measures.

Uniformly Hyperbolic Maps (Sinai, Ruelle, Bowen)

X a compact smooth manifold of dimension ≥ 2;
$f : X \to X$ a diffeomorphism;
$\Lambda \subset X$ a locally maximal hyperbolic set for f and assume that $f|\Lambda$ is topologically transitive.
φ a Hölder continuous potential φ.

1. Existence: there is an equilibrium measure μ_φ;
2. Uniqueness: μ_φ is a unique equilibrium measure;
3. Ergodic properties: μ_φ is Bernoulli (up to a discrete spectrum);
4. Decay of correlations: μ_φ has exponential decay of correlations and satisfies the CLT.

The proof uses symbolic representation of $f|\Lambda$ by a subshift of finite type via a Markov partition. This leads to developing thermodynamic formalism for symbolic subshifts.

The Geometric Potential

Of special interest is the geometric t-potential: a family of potential functions $\varphi_t(x) = -t \log |\mathrm{Jac}(df|E^u(x))|$ for $t \in \mathbb{R}$. Since the subspaces $E^u(x)$ depend Hölder continuously in x for each t the potential φ_t is Hölder continuous and hence, admits a unique equilibrium measure μ_t.
The pressure function $P(t) := P(\varphi_t)$ is convex, decreasing and real analytic in t. Note that $P(0)$ is the topological entropy $h_{\mathrm{top}}(f)$ of f and μ_0 is the unique measure of maximal entropy. Furthermore, there is a number $0 < t_0 \leq 1$ for which $P(t_0) = 0$. In the two-dimensional case the equilibrium measure μ_{t_0} is the measure of maximal Hausdorff dimension of $\Lambda \cap V^u(x)$. In the particular case when Λ is a topological attractor for f (i.e., there is a neighborhood U of Λ such that $\overline{f(U)} \subset U$ implying that $\Lambda = \bigcap_{n\geq 0} f^n(U)$) we have that $t_0 = 1$ and μ_1 is an SRB measure for f.

Away from uniform hyperbolicity there are classes of dynamical systems for which thermodynamical formalism is rather well understood. In particular, existence and uniqueness of equilibrium measures for the geometric t-potential is established for some intervals in t and ergodic properties of these measures are known up to the decay of correlations. Moreover, phase transitions where the pressure function is non-diffrentiable are found.
This include (the list is far from being complete):

- One-dimensional maps: unimodal and multimodal maps (Keller, Bruin, Todd, Dobbs, Iommi, Senti, Pesin, etc.); maps with an indifferent fixed point (Hu, Young, Sarig, etc.);
- Polynomial and rational maps (Przytycki, Letellier, Makarov, Smirnov, etc.).
- (Piecewise) non-uniformly expanding maps (Buzzi, Sarig, Alves, Luzzatto, Pinheiro, Oliveira, Varandas, Viana).

Non-uniformly Hyperbolic Maps

X a compact smooth manifold of dimension ≥ 2;
$f: X \to X$ a $C^{1+\alpha}$-diffeomorphism;
Γ the set of Lyapunov-Perron regular points with nonzero Lyapunov exponents;
$\mathcal{M}^e(f,\Gamma) \subset \mathcal{M}^e(f)$ the set of all ergodic measures that give full weight to the set Γ that is the set of hyperbolic ergodic measures;
φ a measurable potential;
μ an equilibrium measure if

$$P_\Gamma(\varphi) := -\inf_{\mu \in \mathcal{M}^e(f,\Gamma)} E_\mu(\varphi) = -E_{\mu_\varphi}(\varphi).$$

Thus we only allow hyperbolic equilibrium measures.

There are situations where it makes sense to define equilibrium measures using the subset $\mathcal{M}^e(f, \Gamma, h)$ of all measures in $\mathcal{M}^e(f, \Gamma)$ whose entropies are $> h$.

Usually $h = h_{\mathrm{top}}(f, S)$ is the topological entropy of f restricted to a not necessarily invariant subset $S \subset M$ of bad points. This concept originated in the work of Buzzi on piecewise invertible continuous maps of compact metric spaces.

More generally, one can define equilibrium measures using the subset $\mathcal{M}^e(f, \Gamma, \varphi, p)$ of all measures in $\mathcal{M}^e(f, \Gamma)$ for which

$$-E_\mu(\varphi) = h_\mu(f) + \int_\Lambda \varphi \, d\mu > p.$$

Usually $p = P_S(\varphi)$ is the topological pressure of f on a not necessarily invariant subset $S \subset M$ of bad points.

Since the set S may not be invariant, one can use the definition of the topological entropy and topological pressure on S which is based on the Carathéodory construction of dimension-like characteristics for dynamical systems.

Symbolic models I: Countable Markov Shifts, Aaronson–Denker, Mauldin–Urbanski, Sarig, Yuri

$S^{\mathbb{Z}}$ the symbolic space, where $S = \{1, 2, \dots\}$ the alphabet;
(Σ_A, σ) a subshift of countable type, A a transition matrix;
φ a potential;
Following Sarig we have that equilibrium (Gibbs) measures exist and unique for potentials that are

(O) bounded from above, have summable variations, finite Gurevich-Sarig pressure and are positive recurrent.

Smooth examples: $C^{1+\alpha}$ surface diffeomorphisms for which symbolic representations by countable Markov shifts are constructed via countable Markov partitions (Sarig). Existence and uniqueness of equilibrium measures for the geometric t-potential is not known.

Symbolic models II: Inducing Schemes of Hyperbolic Type, Senti, Zhang, P.

For inducing schemes of hyperbolic type and associated towers one has:

- the inducing domain – the base of the tower – $W = \bigcup_{J \in S} J$ where the union is taken over a countable collection of disjoint Borel sets J – basic elements;
- the inducing time $\tau = \tau(J)$ is the hight of the tower;
- the induced map $F : W \to W$ is conjugate to the full shift σ restricted to an invariant subset A of the symbolic space $S^{\mathbb{Z}}$ which is almost the whole space: the complement of A supports no invariant measures which gives positive weight to open sets;
- there is $h > 0$ such that

$$S_n = \#\{J \in S : \tau(J) = n\} \leq ce^{hn}.$$

Equilibrium measures exist and unique for

- the class of potentials that satisfy Condition (O);
- the class of invariant measures $\mathcal{M}^e(f, \Gamma, h)$; these measures can be lifted to the tower.

Smooth examples:

Young's diffeomorphisms:

- the base of the tower is a set of positive volume that has direct product structure of local stable and unstable manifolds; each basic element is a Cantor set of positive volume which consists of local stable manifolds;
- the inducing time is a (not necessarily the first) return time to the base;
- the induced map has the Markov property;
- the potential is the geometric t-potential for t in some interval $(t_0, 1]$ with $t_0 < 0$;
- equilibrium measures have exponential decay of correlations and satisfy CLT.

Particular examples include:

1. The Henón map at the first bifurcation (Senti, Takahashi);
2. The Katok map: a slow-down of a linear automorphism of the 2-torus near the origin (Senti, Zhang, P.).

Symbolic models III: Nonuniform Specification, Climenhaga, Thompson

The symbolic space is $(S^{\mathbb{Z}}, \sigma)$ where $S = \{1, \dots, p\}$;
$\mathcal{A}$ is an invariant subset;
a word $w = (w_1, \dots, w_n)$ of symbols is $\mathcal{A}$-admissible if there is a point $w \in \mathcal{A}$ which contains a segment of symbols $(w_1, \dots, w_n)$.
$\mathcal{A}$ has the uniform specification property if any two $\mathcal{A}$-admissible words can be connected by an $\mathcal{A}$-admissible word whose length is uniformly bounded from above.
$\mathcal{A}$ has the non-uniform specification property if almost any two $\mathcal{A}$-admissible words can be connected by an $\mathcal{A}$-admissible word.
Existence and uniqueness of equilibrium measures are shown for systems satisfying the uniform specification property and for Hölder continuous potentials and the results can be extended to certain degree to systems with non-uniform specification.

Smooth examples:

1. the Mañé example on the 3-torus (the map is partially hyperbolic) and the Bonatti-Viana example on the 4-torus (the map has a dominated splitting but is not partially hyperbolic); both examples are derived from Anosov and are robustly transitive; and a Hölder continuous potential; in fact, given a Hölder continuous potential φ, there is a C^1-open set of such examples with the unique equilibrium measure for φ (Climenhaga, Fisher, Thompson);
2. the C^2 Mañé example or the Bonatti-Viana example and the geometric t-potential $-t\log|\det(df|E^{cu})|$ with t in some interval $(t_0, t_1) \supset [0, 1]$ (Climenhaga, Fisher, Thompson);
3. the geodesic flow on a rank 1 surface of nonpositive curvature and the geometric t-potential for every t in $(-\infty, 1)$ (Burns, Climenhaga, Fisher, Thompson).

Constructing Equilibrium Measures For the Geometric t-potential: Non-symbolic Approach

For all $t \in \mathbb{R}$ the geometric t-potential φ_t is well defined on Γ and is a measurable function on M. Furthermore, the pressure function $P(t, \Gamma)$ is well defined, monotonically decreasing and continuous. We consider the situations where there is a (hyperbolic) SRB measure $\mu \in \mathcal{M}^e(f, \Gamma)$ that is the set $\Lambda = \overline{\Gamma} = \mathrm{supp}(\mu)$ is a topological attractor for f. It follows from the results of Ledrappier and Strelcyn on characterization of measures satisfying the entropy formula that $\mu = \mu_1$ is the unique equilibrium measure for φ_1. At this point our approach is to

- construct SRB measures for f;
- construct equilibrium measures for φ_t for $t \in (t_0, t_1)$ with $t_0 < 0 < 1 \leq t_1$;
- study differentiability of the pressure function $P(t)$.

Ergodic Properties: SRB Measures

By a result of Ledrappier, a hyperbolic SRB measure has at most countably many ergodic components and every hyperbolic SRB measure is Bernoulli up to a discrete spectrum. It follows that there may exist at most countably many ergodic SRB measures on Λ. One way to ensure uniqueness of SRB measures is to show that its every ergodic component is open (mod 0) in the topology of Λ and that $f|\Lambda$ is topologically transitive.

Ergodic Properties: Equilibrium Measures

Let μ be a hyperbolic ergodic measure for a $C^{1+\alpha}$ diffeomorphism f. Given $\ell > 0$, consider the regular set Γ_ℓ, which consists of points $x \in \Gamma$ whose local stable $V^s(x)$ and unstable $V^u(x)$ manifolds have size at least $\frac{1}{\ell}$. For $x \in \Gamma_\ell$ and some sufficiently small $r > 0$ let $R_\ell(x, r) = \cup_{y \in A^u(x)} V^s(y)$ be a rectangle at x, where $A^u(x)$ is the set of points of intersection of $V^u(x)$ with local stable manifolds $V^s(z)$ for $z \in \Gamma_\ell \cap B(x, r)$.

We denote by

- $\pi\colon V^u(z_1) \to V^u(z_2)$ with $z_1, z_2 \in R_\ell(x, r) \cap \Gamma_\ell$ the holonomy map generated by local stable manifolds;
- $\mu^u(z)$ the conditional measure generated by μ on local unstable manifolds $V^u(z)$.

We say that μ has a direct product structure if the holonomy map π is absolutely continuous that is for any $z_1, z_2 \in R_\ell(x, r) \cap \Gamma_\ell$ the measure $\pi_*\mu(z_1)$ on $V^u(z_2)$ is absolutely continuous with respect to $\mu^u(z_2)$ with the Jacobian to be uniformly bounded from above and uniformly away from zero on $R_\ell(x, r) \cap \Gamma_\ell$.

Conjecture: If μ is a hyperbolic ergodic equilibrium measure for the geometric t-potential for a $C^{1+\alpha}$ diffeomorphism f, then μ has a direct product structure.

If true this would imply that μ has some "nice" ergodic properties, e.g. it has at most countably many ergodic components.

Constructing SRB measures: General Comments

There are currently two ways to construct SRB measures.
The first one is based on choosing an appropriate natural measure and then pushing it forward under the dynamics. A limit measure resulting from this procedure is a natural candidate for an SRB measure.
Pros: allows to construct SRB measures under rather general requirements on the system.
Cons: establishing exponential (or polynomial) decay of correlations and the CLT is currently out of reach.
The second way is to construct a symbolic representation of the map by either a subshift of finite type (Sinai, Bowen, Ruelle), or a subshift of countable type (Sarig), or a tower (L.-S. Young).
Pros: allows to obtain exponential (or polynomial) decay of correlations and the CLT.
Cons: there are many examples when SRB measures can be constructed using the first approach but for which symbolic representation is not known or may not exist.

Constructing SRB Measures: Chaotic Attractors

Consider the set $D \subset U$ (U is a neighborhood of the attractor Λ) of points such that

- $f(D) \subset D$, i.e., D is forward invariant;
- there are two measurable cone families $K^s(x) = K^s(x, E^s(x), \theta)$ and $K^u(x) = K^u(x, E^s(x), \theta)$, which are invariant, i.e.,

$$\overline{df(K^u(x))} \subset K^u(f(x)), \quad \overline{df^{-1}(K^s(f(x)))} \subset K^s(x)$$

  and transverse, i.e., $T_xM = E^s(x) \oplus E^u(x)$.

Set

1. $\lambda^u(x) = \inf\{\log \|df(v)\| \colon v \in K^u(x), \|v\| = 1\}$,
   $\lambda^s(x) = \sup\{\log \|df(v)\| \colon v \in K^s(x), \|v\| = 1\}$
   – expansion and contraction coefficients at x;
2. $d(x) = \max(0, (\lambda^s(x) - \lambda^u(x))$ – defect of hyperbolicity;
3. $\lambda(x) = \lambda^u(x) - d(x)$ – coefficient of effective hyperbolicity;
4. $\alpha(x) = \measuredangle(K^s(x), K^u(x)) > 0$ – angle between the cones;
5. $\rho_{\hat{\alpha}}(x) = \lim_{n\to\infty} \frac{1}{n}\#\{0 \le k < n \colon \alpha(f^k(x)) < \hat{\alpha}\}$ – average time the angle between the cones is below a given threshold $\hat{\alpha} > 0$.

We further assume that for every $x \in D$

- $\liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \min(\lambda(f^k(x)), -\lambda^s(f^k(x))) > 0$;
- $\lim_{\bar\alpha\to 0} \rho_{\bar\alpha}(x) = 0$.

Theorem (Climenhaga, Dolgopyat, P.)

$$Leb(D) > 0 \Rightarrow SRB\ measure.$$

We call the attractor Λ chaotic if $\mathrm{Leb}(D) > 0$.

Idea of the Proof

Fix $n > 0$ and call a local manifold V through x n-admissible if

$$d(f^{-k}(x), f^{-k}(y)) \leq Ce^{-\lambda k} d(x,y) \text{ for all } 0 \leq k \leq n \text{ and } x, y \in V.$$

Consider the sets:

$$H = \{x \in U\colon \text{all Lyapunov exponents nonzero at } x\},$$

$$\mathcal{M}_n = \{\nu\colon \text{supp. on and a.c. on } n\text{-admissible manifolds}, \nu(H) = 1\}.$$

Each $\nu \in \mathcal{M}_n$ corresponds to a standard pair (V, ρ) (Chernov, Dolgopyat) where V is an n-admissible manifold and ρ the density of ν with respect to the leaf-volume on V.

The set $\mathcal{M}_n$ has various non-uniformities.

1. Value of C, λ in the definition of n-admissibility;
2. Size and curvature of admissible manifolds;
3. $\|\rho\| = \max_{x \in V(x)} \rho(x)$.

Given $K > 0$, let $\mathcal{M}_n(K)$ be the set of measures for which these non-uniformities are all controlled by K.

The set $\mathcal{M}_n(K)$ is compact, but not f_*-invariant.
Choose $m \in \mathcal{M}_n(K)$ and let

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k m. \tag{1}$$

Fix $K > 0$ and write $\mu_n = \nu_n + \zeta_n$, where $\nu_n \in \mathcal{M}_n(K)$.

Theorem (Climenhaga–Dolgopyat–P.)

Assume that for some sequence n_k and some measure μ we have $\mu_{n_k} \to \mu$. Assume also that

$$\overline{lim}_{n_k \to \infty} \|\nu_{n_k}\| > 0. \tag{2}$$

Then some ergodic component of μ is an SRB measure for f.

The idea is to replace the push-forward procedure (**??**) with a more elaborative one that takes care of a given potential φ. Namely, for $m \in \mathcal{M}_n(K)$ let

$$\mu_n^\varphi = \frac{1}{\mathcal{L}_n(\varphi, W)} \int_W e^{S_n\varphi(x)} \mathcal{E}_n(x) d(m \circ f^n),$$

where $\mathcal{E}_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)}$ is the nth empirical measure for the orbit of x and

$$\mathcal{L}_n(\varphi, W) = \int_W e^{S_n\varphi(x)} d(m \circ f^n),$$

is the normalizing factor (so that μ_n^φ is a probability measure). In the case when φ is the geometric 1-potential $\Lambda_n(\varphi, W) = 1$ and $\mu_n^\varphi = \mu_n$ is given by (**??**) thus recovering the push-forward procedure for the SRB measure.

Equilibrium Measures For the Geometric t-potential.

In the case of the geometric t-potential φ_t let $\mu_{t,n} = \mu_n^{\varphi_t}$. For $K > 0$ write $\mu_{t,n} = \nu_{t,n} + \zeta_{t,n}$, where $\nu_{t,n} \in \mathcal{M}_n(K)$.
Problem: Assume that for some sequence n_k and some measure μ we have $\mu_{t,n_k} \to \mu$. Assume also that

$$\overline{lim}_{n_k \to \infty} \|\nu_{t,n_k}\| > 0.$$

Show that there is a number $t_0 < 0$ such that for every $t \in (t_0, 1)$ some ergodic component of μ is an equilibrium measure for φ_t.

I will describe a somewhat different approach for constructing equilibrium measures, which is a generalization of the original Sinai's method. Let f be a $C^{1+\alpha}$ diffeomorphism of a compact smooth manifold M and μ an invariant probability measure on M, which we call a reference measure. Given a potential function φ, consider the sequence of measures $\mu_{m,n}(\varphi)$ which are absolutely continuous with respect to μ with the density

$$\frac{d\mu_{m,n}(\varphi)(x)}{d\mu(x)} = \frac{\exp\left(\sum_{k=-n}^{m}\varphi(f^k(x))\right)}{\int_M \exp\left(\sum_{k=-n}^{m}\varphi(f^k(x))\right) d\mu}.$$

In the case when f is an Anosov diffeomorphism, one chooses μ to be the measure of maximal entropy. Then for any Hölder continuous function φ the sequence of measures $\mu_{m,n}$ converges to the unique equilibrium measure for φ as $m, n \to \infty$.

Problem: Assume that M is a chaotic attractor for f, i.e., almost every point $x \in M$ with repsct to the Riemannian volume lies in the set D described above. Further assume that f has a measure μ of maximal entropy. Show that for every t in some interval $(t_0, 1)$ with $t_0 < 0$ the sequence of measures $\mu_{m,n}$ converges to unique equilibrium measure μ_t for the geometric t-potential φ as $m, n \to \infty$.