# COMPLETE LORENTZIAN 3-MANIFOLDS

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*To Ravi Kulkarni, for his seventieth birthday.*

ABSTRACT. Based on four lectures the authors gave in Almora on flat Lorentzian manifolds, these notes are an introduction to Lorentzian three-manifolds. In particular, we provide examples of quotients of Minkowski space by the actions of groups acting freely and properly discontinuously. Most of these notes deal with complete Lorentz manifolds whose fundamental groups are both *free* and *non-abelian*. We shall also look at Lorentz manifolds whose fundamental groups are *solvable* in some detail.

## CONTENTS



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## 1. Introduction

This is an expanded version of the four lectures the authors gave in Almora, Uttarakhand, India on flat Lorentzian manifolds. We will discuss how to obtain examples of such manifolds by taking quotients of Minkowski space by the actions of groups acting freely and properly discontinuously. We will spend most of our time on groups that are both **free** and **non-abelian**.

As the reader may know, showing that a group acts properly can be quite tricky. One technique is to display a fundamental domain for the action, but sometimes showing that what we have is indeed a fundamental domain might itself require some work. In particular, our examples of actions by free groups will use fundamental domains bounded by disjoint objects. This is why we need *crooked planes*, or at least, something like them.

We will also discuss another criterion for properness (of free groups), called the Margulis invariant. This measure of "signed Lorentzian displacement" may be used to detect whether a group acts freely, and even properly. In truth, the Margulis invariant as we present it yields a **necessary** condition for properness. (A generalized Margulis invariant introduced by Goldman, Labourie and Margulis yields a **sufficient** condition as well [19].)

The paper is structured to follow the order of our talks. Section 2 introduces Minkowski space as an affine space and describes its isometries. We rely on the association with the hyperbolic plane to describe these isometries. Section 3 defines proper actions and fundamental domains; we then proceed to present examples, ranging from cyclic groups to free non-abelian groups and the solvable groups "in between". This is the section where we discuss crooked planes and crooked fundamental domains.

Section 4 is devoted to the Margulis invariant. In particular, we use the Margulis invariant to relate our free groups, which are "affine deformations" of linear groups, to infinitesimal deformations of hyperbolic structures on surfaces. Here the presence of the hyperbolic plane in Minkowski space will play a key role. Finally, Section 5 launches us into the Einstein Universe, which can be seen as the conformal compactification of Minkowski space. We will revisit actions of free groups, using objects called *crooked surfaces*.

This text is really an overview of the field from where we stand. The reader interested in the finer details will want to consult the papers we reference. For instance, the lecture notes by the second author [12] overlap a little with these notes, but are mostly complementary. The survey by Abels [1] offers an excellent discussion of proper actions of groups of affine transformations; see also [9]. For a more comprehensive guide to Lorentzian and hyperbolic geometry, we suggest Ratcliffe [26].

## 2. Basic Lorentzian geometry

We first introduce elementary notions about 3-dimensional Minkowski space, its relationship to the hyperbolic plane, and its isometries.

### 2.1. Affine space and its tangent space.

We define *$n$-dimensional affine space* $\mathsf{A}^n$ to be the set of all $n$-tuples of real numbers $(p_1, \dots, p_n)$. An affine space could be defined over any field, but we will restrict to the field of real numbers. Elements of affine space will be called *points*.

Affine space $\mathsf{A}^n$ should not be confused with the vector space $\mathbb{R}^n$, even though one often identifies the two. In these notes, we use plain font to denote points in affine space: $p, q, r$, etc. and bold font to denote vectors in a vector space: $\mathbf{t}, \mathbf{u}, \mathbf{v}$, etc.

A vector space does act on its corresponding affine space. For a point $p = (p_1, \dots, p_n) \in \mathsf{A}^n$ and a vector $\mathbf{t} = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$, we define:

$$p + \mathbf{t} = (p_1 + t_1, \dots, p_n + t_n) \in \mathsf{A}^n.$$

The vector space $\mathbb{R}^n$, considered as a Lie group, acts transitively on $\mathsf{A}^n$ by translations, which we denote as follows:

$$\begin{aligned} \tau_{\mathbf{t}} : \mathbb{R}^n \times \mathsf{A}^n &\longrightarrow \mathsf{A}^n \\ (\mathbf{t}, p) &\longmapsto p + \mathbf{t}. \end{aligned}$$

The stabilizer of a point is the trivial subgroup $\{\mathbf{0}\}$; as a homogeneous space, $\mathsf{A}^n$ identifies with $\mathbb{R}^n$. But the homogeneity of $\mathsf{A}^n$ means that all points look the same, including $(0, \dots, 0)$.

Affine space, as opposed to a vector space, lacks a notion of sum, but the action of translations on the affine space yields a notion of *difference*:

$$p - q = \mathbf{t} \quad \text{if and only if } p = q + \mathbf{t}$$

where, of course, $p, q \in \mathsf{A}^n$ and $\mathbf{t} \in \mathbb{R}^n$.

Affine space is an $n$-dimensional manifold with trivial tangent bundle. The action by translation allows a precise description of the tangent space to a point in $\mathsf{A}^n$, which is $\mathbb{R}^n$. Adding additional structure to $\mathbb{R}^n$, such as an inner product, allows us to endow the tangent bundle with a corresponding structure – as we will do next. We then say that the affine space is *modeled* on the inner product space.

2.2. **Light, space and time: the causal structure of Minkowski space.** Given $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, set:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 - u_3 v_3.$$

This is a symmetric, non-degenerate bilinear form of signature (2,1). Let $\mathsf{V}$ denote the vector space $\mathbb{R}^3$ endowed with this inner product. We will further assume $\mathsf{V}$ to have the usual orientation, by requiring that the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be positively oriented.

A vector $\mathbf{v} \neq \mathbf{0} \in \mathsf{V}$ is called

- *timelike* if $\mathbf{v} \cdot \mathbf{v} < 0$,
- *null* (or *lightlike*) if $\mathbf{v} \cdot \mathbf{v} = 0$,
- *spacelike* if $\mathbf{v} \cdot \mathbf{v} > 0$; when $\mathbf{v} \cdot \mathbf{v} = 1$, it is called *unit-spacelike*.

For instance, $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit-spacelike and $\mathbf{e}_3$ is timelike. The set of null vectors is called the *lightcone*.

Say that vectors $\mathbf{u}, \mathbf{v} \in \mathsf{V}$ are *Lorentz-orthogonal* if $\mathbf{u} \cdot \mathbf{v} = 0$. Denote the linear subspace of vectors Lorentz-orthogonal to $\mathbf{v}$ by $\mathbf{v}^{\perp}$.

Typically, to speak of a *causal structure* we also need a *time orientation* on $\mathsf{V}$, which consists of choosing one of the two connected components of the set of timelike vectors. Here we choose, as usual, the component containing $\mathbf{e}_3$ and denote it by $\mathsf{Future}$. Call a non-spacelike vector $\mathbf{v} \neq \mathbf{0}$ and its corresponding ray *future-pointing* if $\mathbf{v}$ lies in the closure of $\mathsf{Future}$.

*Minkowski space*, denoted $\mathsf{E}$, is the affine space modeled on $\mathsf{V}$. In fact, $\mathsf{E}$ is a smooth manifold with a Lorentzian (or semi-Riemannian) metric, meaning that it is a *Lorentzian manifold*. As we will see later, it is a *flat Lorentzian manifold*. The orientation and time-orientation on $\mathsf{V}$ endow $\mathsf{E}$ with these orientations as well.

2.2.1. *Cross-product.* The orientation on $\mathsf{V}$ allows us to define a determinant; we adopt the usual definition so that:

$$\mathsf{Det}\,[\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3] = 1.$$

This in turn determines a unique alternating bilinear mapping $\mathsf{V}\times\mathsf{V}\to\mathsf{V}$, called the *Lorentzian cross-product,* such that:

$$\mathsf{Det}(\mathbf{u},\mathbf{v},\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}. \tag{1}$$

2.2.2. *Null frames.* Let $\mathbf{s}\in\mathsf{V}$ be a unit-spacelike vector. The restriction of the inner product to the orthogonal complement $\mathbf{s}^\perp$ is also an inner product, of signature $(1,1)$. The intersection of the lightcone with $\mathbf{s}^\perp$ consists of two null lines intersecting transversely at the origin. Choose a linearly independent pair of future-pointing null vectors $\mathbf{s}^\pm\in\mathbf{s}^\perp\cap\mathsf{Future}$ such that $\{\mathbf{s},\mathbf{s}^-,\mathbf{s}^+\}$ is a positively oriented basis for $\mathsf{V}$. Call such a basis a *null frame* associated to $\mathbf{s}$. See Figure 1.

Alternatively, given a spacelike line $\mathbb{R}\mathbf{v}$, an associated null frame is a basis $\{\mathbf{u},\mathbf{n}_1,\mathbf{n}_2\}$, where $\mathbf{n}_1,\mathbf{n}_2$ are a pair of future-pointing null vectors spanning $\mathbf{v}^\perp$, and $\mathbf{u}$ is a unit-spacelike vector spanning $\mathbb{R}\mathbf{v}$ that is a positive scalar multiple of $\mathbf{n}_1\times\mathbf{n}_2$.

The null vectors $\mathbf{s}^-$ and $\mathbf{s}^+$ are defined only up to positive scaling. Margulis [22, 23] takes them to have unit Euclidean length.

Given a null frame, the Gram matrix, the symmetric matrix of inner products, has the form:

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & -k^2\\ 0 & -k^2 & 0\end{bmatrix}.$$

We can show that the off-diagonal entry, $\frac{1}{2}\mathbf{s}^-\cdot\mathbf{s}^+$, is negative since both $\mathbf{s}^-$ and $\mathbf{s}^+$ are future-pointing.

The null frame defines linear coordinates $(a,b,c)$ on $\mathsf{V}$:

$$\mathbf{v} = a\mathbf{s} + b\mathbf{s}^- + c\mathbf{s}^+.$$

If we choose $\mathbf{s}^-$ and $\mathbf{s}^+$ such that $k=\frac{1}{\sqrt{2}}$, then the corresponding Lorentzian metric on $\mathsf{E}$ is:

$$da^2 - db\ dc. \tag{2}$$

We close this section with a useful identity which we call the *null basis identity* [13]:

$$\begin{aligned}\mathbf{s}\times\mathbf{s}^- &= \mathbf{s}^-\\ \mathbf{s}\times\mathbf{s}^+ &= -\mathbf{s}^+.\end{aligned} \tag{3}$$

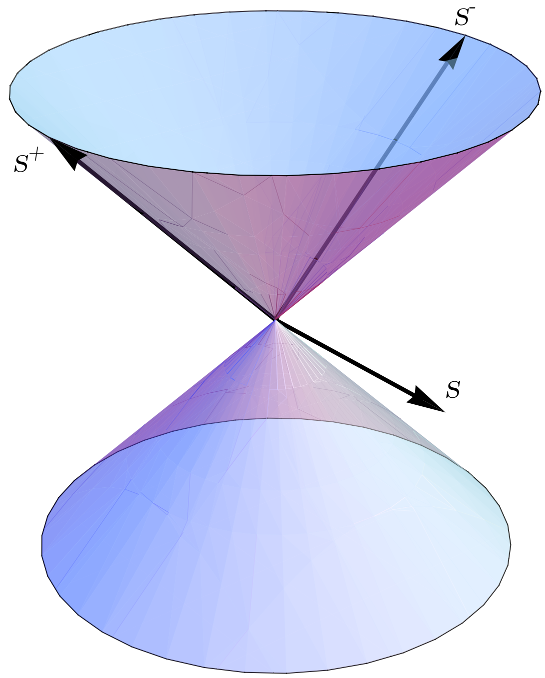


FIGURE 1. A null frame.

2.3. **How the hyperbolic plane sits in Minkowski space.** Let $\mathsf{H}^2$ denote the set of unit future-pointing timelike vectors:

$$\mathsf{H}^2 = \{\mathbf{v} \in \mathsf{Future} \,|\, \mathbf{v}\cdot\mathbf{v} = -1\}.$$

The restriction of the Lorentzian metric to $\mathsf{H}^2$ is positive definite. We can thus define a metric, denoted $d_{\mathsf{H}^2}$, by setting:

$$\cosh\left(d_{\mathsf{H}^2}(\mathbf{u},\mathbf{v})\right) = \mathbf{u}\cdot\mathbf{v}$$

for $\mathbf{u},\mathbf{v} \in \mathsf{H}^2$. This is a Riemannian metric with constant curvature $-1$, allowing us to identify $\mathsf{H}^2$ with the hyperbolic plane.

Geodesics in the hyperbolic plane correspond to indefinite planes (through the origin) in $\mathsf{V}$, which are precisely the planes that intersect $\mathsf{H}^2$. Equivalently, these are Lorentz-orthogonal planes to spacelike vectors. Thus each spacelike vector $\mathbf{s}$ is identified with a geodesic in $\mathsf{H}^2$:

$$\ell_{\mathbf{s}} = \mathbf{s}^{\perp} \cap \mathsf{H}^2.$$

Also, we identify the vector $\mathbf{s}$ with one of the open halfplanes bounded

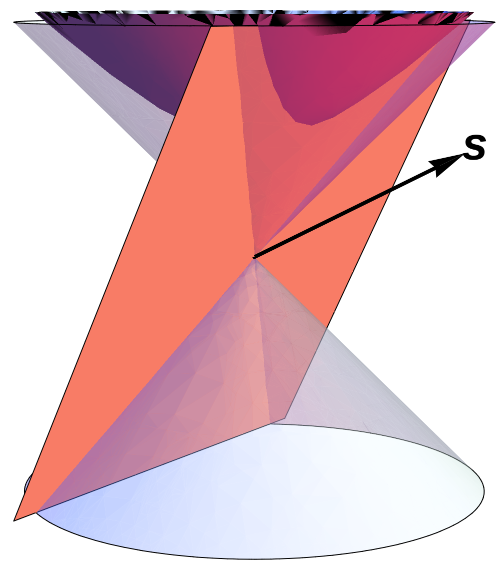


FIGURE 2. The identification between a spacelike vector $\mathbf{s}$ and a line in $\mathsf{H}^2$. The part of $\mathsf{H}^2$ in front of the plane is $\mathfrak{h}_{\mathbf{s}}$.

by the $\ell_{\mathbf{s}}$. Namely, the halfplane $\mathfrak{h}_{\mathbf{s}}$ is the set of vectors $\mathbf{v} \in \mathsf{H}^2$ such that $\mathbf{v} \cdot \mathbf{s} > 0$. See Figure 2.

Thinking of the upper halfplane model of the hyperbolic plane, we will denote the orientation-preserving isometries of $\mathsf{H}^2$ by $\mathrm{PSL}(2, \mathbb{R})$. The identification of the hyperbolic plane with $\mathsf{H}^2$ induces an isomorphism between $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{SO}^0(2, 1)$.

A more detailed and complete correspondence between Lorentzian geometry and hyperbolic spaces can be found in [26].

2.4. **Isometries and similarities of Minkowski space.** Identify $\mathsf{E}$ with $\mathsf{V}$ by choosing a distinguished point $o \in \mathsf{V}$, which we call an *origin.* For any point $p \in \mathsf{E}$ there is a unique vector $\mathbf{v} \in \mathsf{V}$ such that $p = o + \mathbf{v}$. Thus the choice of origin defines a bijection :

$$\mathsf{V} \xrightarrow{A_o} \mathsf{E}$$
$$\mathbf{v} \longmapsto o + \mathbf{v}.$$

For any $o_1, o_2 \in \mathsf{E}$,

$$A_{o_1}(\mathbf{v}) = A_{o_2}\big(\mathbf{v} + (o_1 - o_2)\big)$$

where $o_1 - o_2 \in \mathsf{V}$ is the unique vector translating $o_2$ to $o_1$.

A transformation $\mathsf{E} \xrightarrow{\gamma} \mathsf{E}$ is called *affine* if and only if there exists a linear map $\mathsf{L}(\gamma)$ such that, for every $p, q \in \mathsf{E}$ :

$$\gamma(p) - \gamma(q) = \mathsf{L}(\gamma)(p - q).$$

Equivalently, $\gamma$ is an affine transformation if there is a linear transformation $\mathsf{L}(\gamma)$, called its *linear part*, and $\mathbf{u} \in \mathsf{V}$, called its *translational part*, such that :

$$\gamma(p) = o + \mathsf{L}(\gamma)(p - o) + \mathbf{u}.$$

Note that the linear part does not depend on the choice of origin but the translational part does.

The group of orientation-preserving affine automorphisms of $\mathsf{E}$ thus decomposes as a semidirect product :

$$\mathrm{Aff}^+(\mathsf{E}) \;=\; \mathsf{V} \rtimes \mathrm{GL}^+(3, \mathbb{R}).$$

The elements of $\mathrm{GL}(3, \mathbb{R})$ which preserve the inner product "$\cdot$" will be called *linear Lorentzian isometries*. The group of linear Lorentzian isometries is denoted $\mathrm{O}(2, 1)$ and its subgroup of orientation-preserving elements, $\mathrm{SO}(2, 1)$ :

$$\mathrm{SO}(2, 1) = \mathrm{O}(2, 1) \cap \mathrm{GL}^+(3, \mathbb{R}).$$

Thus the group of orientation-preserving *Lorentzian isometries* of $\mathsf{E}$ decomposes as follows :

$$\mathrm{Isom}^+(\mathsf{E}) \;=\; \mathsf{V} \rtimes \mathrm{SO}(2, 1).$$

The one-parameter group $\mathbb{R}^+$ of *positive homotheties* :

$$\begin{bmatrix} e^s & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^s \end{bmatrix}$$

(where $s \in \mathbb{R}$) acts conformally on $\mathsf{V}$, preserving orientation. We obtain the group of orientation-preserving conformal automorphisms of $\mathsf{E}$ by including homotheties :

$$\mathrm{Conf}^+(\mathsf{E}) \;=\; \mathsf{V} \rtimes \big(\mathrm{SO}(2, 1) \times \mathbb{R}^+\big).$$

2.4.1. *Components and elements of the isometry group.* The full group of linear Lorentzian isometries $O(2,1)$ divides into four connected components. The identity component $SO^0(2,1)$ consists of orientation-preserving linear isometries preserving time orientation. Recall from §2.3 that it is isomorphic to the group $PSL(2,\mathbb{R})$ of orientation-preserving isometries of the hyperbolic plane. Such isometries come in three flavors : hyperbolic, parabolic or elliptic, for which we describe their Lorentzian counterparts.

*Hyperbolic elements*, or *boosts*, fix a line spanned by a spacelike vector $\mathbf{s}$ and preserve $\mathbf{s}^\perp$. In the null frame coordinates of §2.2.2, associated to $\mathbf{s}$, the matrix of a hyperbolic element is of the form :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^t \end{bmatrix} \tag{4}$$

for $t > 0$. They constitute the identity component $SO^0(1,1)$ of the isometry group of $\mathbf{s}^\perp$.

*Parabolic elements* have a single eigenvalue, 1, with a 1-dimensional eigenspace spanned by a null vector, $\mathbf{n}$. Let $\mathbf{s}_1 \in \mathbf{n}^\perp$ and $\mathbf{s}_2 \in \mathbf{s}_1^\perp$ be spacelike vectors such that $\{\mathbf{n}, \mathbf{s}_1, \mathbf{s}_2\}$ is a positively oriented basis; then the corresponding matrix is of the following form :

$$\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

(The curious reader should work out expressions for the starred entries.)

Finally, *elliptic elements* are "rotations" around timelike axes; specifically, they are conjugate to a Euclidean rotation around $\mathbb{R}\mathbf{e}_3$, which happens to be a Lorentzian rotation as well.

**Remark 2.4.1.** An element of $\mathrm{Isom}^+(\mathsf{E})$ may be called hyperbolic, parabolic or elliptic if its linear part is as well.

The group $O(2,1)$ is a semidirect product :

$$O(2,1) \cong (\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes SO^0(2,1).$$

Here $\pi_0\big(O(2,1)\big) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ is generated by the antipodal map and a *spine reflection* in a spacelike line $\mathbb{R}\mathbf{s}$. The antipodal map reverses orientation, while the spine reflection does not. More precisely, the spine reflection has the following matrix in the null basis of $\mathbf{s}$ :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

This reflection does reverse time orientation. It corresponds to a reflection in the hyperbolic plane. Glide reflections take the form :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -e^{-t} & 0 \\ 0 & 0 & -e^{t} \end{bmatrix}$$

for $t > 0$.

## 3. Proper actions and locally homogeneous Lorentzian 3-manifolds

In this section, we use what we know about Lorentzian isometries to construct manifolds which are modeled on $\mathsf{E}$. More specifically, we will consider manifolds of the form $\mathsf{E}/G$, where $G < \mathrm{Isom}^{+}(\mathsf{E})$ acts "nicely". The fact that $G$ consists of isometries means that the quotient space inherits a causal structure from $\mathsf{E}$. Some of the features of homogeneity survive as well : this is what we mean by "locally homogeneous".

We will pay particular attention to the case where the linear part $\mathsf{L}(G)$ is a free group : these manifolds will be called *Margulis spacetimes*.

### 3.1. **Group actions by isometries.**

We start by making precise what we mean by a "nice action".

**Definition 3.1.1.** Let $X$ be a locally compact space and $G$ a group acting on $X$. We say that $G$ acts *properly discontinuously* on $X$ if for every compact $K \subset X$, the set :

$$\{\gamma \in G \mid \gamma K \cap K \neq \emptyset\}$$

is finite.

Recall that a group acts *freely* if it fixes no points.

**Theorem 3.1.2.** *Let $X$ be a Hausdorff manifold and let $G$ be a group that acts freely and properly discontinuously on $X$. Then $X/G$ is a Hausdorff manifold.*

The proof of the theorem is a good exercise. One wants to show that a free action by a discrete group implies that projection onto the quotient yields a covering space; proper discontinuity ensures that the quotient is furthermore Hausdorff.

Kulkarni [21] studied proper actions in the more general context of *pseudo-Riemannian manifolds.*

**Remark 3.1.3.** A group that acts properly discontinuously on $\mathsf{E}$ is discrete. But the converse, which holds for Riemannian manifolds, is false for group actions on $\mathsf{E}$.

We start our trip through Lorentzian manifolds by visiting some group actions which define both a Euclidean and a Lorentzian structure.

**Example 3.1.4.** Let $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3 \in \mathsf{V}$ be three linearly independent vectors. Then $G = \langle \tau_{\mathbf{t}_1}, \tau_{\mathbf{t}_2}, \tau_{\mathbf{t}_3} \rangle \cong \mathbb{Z}^3$ acts properly discontinuously on $\mathsf{E}$. In fact, one easily sees that $\mathsf{E}/G$ is obtained by taking a parallelepiped generated by the three translations and then gluing opposite sides.

This example is clearly compact. Similar noncompact examples also arise by looking at two-generator or one-generator subgroups of translations.

Example 3.1.4 illustrates the next criterion for a proper action. Denote the closure of a set $A$ by $\mathrm{cl}\,(A)$ and its interior by $\mathrm{int}\,(A)$.

**Definition 3.1.5.** Let $X$ be a topological space and $G$ a group acting on $X$. Let $F \subset X$ be a closed subset such that $\mathrm{cl}\,(\mathrm{int}\,(F)) = F$. We say that $F$ is a *fundamental domain* for the $G$-action on $X$ if :

- $X = \bigcup_{\gamma \in G} \gamma F$;
- for all $\gamma \neq \eta \in G$, $\mathrm{int}\,(\gamma F) \cap \mathrm{int}\,(\eta F) = \emptyset$.

**Theorem 3.1.6.** *Let $X$ be a topological space and $G$ a group acting on $X$. Suppose there exists a fundamental domain $F$ for the $G$-action on $X$. Then $G$ acts properly discontinuously on $X$ and :*

$$X/G = F/G.$$

We now consider a fancier version of Example 3.1.4. In particular, these manifolds are finitely covered by the manifolds defined in Example 3.1.4.

**Example 3.1.7.** Let $\mathbf{t} \in \mathsf{V}$ be a timelike vector, and let $\sigma$ be a *screw motion* of order 4 about $\mathbb{R}\mathbf{t}$. Specifically, the linear part of $\sigma$ is elliptic and its translational part is $\mathbf{t}$. Let $\mathbf{v} \neq \mathbf{0}$ be a vector in $\mathbf{t}^{\perp}$ (it is necessarily spacelike). Consider the following group :

$$G = \langle \tau_{\mathbf{v}}, \tau_{\sigma\mathbf{v}}, \sigma \rangle.$$

This group admits a fundamental domain as in Example 3.1.4 : a parallelepiped generated by $\mathbf{v}, \sigma\mathbf{v}, \mathbf{t}$. The group thus acts properly discontinuously on $\mathsf{E}$ and furthermore acts freely. Thus $\mathsf{E}/G$ is a Lorentzian manifold.

In the special case where the fixed eigendirection of $\mathbf{t}$ is in the direction of $\mathbf{e}_3$, a Euclidean structure can also imposed on the manifold. Similar examples can also be created with screw motions of order 2.

The group $G$ in Example 3.1.7 is a very basic example of a *solvable group*, since it contains a finite-index subgroup of translations. Specifically, $G \cong \mathbb{Z}^3$, the subgroup $\langle\sigma\rangle$ is normal in $G$ and:

$$G/\langle\sigma\rangle \cong \mathbb{Z}^2.$$

Fried and Goldman [17] proved the following important classification result for groups of affine transformations acting properly discontinuously on $\mathbb{R}^3$. If a group of affine transformations $G$ acts properly discontinuously on $\mathbb{R}^3$, then it is either virtually solvable or it does not act cocompactly and its linear part is conjugate to a subgroup of $\mathrm{O}(2,1)$. Furthermore, Mess [24] showed that the linear part of $G$ cannot be the linear holonomy of a closed surface.

Our example has linear part in $\mathrm{O}(2,1)$ *and* is virtually solvable. Are there examples that are not virtually solvable? This was a question posed by Milnor in the 1970s [25]. Margulis discovered such examples [22, 23], where the linear part was a *Schottky group*, or a free, non-abelian discrete subgroup of $\mathrm{SO}^0(2,1)$. (See also §3.5.3.)

**Definition 3.1.8.** A *Margulis spacetime* is a Hausdorff manifold $\mathsf{E}/G$ where $G$ is free and non-abelian.

Given a group $G$ with Schottky linear part, it is difficult to determine whether it acts properly discontinuously on $\mathbb{R}^3$. We would like a "ping-pong" lemma as for Schottky groups acting on the hyperbolic plane. However, the absence of a Riemannian metric makes this challenging. The remedy was the introduction of fundamental domains for these actions [11], bounded by piecewise linear surfaces called *crooked planes*, which we discuss in §3.5.

3.2. **Aside: what the title of the paper means.** Suppose then that $X$ is a manifold with some geometry, say with a transitive action by a group $T$. (Technically, we need something like smoothness, but we will not be this technical here.) If $G < T$ acts freely and properly discontinuously on $X$, then $X/G$ inherits an atlas of charts into $X$ with coordinate changes in $T$. A manifold $M$ equipped with such an atlas is called a *locally homogeneous* structure modeled on $X$, or $(T,X)$-structure. The charts induce a *developing map* from the universal cover of $M$ to $X$. We say that the structure is *complete* if the developing map is bijective.

In the particular case where $X = \mathsf{E}$ and $T = \mathrm{Isom}^+(\mathsf{E})$, we say that $M$ is a *flat Lorentzian manifold*. So a complete flat Lorentzian 3-manifold is isometric to a quotient $\mathsf{E}/G$. Completeness means the usual geodesic completeness: straight lines in $\mathsf{E}/G$, which are projections of straight lines in $\mathsf{E}$, can be infinitely extended.

In the next three paragraphs, we will consider some more examples of Lorentzian 3-manifolds. The list of examples is not exhaustive, but some of the more "interesting" examples are covered. The reader who is interested in a complete classification might start with Fried and Goldman's classification of 3-dimensional crystallographic groups [17] and the references therein. Their classification includes our examples in §3.4.

3.3. **Cyclic group actions.** Consider cyclic Lorentzian group actions. If the group $\langle\gamma\rangle$ acts freely and properly discontinuously, then $\mathsf{E}/\langle\gamma\rangle$ is homeomorphic to (the interior of) a solid handlebody. Importantly, these manifolds are certainly noncompact.

3.3.1. *Hyperbolic transformations.* Assume that $\gamma$ is hyperbolic. Recall that in a null frame, associated to a suitable 1-eigenvector $\mathbf{s}$, the matrix for $\mathsf{L}(\gamma)$ is given by (4) and:

$$\mathbf{s}^{\perp} = \langle \mathbf{s}^{+}, \mathbf{s}^{-} \rangle.$$

The group $G$ acts freely if and only if for any $p \in \mathsf{E}$:

$$\gamma(p) - p \notin \mathbf{s}^{\perp}.$$

We underscore the fact that this criterion holds for *any* $p$.

Moreover, there is a unique $\gamma$-invariant line: the action of $\gamma$ on the set of lines parallel to $\mathbf{s}$ corresponds to an affine action on $\mathbf{s}^{\perp}$, which must have a fixed point because the restriction of $\mathsf{L}(\gamma)$ to that plane does not have 1 as an eigenvalue. Let $C_\gamma$ be that invariant line.

Choosing the origin to belong to $C_\gamma$, the translational part of $\gamma$ becomes:

$$\mathbf{u} = \alpha \mathbf{s}$$

where $\alpha \neq 0$. This scalar $\alpha$ is the Margulis invariant of $\gamma$, which will be discussed more at length in §4.

**Remark 3.3.1.** Up to conjugation of $G$ by a translation, $G$ is entirely determined by the linear part of $\gamma$ and the value of $\alpha$.

To build a fundamental domain for $G$, choose for instance a point $p \in C_\gamma$. The point $p$ could be chosen quite arbitrarily, but picking a point on the invariant line is easier to visualize. Then:

$$\gamma\left(p + \mathbf{s}^{\perp}\right) = p + \alpha\mathbf{s} + \mathbf{s}^{\perp}.$$

The plane $p + \mathbf{s}^{\perp}$ bounds two closed halfspaces; one of the two, $H_p$, satisfies:

$$\gamma\left(H_p\right) \subset H_p. \tag{6}$$

Set :

$$F = H_p \setminus \operatorname{int}\left(\gamma\left(H_p\right)\right). \tag{7}$$

Then $F$ is a fundamental domain for $G$, confirming that $G$ acts properly discontinuously on $E$ as long as $\alpha \neq 0$.

3.3.2. *Elliptic transformations.* The above construction is easily adapted to any screw motion. The eigenvector corresponding to the eigenvalue 1 will now be timelike. As long as the translational part of the screw motion is not Lorentz-perpendicular to this fixed eigendirection, the screw motion acts freely and the group acts properly discontinuously.

Within these examples we find groups which act freely and properly discontinuously but whose linear part is not discrete. In particular, if the linear part of the elliptic transformation is a rotation of infinite order, the linear group is not discrete.

3.3.3. *Parabolic transformations.* Suppose as above that $G = \langle\gamma\rangle$ acts freely on $\mathsf{E}$, but that $\gamma$ is parabolic. We use the same notation as for (5), so that $\mathbf{n} \neq \mathbf{0}$ is a 1-eigenvector for $\mathsf{L}(\gamma)$.

In contrast to the hyperbolic case, there is no $\gamma$-invariant line. Nevertheless, $\mathbf{n}^{\perp}$ is $\mathsf{L}(\gamma)$-invariant. So choose any $p \in \mathsf{E}$ and as before, choose a closed halfplane $H_p$ bounded by $p + \mathbf{n}^{\perp}$ such that Equation (6) holds; then $F$ as in Equation (7) remains a fundamental domain for the action of $G$ in this case, with quotient the interior of a solid handlebody.

Again, $\gamma$ acts freely if and only if the translational part is not parallel to $\mathbf{n}^{\perp}$. This is equivalent to a generalized version of the Margulis invariant being non-zero [4].

3.4. **More solvable group actions.** Let us now consider an example of a proper action on $\mathsf{E}$ by a solvable group $G$, beyond those encountered in §3.1 and with the following property : $G$ admits a normal subgroup $H$ of translations, $H \cong \mathbb{Z}^2$, such that $G/H \cong \mathbb{Z}$, and the projection of $G$ onto its linear part is a cyclic group generated by a hyperbolic or parabolic element. In the following example, we will work with the standard basis on the vector space $\mathbb{R}^3$, allowing us to identify any linear map with its matrix in that basis.

**Example 3.4.1.** Let $N = \langle\tau_{\mathbf{e}_2}, \tau_{\mathbf{e}_3}\rangle$, generated by the standard orthogonal translations of unit length, so that the corresponding plane is orthogonal to the spacelike vector $\mathbf{e}_1$. Choose any hyperbolic matrix $B \in \mathrm{SL}(2,\mathbb{Z})$. The group $N$ is invariant under the action of the matrix

$$M = \begin{bmatrix} 1 & \\ & B \end{bmatrix}.$$

Let $\eta$ be the affine transformation with linear part $M$ and translational part $\mathbf{e}_1$. Let $G' = \langle N, \eta \rangle$. We are going to conjugate $G'$ by a suitable linear map, so that the resulting group is Lorentzian.

Specifically, let $\mathbf{e}_x$ and $\mathbf{e}_c$ to be expanding and contracting eigenvectors for the matrix $M$. Choose $A \in \mathrm{GL}(3, \mathbb{R})$ so that :

- $A(\mathbf{e}_1) = \mathbf{e}_1$,
- $A(\mathbf{e}_c) = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and
- $A(\mathbf{e}_x) = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

At this point, we identify $\mathbb{R}^3$ with $\mathsf{V}$ and note that if $\mathbf{s} = \mathbf{e}_1$, then $\mathbf{s}^- = A(\mathbf{e}_c)$ and $\mathbf{s}^+ = A(\mathbf{e}_x)$.

The group $G = AG'A^{-1} < \mathrm{Isom}^+(\mathsf{E})$ has a normal subgroup of translations $H = ANA^{-1}$, and the projection of $G$ onto its linear part is just a cyclic group $\langle C \rangle$, where $C = AMA^{-1}$.

A fundamental domain for $G$ is obtained by taking a parallelogram with vertex $p$ with adjacent sides defined by $A(\mathbf{e}_1)$ and $A(\mathbf{e}_2)$ at one end, another parallelogram with vertex $p+\mathbf{e}_1$ and adjacent sides defined by $CA(\mathbf{e}_1)$ and $CA(\mathbf{e}_2)$ at the other end, and filling in between by a continuous path of polygons between the two, with each a fundamental domain for $T$ acting on $\mathbf{s}^\perp$. An example is depicted in Figure 3.[1]

Example 3.4.1 can easily be adapted for parabolic transformations. In this case, the normal subgroup of translations lies in a plane which is tangent to the null cone.

3.5. **Margulis spacetimes and crooked fundamental domains.** In §3.3, for the action of cyclic groups, we were able to bound fundamental domains by parallel planes, which are sometimes referred to as "slabs". But for actions of non-abelian free groups "slabs don't work". This was the motivation for introducing crooked planes [11]. The mechanics of getting crooked planes disjoint from each other was thoroughly studied in [13]; it has been recently recast in terms of the crooked halfspaces they bound in [3], in order to further study the dynamics of geodesics in a Margulis spacetime.

[1]The figure was produced by Yannick Lebrun during an undergraduate research internship with the first author.

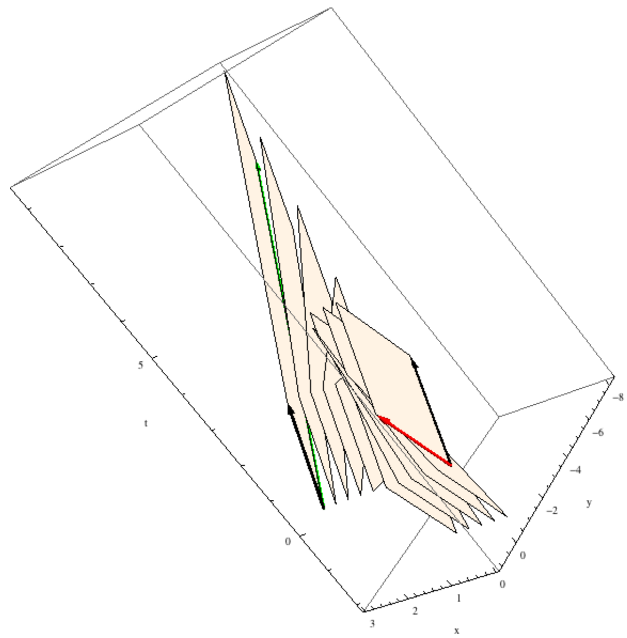


FIGURE 3. A fundamental domain for a solvable group generated by a hyperbolic element and a translation.

**Definition 3.5.1.** Let $\mathbf{x} \in \mathbb{R}^{2,1}$ be a future-pointing null vector. Then the closure of the following halfplane:

$$\mathrm{Wing}(\mathbf{x}) = \{\mathbf{u} \in \mathbf{x}^{\perp} \mid \mathbf{x} = \mathbf{u}^{+}\}$$

is called a *positive linear wing*.

In the affine setting, given $p \in \mathsf{E}$, $p + \mathrm{Wing}(\mathbf{x})$ is called a *positive wing*.

Observe that if $\mathbf{u} \in \mathbb{R}^{2,1}$ is spacelike:

$$\mathbf{u} \in \mathrm{Wing}(\mathbf{u}^{+})$$
$$-\mathbf{u} \in \mathrm{Wing}(\mathbf{u}^{-})$$
$$\mathrm{Wing}(\mathbf{u}^{+}) \cap \mathrm{Wing}(\mathbf{u}^{-}) = \mathbf{0}.$$

The set of positive linear wings is $\mathrm{SO}(2,1)$-invariant.

**Definition 3.5.2.** Let $\mathbf{u} \in \mathbb{R}^{2,1}$ be spacelike. Then the following set:

$$\mathrm{Stem}(\mathbf{u}) = \{\mathbf{x} \in \mathbf{u}^{\perp} \mid \mathbf{x} \cdot \mathbf{x} \leq 0\}$$

is called a *linear stem*. For $p \in \mathsf{E}$, $p + \mathrm{Stem}(\mathbf{u})$ is called a *stem*.

Observe that $\mathrm{Stem}(\mathbf{u})$ is bounded by the lines $\mathbb{R}\mathbf{u}^{+}$ and $\mathbb{R}\mathbf{u}^{-}$ and thus respectively intersects the closures of $\mathrm{Wing}(\mathbf{u}^{+})$ and $\mathrm{Wing}(\mathbf{u}^{-})$ in these lines.

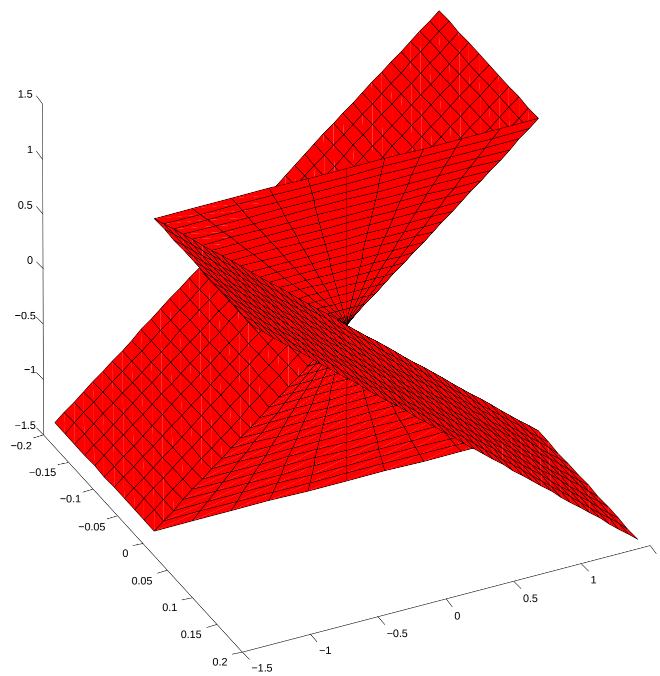


FIGURE 4. A crooked plane.

**Definition 3.5.3.** Let $p \in \mathsf{E}$ and $\mathbf{u} \in \mathbb{R}^{2,1}$ be spacelike. The *positively extended crooked plane* with *vertex* $p$ and *director* $\mathbf{u}$ is the union of:

- the stem $p + \mathrm{Stem}(\mathbf{u})$;
- the positive wing $p + \mathrm{Wing}(\mathbf{u}^+)$;
- the positive wing $p + \mathrm{Wing}(\mathbf{u}^-)$.

It is denoted $\mathcal{C}(p, \mathbf{u})$.

A crooked plane is depicted in Figure 4.

**Remark 3.5.4.** A negatively extended crooked plane is obtained by replacing positive wings with negative wings. One obtains a *negative wing* by choosing the other connected component of $\mathbf{x}^\perp \setminus \mathbb{R}\mathbf{x}$. Without going into details, we will simply state that one can avoid resorting to negatively extended crooked planes by changing the orientation on $\mathsf{V}$ and $\mathsf{E}$. Thus we will simply write "wing" to mean positive wing and "crooked plane" to mean positively extended crooked plane. The same convention is used in [3].

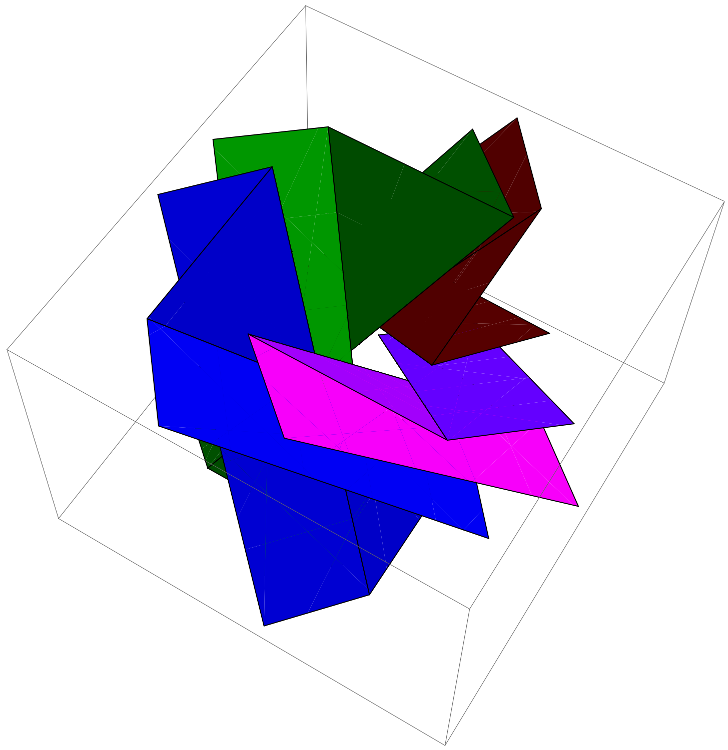

Figure 5. Four pairwise disjoint crooked planes. Identifying pairs of "adjacent" crooked planes yields a fundamental domain in the case where the linear part corresponds to a three-holed sphere. Identifying pairs of "opposite" crooked planes yields one in the case of a one-holed torus.

**Theorem 3.5.5** (Drumm [11])**.** *Let* $G = \langle \gamma_1, \ldots, \gamma_n \rangle < \mathrm{Isom}^+(\mathsf{E})$ *with linear part in* $\mathrm{SO}^0(2,1)$*, such that for each* $i = 1, \ldots, n$*,* $\mathsf{L}(\gamma_i)$ *is non-elliptic. Suppose there exists a simply connected region* $\Delta$ *bounded by* $2n$ *pairwise disjoint crooked planes* $\mathcal{C}_1^-, \mathcal{C}_1^+, \ldots, \mathcal{C}_n^-, \mathcal{C}_n^+$ *such that:*

$$\gamma_i \mathcal{C}_i^- = \mathcal{C}_i^+, \ i = 1, \ldots, n.$$

*Then* $\Delta$ *is a fundamental domain for* $G$*, which acts freely and properly discontinuously on* $\mathsf{E}$*.*

Moreover, the quotient can be seen to be (the interior of) a solid handlebody. Figure 5 shows four pairwise disjoint crooked planes; these bound a fundamental domain for a group whose linear part is the holonomy of a one-holed torus or a three-holed sphere. A fundamental domain bounded by (disjoint) crooked planes is called a *crooked fundamental domain.*

The conditions on $\mathsf{L}(G)$ stated in Theorem 3.5.5, along with the disjointness of the crooked planes, mean that $\mathsf{L}(G)$ is a generalized *Schottky subgroup* of isometries of the hyperbolic plane. Roughly speaking, a Schottky group admits a fundamental domain bounded by pairwise disjoint halfplanes, such that any given generator maps one of these halfplanes to the complement of another. The halfplanes in question correspond to directors of the crooked planes. (More on this in §3.5.3.)

3.5.1. *Crooked halfspaces and disjointness.* We will discuss here criteria for disjointness of crooked planes, as it plays a vital role in constructing crooked fundamental domains.

The complement of a crooked plane in $\mathcal{C}(p,\mathbf{u}) \in \mathsf{E}$ consists of two *crooked halfspaces*, respectively corresponding to $\mathbf{u}$ and $-\mathbf{u}$. A crooked halfspace will be determined by the appropriate *stem quadrant*, which we introduce next. Our notation for the stem quadrant is slightly different from that adopted in [3], where it is defined in terms of the crooked halfspace.

**Definition 3.5.6.** Let $\mathbf{u} \in \mathsf{V}$ be spacelike and $p \in \mathsf{E}$. The associated *stem quadrant* is:

$$\mathsf{Quad}(p,\mathbf{u}) = p + \{a\mathbf{u}^- - b\mathbf{u}^+ \;\mid\; a,b \geq 0\}.$$

The stem quadrant $\mathsf{Quad}(p,\mathbf{u})$ is bounded by light rays parallel to $\mathbf{u}^-$ and $-\mathbf{u}^+$.

**Definition 3.5.7.** Let $p \in \mathsf{E}$ and $\mathbf{u} \in \mathsf{V}$ be spacelike. The *crooked halfspace* $\mathcal{H}(p,\mathbf{u})$ is the component of the complement of $\mathcal{C}(p,\mathbf{u})$ containing $\mathrm{int}\,(\mathsf{Quad}(p,\mathbf{u}))$.

By definition, crooked halfspaces are open. While the crooked planes $\mathcal{C}(p,\mathbf{u})$, $\mathcal{C}(p,-\mathbf{u})$ are equal, the crooked halfspaces $\mathcal{H}(p,\mathbf{u})$, $\mathcal{H}(p,-\mathbf{u})$ are disjoint, sharing $\mathcal{C}(p,\mathbf{u})$ as a common boundary.

**Definition 3.5.8.** Let $o \in \mathsf{E}$ and $\mathbf{u}_1,\mathbf{u}_2 \in \mathsf{V}$ be spacelike. The vectors are said to be *consistently oriented* if the closures of the crooked halfspaces $\mathcal{H}(o,\mathbf{u}_1)$ and $\mathcal{H}(o,\mathbf{u}_2)$ intersect only in $o$.

Consistent orientation is in fact a linear property and independent of the choice of $o$. An equivalent definition, originally stated in [13], requires that $\mathbf{u}_1,\mathbf{u}_2$ be spacelike vectors such that $\mathbf{u}_1 \times \mathbf{u}_2$ is also spacelike, and whose associated null frames satisfy certain conditions on the inner product.

**Definition 3.5.9.** Let $\mathbf{u}_1,\mathbf{u}_2 \in \mathsf{V}$ be a pair of consistently oriented ultraparallel spacelike vectors. The set of *allowable translations* for

$\mathbf{u}_1, \mathbf{u}_2$ is :

$$\mathsf{A}(\mathbf{u}_1, \mathbf{u}_2) = \mathrm{int}\,(\mathsf{Quad}(p, \mathbf{u}_1) - \mathsf{Quad}(p, \mathbf{u}_2)) \subset \mathsf{V}$$

where $p \in \mathsf{E}$ can be arbitrarily chosen.

**Theorem 3.5.10** (Drumm-Goldman [13])**.** *Let $\mathbf{u}_1, \mathbf{u}_2 \in \mathsf{V}$ be a pair of consistently oriented ultraparallel spacelike vectors. Then the closures of the crooked halfspaces $\mathcal{H}(p_1, \mathbf{u}_1)$ and $\mathcal{H}(p_2, \mathbf{u}_2)$ are disjoint if and only if $p_1 - p_2 \in \mathsf{A}(\mathbf{u}_1, \mathbf{u}_2)$.*

Choosing $p_1 - p_2 \in \mathsf{A}(\mathbf{u}_1, \mathbf{u}_2)$ means that the complements of the crooked halfspaces $\mathcal{H}(p_1, \mathbf{u}_1)$ and $\mathcal{H}(p_2, \mathbf{u}_2)$ intersect nicely, in a set which we call a *crooked slab*. A crooked slab is a crooked fundamental domain for a suitable cyclic group. Indeed, let $\gamma \in \mathrm{Isom}^+(\mathsf{E})$ be any non-elliptic isometry whose linear part maps $\mathbf{u}_1$ to $-\mathbf{u}_2$ and such that $\gamma(p_1) = p_2$. (We stress here that we want $-\mathbf{u}_2$, rather than $\mathbf{u}_2$; on one hand $\mathbf{u}_1$ and $\mathbf{u}_2$ must be consistently oriented, but on the other hand, $\gamma$ is non-elliptic.) There is one degree of freedom in choosing the linear part, but the translational part is entirely determined by the condition on $p_1$ and $p_2$. The crooked slab :

$$\mathsf{E} \setminus (\mathcal{H}(p_1, \mathbf{u}_1) \cup \mathcal{H}(p_2, \mathbf{u}_2))$$

is a crooked fundamental domain for $\langle\gamma\rangle$. See Figure 6.

3.5.2. *Cyclic example revisited.* Let's make this construction more specific in the case where $\gamma$ is hyperbolic. Using the notation in §3.3, we may choose $\mathbf{u}_1 \in \mathbf{s}^\perp$ to be a spacelike vector and then set $\mathbf{u}_2 = -\mathsf{L}(\gamma)\mathbf{u}_1$. Let $p_1 \in C_\gamma$, then set $p_2 = \gamma(p_1) = p_1 + \alpha\mathbf{s}$. It is easy to check that $p_1 - p_2 \in \mathsf{A}(\mathbf{u}_1, \mathbf{u}_2)$ and thus the crooked planes are disjoint.

By Remark 3.3.1, this yields a crooked fundamental domain for any such cyclic group, regardless of translational part, since $\alpha$ is arbitrary.

3.5.3. *Higher rank groups.* Crooked fundamental domains for higher rank groups are obtained by intersecting crooked slabs whose boundary components are pairwise disjoint. Given a generalized Schottky subgroup of $G_0 < \mathrm{PSL}(2, \mathbb{R})$, here is how we might "build" a crooked fundamental domain for an affine group obtained by adding suitable translational parts to the generators. (By "generalized" Schottky group, we mean than $G_0$ may contain parabolic elements, as well as hyperbolic elements.)

Write $G_0 = \langle g_1, \ldots, g_n\rangle$. Being a generalized Schottky group, $G_0$ admits a fundamental domain in $\mathsf{H}^2$ bounded by pairwise disjoint open

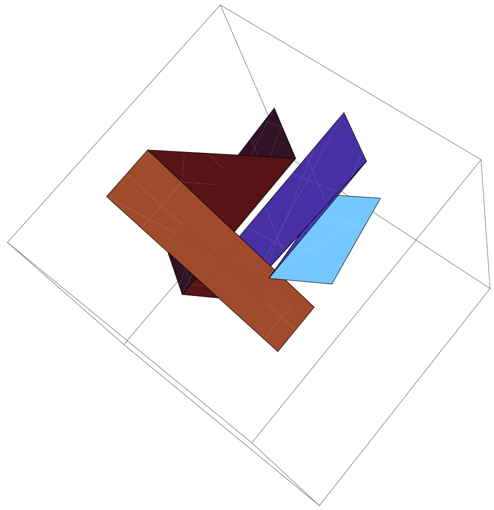

FIGURE 6. A crooked slab bounded by a pair of disjoint crooked planes.

halfplanes, $\mathfrak{h}_i^\pm$, $i = 1, \dots, n$, satisfying a pairing condition :

$$g_i(\mathfrak{h}_i^-) = \mathsf{H}^2 \setminus \mathrm{cl}\left(\mathfrak{h}_i^+\right), \ i = 1, \dots, n.$$

In fact, the closures of these halfplanes are disjoint as well; but if $g_i$ is parabolic, then these must intersect in the fixed point on the boundary. Recall from §2.3 that a halfplane in $\mathsf{H}^2$ corresponds to a spacelike vector in $\mathsf{V}$. Disjointness of the halfplanes corresponds to consistent orientation of the corresponding spacelike vectors. So choosing some point $o \in \mathsf{E}$, we have $2n$ pairwise disjoint crooked halfspaces $\mathcal{H}(o, \mathbf{s}_i^\pm)$, where $\mathbf{s}_i^\pm$ is the unit-spacelike vector such that :

$$\mathfrak{h}_i^\pm = \mathfrak{h}_{\mathbf{s}_i^\pm}.$$

The next step is to choose points $p_i^\pm$, $i = 1, \dots, n$, such that $p_i^j - p_k^l \in \mathsf{A}(\mathbf{s}_i^j, \mathbf{s}_k^l)$, for all possible choices of $1 \le i, k \le n$ and $j, l \in \{+, -\}$. An easy way to choose such allowable translations, as seen from Definition 3.5.9, is to pick a point in the stem quadrant :

$$p_i^j \in \mathsf{Quad}(o, \mathbf{s}_i^j).$$

Then the crooked planes $\mathcal{C}(p_i^j, \mathbf{s}_i^j)$ are pairwise disjoint. Now for each $i = 1, \dots, n$, let $\gamma_i \in \mathrm{Isom}^+(\mathsf{E})$ such that :

$$\mathsf{L}(\gamma_i) = g_i$$
$$\gamma_i(p_i^-) = p_i^+.$$

Then by construction, $G = \langle \gamma_1, \ldots, \gamma_n \rangle$ is a group admitting a crooked fundamental domain and thus acts freely and properly discontinuously on $\mathsf{E}$.

Two remarks are in order. First, one might expect that there are many more possible choices for the points $p_i^j$. Indeed, sets of allowable translations may be larger. It suffices to check the neighbors of a crooked plane to ensure that all crooked planes are disjoint.

Second, it may still not be obvious that we get all possible proper actions of a free group on $\mathsf{E}$ in this manner. Indeed, it is conjectured that every Margulis spacetime admits a crooked fundamental domain. So far it has been proved in the rank two case, using "stretched" versions of crooked fundamental domains [8, 6, 7]. In these stretched versions, corresponding to geodesic laminations on the underlying hyperbolic surface, we are able to recover all possible Margulis spacetimes while choosing points in the stem quadrants as above.

## 4. Affine deformations and the Margulis invariant

In this section, we consider affine groups as deformations of Fuchsian groups. We also formally introduce the Margulis invariant, mentioned in §3.3. We will interpret properness of an affine action in terms of paths of Fuchsian representations, and the Margulis invariant as a derivative of length on such paths. (This was first studied by Goldman and Margulis in [20, 18]).

The reader consulting the references may despair at how much the notation changes from paper to paper; we feel we should apologize for this, as we are partially responsible for this state of affairs. In this section, we have tried to stay consistent with the notation used in the Almora lectures, as well as [5], since it seemed the most suited to our present focus.

Let $G_0 \subset \mathrm{SO}(2,1)$ be a subgroup. An *affine deformation* of $G_0$ is a representation:

$$\rho : G_0 \longrightarrow \mathrm{Isom}^+(\mathsf{E}).$$

For the remainder of this section, fix $o \in \mathsf{E}$ so that translational parts are well-defined. For $g \in G_0$, set $u(g) \in \mathsf{V}$ to be the translational part of $\rho(g)$; in other words, for $x \in \mathsf{E}$:

$$\rho(g)(x) = o + g(x - o) + u(g).$$

Then $u$ is a cocycle of $G_0$ with coefficients in the $G_0$-module $\mathsf{V}$ corresponding to the linear action of $G_0$. In this way affine deformations of $G_0$ correspond to cocycles in $\mathrm{Z}^1(G_0, \mathsf{V})$ and translational conjugacy classes of affine deformations correspond to cohomology classes in $\mathrm{H}^1(G_0, \mathsf{V})$.

4.1. **The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ as $\mathsf{V}$.** The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is the tangent space to $\mathrm{PSL}(2,\mathbb{R})$ at the identity and consists of the set of traceless $2 \times 2$ matrices. The three-dimensional vector space has a natural inner product, the Killing form, defined to be:

$$\langle X, Y\rangle = \frac{1}{2}\,\mathrm{Tr}(XY). \tag{8}$$

A basis for $\mathfrak{sl}(2,\mathbb{R})$ is given by:

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ E_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{9}$$

Evidently, $\langle E_1, E_1\rangle = \langle E_2,\ E_2\rangle = 1$, $\langle E_3, E_3\rangle = -1$ and $\langle E_i, E_j\rangle = 0$ for $i \neq j$. That is, $\mathfrak{sl}(2,\mathbb{R})$ is isomorphic to $\mathsf{V}$ as a vector space:

$$\left\{ \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\} \longleftrightarrow \{aE_1 + bE_2 + cE_3 = X\}.$$

The adjoint action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathfrak{sl}(2,\mathbb{R})$:

$$g(X) = gXg^{-1}$$

corresponds to the linear action of $\mathrm{SO}^0(2,1)$ on $\mathsf{V}$.

In what follows, we will identify the linear action of a discrete group $G_0 < \mathrm{SO}^0(2,1)$ on $\mathsf{V}$ with the action of the corresponding Fuchsian group on $\mathfrak{sl}(2,\mathbb{R})$.

4.2. **The Margulis invariant.** The Margulis invariant is a measure of an affine transformation's signed Lorentzian displacement in $\mathsf{E}$, originally defined by Margulis for hyperbolic transformations [22, 23].

**Definition 4.2.1.** Let $\gamma$ be a hyperbolic Lorentzian transformation, $\mathbf{s}$ be a suitably chosen $\gamma$-invariant unit-spacelike vector and $q$ any point the unique $\gamma$-invariant line $C_\gamma$. The *Margulis invariant* of $\gamma$ is

$$\alpha(\gamma) = (\gamma(p) - p) \cdot \mathbf{s}.$$

The geometric interpretation of the Margulis invariant comes directly from this definition. The line $C_\gamma$ projects to the unique closed geodesic in $\mathsf{E}/\langle\gamma\rangle$. The Lorentzian length of this closed geodesic is just the Lorentzian distance that any point on $C_\gamma$ is moved by $\gamma$. To get the

Lorentzian length, take the Lorentzian inner product of the vector between the identified points on $C_\gamma$ and the uniquely defined $\gamma$-invariant unit-spacelike vector $\mathbf{s}$. The direction of $\mathbf{s}$ is well-defined by the linear part of $\gamma$, so the Lorentzian length has a sign.

The "sign" of the Margulis invariant is of utmost importance. In [22, 23], Margulis proved his "Opposite Sign Lemma."

**Lemma 4.2.2.** *If Margulis invariants of $\gamma$ and $\eta$ have opposite signs, then $\langle \gamma, \eta \rangle$ does not act properly discontinuously on $\mathsf{E}$.*

The Margulis invariant was adapted to parabolic transformations in [4]. The Opposite Sign Lemma was also shown to hold for groups with parabolic transformations.

We now recast the Margulis invariant in terms of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$.

Let $g \in \mathrm{PSL}(2,\mathbb{R})$ be a hyperbolic element. Lift $g$ to a representative in $\mathrm{SL}(2,\mathbb{R})$; then the following element of $\mathfrak{sl}(2,\mathbb{R})$ is a $g$-invariant vector which is independent of choice of lift :

$$F_g = \mathsf{sgn}(g)\left(g - \frac{\mathrm{Tr}(g)}{2} I\right)$$

where $\mathsf{sgn}(g)$ is the sign of the trace of the lift.

Now let $G_0 \subset \mathrm{PSL}(2,\mathbb{R})$ such that every element other than the identity is hyperbolic. Let $\rho$ be an affine deformation of $G_0$, with corresponding $u \in \mathrm{Z}^1(G_0, \mathsf{V})$. We define the *non-normalized Margulis invariant* of $\rho(g) \in \rho(G_0)$ to be :

$$\tilde{\alpha}_\rho(g) = \langle u(g), F_g \rangle. \tag{10}$$

Since $\rho(g)$ is hyperbolic, then the vector $F_g$ is spacelike and we may replace it by the unit-spacelike vector :

$$X_g^0 = \frac{2\ \mathsf{sgn}(g)}{\sqrt{\mathrm{Tr}(g)^2 - 4}}\left(g - \frac{\mathrm{Tr}(g)}{2} I\right)$$

obtaining the *normalized Margulis invariant*:

$$\alpha_\rho(g) = \langle u(g), X_g^0 \rangle. \tag{11}$$

This is exactly $\alpha$ encountered in §3.3 and defined above. Furthermore, this definition can be also adapted to parabolic elements.

As a function of word length in the group $G_0$, normalized $\alpha_\rho$ behaves better than non-normalized $\tilde{\alpha}_\rho$. Nonetheless, the sign of $\tilde{\alpha}_\rho(g)$ is well defined and is equal to that of $\alpha_\rho(g)$.

**Theorem 4.2.3.** [8] *Let $G_0$ be a Fuchsian group such that the surface $\Sigma = \mathsf{H}^2/G_0$ is homeomorphic to a three-holed sphere. Denote the generators of $G_0$ corresponding to the three components of $\partial\Sigma$ by $\partial_1, \partial_2, \partial_3$. Let $\rho$ be an affine deformation of $G_0$.*

*If $\alpha_\rho(\partial_i)$ is positive (respectively, negative, nonnegative, nonpositive) for each $i$ then for all $\gamma \in G_0 \setminus \{1\}$, $\alpha_\rho(\gamma)$ is positive (respectively, negative, nonnegative, nonpositive).*

The proof of Theorem 4.2.3 relies upon showing that the affine deformation $\rho$ of the Fuchsian group $G_0$ acts properly on $\mathsf{E}$, because it admits a crooked fundamental domain as discussed in §3.5.

By a fundamental lemma due to Margulis [22, 23] and extended in [4], if $\rho$ is proper, then $\alpha_\rho$ applied to every element has the same sign. Moreover,

- if $\alpha_\rho(\partial_1) = 0$ and $\alpha_\rho(\partial_2), \alpha_\rho(\partial_3) > 0$ then specifically $\alpha_\rho(\gamma) = 0$ only if $\gamma \in \langle \partial_1 \rangle$, and
- if $\alpha_\rho(\partial_1) = \alpha_\rho(\partial_2) = 0$ and $\alpha_\rho(\partial_3) > 0$ then specifically $\alpha_\rho(\gamma) = 0$ only if $\gamma \in \langle \delta_1 \rangle \cup \langle \delta_2 \rangle$.

4.3. **Length changes in deformations.** An *affine* deformation of a holonomy representation corresponds to an *infinitesimal* deformation of the holonomy representation, or a tangent vector to the holonomy representation. In this section, we will further explore this correspondence, relating the affine Margulis invariant to the derivative of length along a path of holonomy representations. We will then prove Theorem 4.3.1 by applying Theorem 4.2.3, which characterizes proper deformations in terms of the Margulis invariant, to the study of length changes along a path of holonomy representations.

Let $\rho_0 : \pi_1(\Sigma) \to G_0 \subset \mathrm{PSL}(2, \mathbb{R})$ be a holonomy representation for a surface $\Sigma$ and let $\rho : G_0 \to \mathrm{Isom}^+(\mathsf{E})$ be an affine deformation of $G_0$, with corresponding cocycle $u \in \mathrm{Z}^1(G_0, \mathsf{V})$. By extension we will call $\rho$ an affine deformation of $\rho_0$.

The affine deformation $\rho$ induces a path of holonomy representations $\rho_t$ as follows:

$$\begin{aligned} \rho_t : \pi_1(\Sigma) &\longrightarrow G_0 \\ \sigma &\longmapsto \exp(tu(g))g \end{aligned}$$

where $g = \rho_0(\sigma)$, and $u$ is the tangent vector to this path at $t = 0$. Conversely, for any path of representations $\rho_t$:

$$\rho_t(\sigma) = \exp(tu(g) + O(t^2))g$$

where $u \in \mathrm{Z}^1(G_0, \mathsf{V})$ and $g = \rho_0(\sigma)$.

Suppose $g$ is hyperbolic. Then the length of the corresponding closed geodesic in $\Sigma$ is :

$$l(g) = 2\cosh^{-1}\left(\frac{|\operatorname{Tr}(\tilde{g})|}{2}\right)$$

where $\tilde{g}$ is a lift of $g$ to $\mathrm{SL}(2,\mathbb{R})$. With $\rho,\ \rho_t$ as above and $\rho_0(\sigma) = g$, set :

$$l_t(\sigma) = l(\rho_t(\sigma)).$$

Consequently :

$$\left.\frac{d}{dt}\right|_{t=0} l_t(\sigma) = \frac{\alpha_\rho(g)}{2}$$

so we may interpret $\alpha_u$ as the change in length of an affine deformation, up to first order [20, 18].

Although $l_t(\sigma)$ is not differentiable at 0 for parabolic $g$ :

$$\left.\frac{d}{dt}\right|_{t=0} \frac{\mathsf{sgn}(g)}{2}\operatorname{Tr}(\rho_t(\sigma)) = \tilde{\alpha}_\rho(g).$$

We obtain a infinitesimal version of a theorem due to Thurston [27], by reinterpreting Theorem 4.2.3.

**Theorem 4.3.1.** [5] *Let $\Sigma$ be a three-holed sphere with a hyperbolic structure. Consider any deformation of the hyperbolic structure of $\Sigma$ where the lengths of the three boundary curves are increasing up to first order, then the lengths of all of the remaining geodesics are also increasing up to first order.*

*Proof.* Let $\rho_t$, $-\epsilon \leq t \leq \epsilon$ be a path of holonomy representations. Since we assume the boundary components to be lengthening, they must have hyperbolic holonomy on $(-\epsilon, \epsilon)$.

Suppose there exists $\sigma \in \pi_1(\Sigma)$ and $T \in (-\epsilon, \epsilon)$ such that the length of $\rho_t(\sigma)$ decreases in a neighborhood of $T$. Reparameterizing the path if necessary, we may assume $T = 0$, so that the tangent vector at $T$ corresponds to an affine deformation $\rho$ and :

$$\alpha_\rho(\sigma) < 0.$$

Theorem 4.2.3 implies that for some $i = 1, 2, 3$ :

$$\alpha_\rho(\partial_i)) < 0.$$

but then the length of the corresponding end must decrease, contradicting the hypothesis.

□

4.3.1. *Deformed hyperbolic transformations.* In this and the next paragraph, we explicitly compute the trace of some deformations, to understand first order length changes.

Let $g \in \mathrm{SL}(2,\mathbb{R})$ be a hyperbolic element, thus a lift of a hyperbolic isometry of $\mathsf{H}^2$. Given a tangent vector in $X \in \mathfrak{sl}(2,\mathbb{R})$, consider the following two actions on $\mathrm{SL}(2,\mathbb{R})$ :

$$\pi_X : g \to \exp(X) \cdot g \tag{12}$$

and

$$\pi'_X : g \to g \cdot (\exp(X)^{-1}) = g \cdot \exp(-X). \tag{13}$$

All of our quantities are conjugation-invariant. Therefore, all of our calculations reduce to a single hyperbolic element of $\mathrm{SL}(2,\mathbb{R})$ :

$$g = \begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix} = \exp\left(\begin{bmatrix} s & 0 \\ 0 & -s \end{bmatrix}\right)$$

whose trace is $\mathrm{Tr}(g) = 2\cosh(s)$. The eigenvalue frame for the action of $g$ on $\mathfrak{sl}(2,\mathbb{R})$ is :

$$X_g^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, X_g^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, X_g^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

where :

$$\begin{aligned} gX_g^0 g^{-1} &= X_g^0, \\ gX_g^- g^{-1} &= e^{-2s} X_g^- \\ gX_g^+ g^{-1} &= e^{2s} X_g^+. \end{aligned}$$

Write the vector $X \in \mathfrak{sl}(2,\mathbb{R})$ as :

$$X = aX^0(g) + bX^-(g) + cX^+(g) = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}.$$

By direct computation, the trace of the induced deformation $\pi_X(g)$ is :

$$\mathrm{Tr}(\pi_X(g)) = 2\cosh s \cosh\sqrt{a^2+bc} + \frac{2a\sinh s \sinh\sqrt{a^2+bc}}{\sqrt{a^2+bc}}.$$

Observe that when $X = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$, which is equivalent to the Margulis invariant being zero :

$$\mathrm{Tr}(\pi_X(g)) = 2\cosh(s)\cosh(\sqrt{bc}).$$

Up to first order, $\mathrm{Tr}(\pi_X(g)) = 2\cosh(s)$.

Alternatively, when $b = c = 0$ :

$$\mathrm{Tr}(\pi_X(g)) = 2\cosh(s + a)$$

whose Taylor series about $a = 0$ does have a linear term. We assumed that $s > 0$, defining our expanding and contracting eigenvectors. As long as $a > 0$, which corresponds to positivity of the Margulis invariant, the trace of the deformed element $\pi_X(g)$ is greater than the original element $g$.

Now consider the deformation $\pi'_X(g) = g \cdot (\exp(X))^{-1}$. When $b = c = 0$ :

$$\mathrm{Tr}(\pi'_X(g)) = 2\cosh(s - a)$$

whose Taylor series about $a = 0$ has a nonzero linear term. As long as $a > 0$, $\mathrm{Tr}(\pi_X(g))$ is now less than the original element $g$. So for this deformation, a positive Margulis invariant corresponds to a decrease in trace of the original hyperbolic element.

**Lemma 4.3.2.** *Consider a hyperbolic $g \in \mathrm{SL}(2,\mathbb{R})$, with corresponding closed geodesic $\partial$ and an affine deformation represented by $X \in \mathfrak{sl}(2,\mathbb{R})$. For the actions of $X$ on $\mathrm{SL}(2,\mathbb{R})$ by*

- *$\pi_X(g) = \exp(X) \cdot g$ then a positive value for the Margulis invariant corresponds to first order lengthening of $\partial$;*
- *$\pi'_X(g) = g \cdot \exp(X)$ then a positive value for the Margulis invariant corresponds to first order shortening of $\partial$.*

4.3.2. *Deformed parabolic transformations.* As before, we are interested in quantities invariant under conjugation. Because of this, all of our calculations can be done with a very special parabolic transformation in $\mathrm{SL}(2,\mathbb{R})$ :

$$p = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} = \exp\left(\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}\right)$$

where $r > 0$ and whose trace is $\mathrm{Tr}(p) = 2$. We choose a convenient frame for the action of $p$ on $\mathfrak{sl}(2,\mathbb{R})$ :

$$X^u(g) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, X^0(g) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, X^c(g) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The trace of the deformation of the element $p$ by the tangent vector $X$ described above is :

$$\mathrm{Tr}(\pi_X(p)) = 2\cosh(\sqrt{a^2 + bc}) + \frac{cr}{\sqrt{a^2 + bc}}\sinh(\sqrt{a^2 + bc}).$$

When the Margulis invariant is zero or, equivalently, when $X = \begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}$:

$$\mathrm{Tr}(\pi_X(p)) = 2\cosh(a).$$

Thus the trace equals 2, in terms of $a$, to first order.

Alternatively, when $a = b = 0$ in the expression for $X$:

$$\mathrm{Tr}(\pi_X(p)) = 2 + cr$$

which is linear and increasing in $c$. As long as $c > 0$, which corresponds to positivity of the Margulis invariant, the trace of the deformed element $\pi_X(p)$ is larger than the original element $p$.

**Lemma 4.3.3.** *Consider a parabolic $g \in \mathrm{SL}(2,\mathbb{R})$, and an affine deformation represented by $X \in \mathfrak{sl}(2,\mathbb{R})$. For the actions of $X$ on $\mathrm{SL}(2,\mathbb{R})$ by*

- $\pi_X(g) = \exp(X)\cdot g$ *then a positive value for the Margulis invariant corresponds to first order increase in the trace of $g$;*
- $\pi'_X(g) = g\cdot\exp(X)$ *then a positive value for the Margulis invariant corresponds to first order decrease in the trace of $g$.*

## 5. Einstein Universe

The Einstein Universe $\mathsf{Ein}_n$ can be defined as the projectivization of the lightcone of $\mathbb{R}^{n,2}$. Our own interest in the Einstein Universe may be traced back to the work of Frances, initiated in his thesis [14], on actions of discrete groups on this space. He also described *Lorentzian Schottky groups* [16]. The dynamics of group actions is quite rich in the Einstein setting. Sequences of maps can go to infinity in a variety of ways, so caution must be exercised when considering the limit sets.

Frances introduced a generalization of crooked planes in order to build compactifications of Margulis spacetimes [15]. We will describe a number of objects in the Einstein Universe, leading up to the notion of a *crooked surface*. We will restrict ourselves to $n = 3$, the setting for compactifying $\mathsf{E}$. This is but a brief introduction; the interested reader is encouraged to read the papers cited above, as well as [2, 10].

5.1. **Definition.** Let $\mathbb{R}^{3,2}$ denote the vector space $\mathbb{R}^5$ endowed with a symmetric bilinear form of signature $(3,2)$. Specifically, for $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix}$

and $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_5 \end{bmatrix} \in \mathbb{R}^5$, set :

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4 - x_5 y_5.$$

As before, let $\mathbf{x}^\perp$ denote the orthogonal hyperplane to $\mathbf{x} \in \mathbb{R}^{3,2}$ :

$$\mathbf{x}^\perp = \{\mathbf{y} \in \mathbb{R}^{3,2} \mid \mathbf{x} \cdot \mathbf{y} = 0\}.$$

Let $\mathcal{N}^{3,2}$ denote the *lightcone* of $\mathbb{R}^{3,2}$ :

$$\mathcal{N}^{3,2} = \{\mathbf{x} \in \mathbb{R}^{3,2} \setminus \mathbf{0} \mid \mathbf{x} \cdot \mathbf{x} = 0\}.$$

Note that to keep the definitions as simple as possible, we do not consider the zero vector to belong to the lightcone.

The Einstein Universe is the quotient of $\mathcal{N}^{3,2}$ under the action of the non-zero reals, $\mathbb{R}^*$, by scaling :

$$\mathsf{Ein}_3 = \mathcal{N}^{3,2}/\mathbb{R}^*.$$

Denote by $\pi(\mathbf{v})$ the image of $\mathbf{v} \in \mathcal{N}^{3,2}$ under this projection. Wherever convenient, for $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_5 \end{bmatrix}$ we will alternatively write :

$$\pi(\mathbf{v}) = (v_1 : v_2 : v_3 : v_4 : v_5).$$

Denote by $\widehat{\mathsf{Ein}_3}$ the orientable double-cover of $\mathsf{Ein}_3$. Alternatively, $\widehat{\mathsf{Ein}_3}$ can also be expressed as a quotient by the action of the positive reals :

$$\widehat{\mathsf{Ein}_3} = \mathcal{N}^{3,2}/\mathbb{R}^+.$$

Any lift of $\widehat{\mathsf{Ein}_3}$ to $\mathcal{N}^{3,2}$ induces a metric on $\mathsf{Ein}_3$ by restricting "$\cdot$" to the image of the lift. For instance, the intersection with $\mathcal{N}^{3,2}$ of the sphere of radius 2, centered at $\mathbf{0}$, consists of vectors $\mathbf{x}$ such that :

$$x_1^2 + x_2^2 + x_3^2 = 1 = x_4^2 + x_5^2.$$

It projects bijectively to $\widehat{\mathsf{Ein}_3}$, endowing it with the Lorentzian product metric $dg^2 - dt^2$, where $dg^2$ is the standard round metric on the 2-sphere $S^2$, and $dt^2$ is the standard metric on the circle $S^1$.

Thus $\mathsf{Ein}_3$ is conformally equivalent to :

$$S^2 \times S^1/\sim, \text{ where } \mathbf{x} \sim -\mathbf{x}.$$

Here $-I$ factors into the product of two antipodal maps.

Any metric on $\widehat{\mathsf{Ein}_3}$ pushes forward to a metric on $\mathsf{Ein}_3$. Thus $\mathsf{Ein}_3$ inherits a conformal class of Lorentzian metrics from the ambient spacetime $\mathbb{R}^{3,2}$. The group of conformal automorphisms of $\mathsf{Ein}_3$ is :

$$\mathrm{Conf}(\mathsf{Ein}_3) \cong \mathrm{PO}(3,2) \cong \mathrm{SO}(3,2).$$

As $\mathrm{SO}(3,2)$ acts transitively on $\mathcal{N}^{3,2}$, $\mathrm{Conf}(\mathsf{Ein}_3)$ acts transitively on $\mathsf{Ein}_3$.

Slightly abusing notation, we will also denote by $\pi(p)$ the image of $p \in \widehat{\mathsf{Ein}_3}$ under projection onto $\mathsf{Ein}_3$.

The antipodal map being orientation-reversing in the first factor (but orientation preserving in the second), $\mathsf{Ein}_3$ is non-orientable. However, it is *time-orientable*, in the sense that a future-pointing timelike vector field on $\mathbb{R}^{3,2}$ induces one on $\mathsf{Ein}_3$.

5.2. **Conformally flat Lorentzian structure on $\mathsf{Ein}_3$.** The Einstein Universe contains a copy of Minkowski space, which we describe here for dimension three. Denote by $\cdot$ the scalar product on $\mathsf{V}$. Set :

$$\begin{aligned} \iota : \mathsf{V} &\longrightarrow \mathsf{Ein}_3 \\ \mathbf{v} &\longmapsto \pi\left(\frac{1-\mathbf{v}\cdot\mathbf{v}}{2}, \mathbf{v}, \frac{1+\mathbf{v}\cdot\mathbf{v}}{2}\right). \end{aligned}$$

This is a conformal transformation that maps $\mathsf{V}$ to a neighborhood of $(1:0:0:0:1)$. In fact, setting :

$$p_\infty = (-1:0:0:0:1)$$

then :

$$\iota(\mathsf{V}) = \mathsf{Ein}_3 \setminus \mathcal{L}(p_\infty)$$

where $\mathcal{L}(p_\infty)$ is the *lightcone* at $p_\infty$ (see Definition 5.3.3). Thus $\mathsf{Ein}_3$ is the *conformal compactification* of $\mathsf{V}$.

Since $\mathrm{Conf}(\mathsf{Ein}_3)$ acts transitively on $\mathsf{Ein}_3$, every point of the Einstein Universe admits a neighborhood that is conformally equivalent to $\mathsf{V}$. In other words, $\mathsf{Ein}_3$ is a *conformally flat Lorentzian manifold*.

Furthermore, identifying $\mathsf{V}$ with $\mathsf{E}$ in the usual manner, we may consider $\iota$ as a map from $\mathsf{E}$ into $\mathsf{Ein}_3$ and as such, $\mathsf{Ein}_3$ is the conformal compactification of $\mathsf{E}$.

5.3. **Light : photons, lightcones (and tori).** We now describe the causal structure of $\mathsf{Ein}_3$, namely photons and lightcones. It is useful to know (see for instance [14]) that conformally equivalent Lorentzian metrics give rise to the same causal structure. In particular, the non-parametrized lightlike geodesics are the same for conformally equivalent Lorentzian metrics, so anything defined in terms of the causal structure

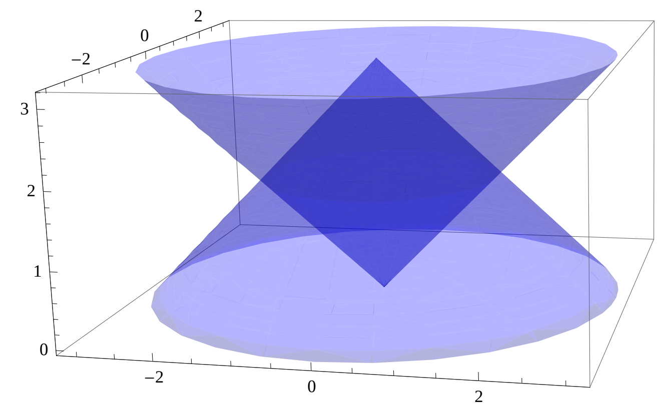


Figure 7. Two lightcones in the Einstein Universe. They intersect in a circle.

of a given metric will in fact be well defined in the conformal class of that metric.

Recall that, given a vector space $V$ endowed with an inner product, a subspace of $W \subset V$ is *totally isotropic* if the restriction of the inner product to $W$ is identically zero. In particular, $W \subset \mathbb{R}^{3,2}$ is totally isotropic if and only if $W \setminus \mathbf{0} \subset \mathcal{N}^{3,2}$.

**Definition 5.3.1.** Let $W \subset \mathbb{R}^{3,2}$ be a totally isotropic plane. Then $\pi(W \setminus \mathbf{0})$ is called a *photon*.

Alternatively, a photon is an unparameterized lightlike geodesic of $\mathsf{Ein}_3$. It can easily be shown that no photon is homotopically trivial. The homotopy class of a photon generates the fundamental group of $\mathsf{Ein}_3$.

**Definition 5.3.2.** Two points $p, q \in \mathsf{Ein}_3$ are said to be *incident* if they lie on a common photon.

**Definition 5.3.3.** Let $p \in \mathsf{Ein}_3$. The *lightcone* at $p$, denoted $\mathcal{L}(p)$, is the union of all photons containing $p$.

In other words, $\mathcal{L}(p)$ is the set of all points incident to $p$. Also:

$$\mathcal{L}(p) = \pi\left(\mathbf{v}^{\perp} \cap \mathcal{N}^{3,2}\right)$$

where $\mathbf{v} \in \mathcal{N}^{3,2}$ is such that $\pi(\mathbf{v}) = p$. Figure 7 shows two lightcones, intersecting in a simple closed curve. In fact, it looks like a circle and is a circle, given the points and the parametrization we used for $\mathsf{Ein}_3$. Note that in this and the remaining figures, we visualize $\mathsf{Ein}_3$ as a quotient of $S^2 \times S^1$, with a copy of $S^2$ and a copy of $S^1$ removed. (See §5.4.2.)

**Lemma 5.3.4.** *Suppose* $p, q \in \mathsf{Ein}_3$ *are non-incident. Then* $\mathcal{L}(p) \cap \mathcal{L}(q)$ *is a simple closed curve.*

Indeed, the intersection in this case is a *spacelike circle*: the tangent vector at every point is spacelike.

*Proof.* Suppose without loss of generality that $p = p_\infty$. The intersection of $\mathcal{L}(q)$ with the Minkowski patch $\mathsf{Ein}_3 \setminus \mathcal{L}(p)$ corresponds to a lightcone in $\mathsf{V}$. Applying a translation if necessary, we may suppose that $q = \iota(0, 0, 0)$. Then $\mathcal{L}(p) \cap \mathcal{L}(q)$ is the so-called circle at infinity:

$$\mathcal{L}(p) \cap \mathcal{L}(q) = \{(0 : \cos t : \sin t : 1 : 0) \;\mid\; t \in \mathbb{R}\}.$$

□

5.3.1. *Einstein torus.*

**Definition 5.3.5.** An *Einstein torus* is a closed surface in $S \subset \mathsf{Ein}_3$ such that the restriction of the conformal class of metrics to $S$ is of signature $(1, 1)$.

Specifically, an Einstein torus is given by a certain configuration of four points $\{p_1, p_2, f_1, f_2\}$, where:

- $p_1, p_2$ are non-incident;
- $f_1, f_2 \in \mathcal{L}(p_1) \cap \mathcal{L}(p_2)$.

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}^{3,2}$ such that:

$$\mathbf{v}_i \in \pi^{-1} p_i$$
$$\mathbf{x}_i \in \pi^{-1} f_i.$$

The restriction of the inner product endows the subspace of $\mathbb{R}^{3,2}$ spanned by the four vectors with a non-degenerate scalar product of signature $(2, 2)$. Its lightcone is a 3-dimensional subset of $\mathcal{N}^{3,2}$. It projects to a torus in $\mathsf{Ein}_3$ that is conformally equivalent to $\mathsf{Ein}_2$.

5.4. **Crooked surfaces.** Originally described as conformal compactifications of crooked planes [15], define crooked surfaces to be any element in the SO(3, 2)-orbit of such an object.

5.4.1. *Crooked surfaces as conformal compactifications of crooked planes.* Recall that $\iota(\mathsf{E})$ consists of the complement of $\mathcal{L}(p_\infty)$, where:

$$p_\infty = (-1 : 0 : 0 : 0 : 1).$$

Let $\mathbf{u} \in \mathsf{V}$ be spacelike and $p \in \mathsf{E}$. The crooked plane $\mathcal{C}(p, \mathbf{u})$ admits a conformal compactification, which we denote by $\overline{\mathcal{C}(p, \mathbf{u})}^{\mathrm{conf}}$. Explicitly, setting $o = (0, 0, 0)$:

$$\overline{\mathcal{C}(o, \mathbf{u})}^{\mathrm{conf}} = \iota(\mathcal{C}(o, \mathbf{u})) \cup \phi \cup \psi$$

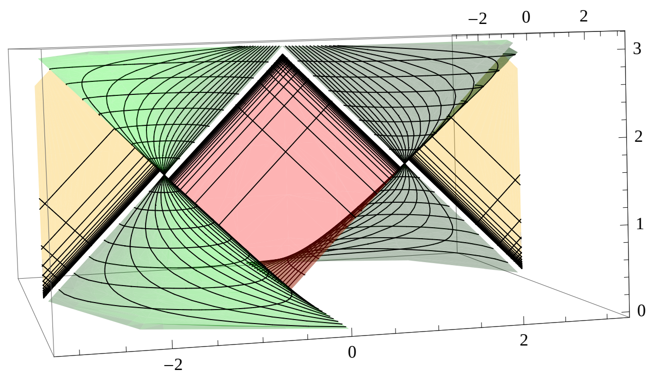


FIGURE 8. A crooked surface in the Einstein Universe. One piece of the stem only appears to be cut in half, due to the removal of a circle in the picture of $\mathsf{Ein}_3$.

where :

- $\phi \subset \mathcal{L}(p_\infty)$ is the photon containing $(0 : \mathbf{u}^+ : 0)$;
- $\psi \subset \mathcal{L}(p_\infty)$ is the photon containing $(0 : \mathbf{u}^- : 0)$.

The wing $o+\mathrm{Wing}(\mathbf{u}^+)$ is in fact a "half lightcone"; specifically it is one of the two components in $\mathcal{L}(0 : \mathbf{u}^+ : 0) \setminus \phi'$, where $\phi' \subset \mathcal{L}(0 : \mathbf{u}^+ : 0)$ is the photon containing $\iota(o)$. A similar statement holds for $o+\mathrm{Wing}(\mathbf{u}^-)$.

**Definition 5.4.1.** A *crooked surface* is any element in the $\mathrm{SO}(3,2)$-orbit of $\overline{\mathcal{C}(p,\mathbf{u})}^{\mathrm{conf}}$, where $p \in \mathsf{E}$ and $\mathbf{u} \in \mathsf{V}$ is spacelike.

Figure 8 shows a crooked surface.

5.4.2. *A basic example.* We will describe $\mathcal{S} = \overline{\mathcal{C}(o,\mathbf{u})}^{\mathrm{conf}}$, where $o = (0,0,0)$ and $\mathbf{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$. We identify $\mathsf{Ein}_3$ with a quotient of $S^2 \times S^1$, admitting the following parametrization (which can be recognized as a permuted version of the usual parametrization) :

(14)
$$(\cos\phi, \sin\phi\cos\theta, \sin\phi\sin\theta, \sin t, \cos t)\,,\ 0 \le \phi \le \pi,\ 0 \le \theta, t \le 2\pi.$$

Since $\mathbf{u}^\pm = (0, \mp 1, 1)$, the compactification of $\iota\left(o + \mathbf{u}^\perp\right)$ is the Einstein torus determined by $\{\iota(o), p_\infty, f_1, f_2\}$, where :

$$f_1 = (0:0:1:1:0)$$
$$f_2 = (0:0:-1:1:0).$$

Thus it is $\pi\left((0,1,0,0,0)^\perp \cap \mathcal{N}^{3,2}\right)$, which can be parametrized as :

$$(\cos s : 0 : \sin s : \sin t : \cos t)\,,\ 0 \le s \le 2\pi,\ 0 \le t \le \pi.$$

We obtain one piece of the stem by restricting $s$ to lie between $-t$ and $t$, and the other, between $\pi - t$ and $\pi + t$.

The wing $o+\mathrm{Wing}(\mathbf{u}^-)$ is a subset of the lightcone $\mathcal{L}(0:0:1:1:0)$, which can be parametrized as follows:

$$(\sin s \cos t : \sin s \sin t : \cos s : \cos s : -\sin s)\,, -\frac{\pi}{2} \leq s \leq \frac{\pi}{2}, 0 \leq t \leq 2\pi.$$

Photons are parametrized as $t =$ constant. The photon incident to $p_\infty$ corresponds to $t = 0$ and the photon incident to $\iota(o)$, to $t = \pi$. This wing contains:

$$\iota(o - \mathbf{u}) = (0 : -1 : 0 : 0 : 1)$$

which lies on the photon $t = \frac{\pi}{2}$. Therefore, the wing is the half lightcone $0 \leq t \leq \pi$.

In a similar way, we find that the wing $o + \mathrm{Wing}(\mathbf{u}^+)$ is the half lightcone parametrized as follows:

$$(\sin s \cos t : -\sin s \sin t : -\cos s : \cos s : -\sin s)$$

where, again, $-\frac{\pi}{2} \leq s \leq \frac{\pi}{2}$ and $0 \leq t \leq 2\pi$.

The following theorem may be proved using a cut and paste argument on the crooked surface; see [2].

**Theorem 5.4.2.** *A crooked surface is homeomorphic to a Klein bottle.*

The reader might check the next theorem, proved in [10], by carefully inspecting Figure 8.

**Theorem 5.4.3.** *A crooked surface separates* $\mathsf{Ein}_3$.

5.5. **Lorentzian Schottky groups.** In closing, we sketch a construction for fundamental domains of Lorentzian Schottky groups, bounded by pairwise disjoint surfaces [10]. Start with $2n$ crooked planes in $\mathsf{E}$ with common vertex, say $o = (0, 0, 0)$, but with pairwise consistently oriented directors. Move them away from each other, using allowable translations as in Definition 3.5.9. More precisely, *move each crooked plane in its stem quadrant*, so that the difference for pairs $\mathcal{C}_i^\pm$ is an allowable translation. This yields $2n$ pairwise disjoint crooked planes:

$$\mathcal{C}_1^-, \mathcal{C}_1^+, \ldots, \mathcal{C}_n^-, \mathcal{C}_n^+$$

bounding pairwise disjoint crooked halfspaces. Figure 9 shows a pair of disjoint crooked surfaces. Now their conformal compactifications in the Einstein Universe share a single point in common, $p_\infty$. Consider the following conformal involution:

$$\nu : (v_1 : v_2 : v_3 : v_4 : v_5) \longmapsto (-v_1 : v_2 : v_3 : v_4 : v_5).$$

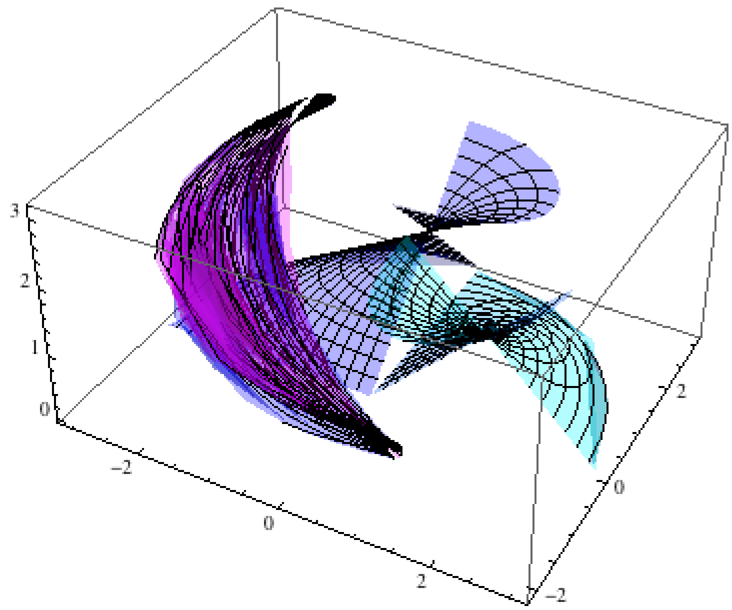


FIGURE 9. A pair of disjoint crooked surfaces, with the stems removed.

This involution permutes $\iota(o)$ and $p_\infty$; moreover it leaves invariant any crooked surface which is the conformal compactification of a crooked plane with vertex $o$.

Conjugating by $\nu$, we may move each crooked surface in its "stem quadrant" – appropriately interpreted in $\mathsf{Ein}_3$ – at $p_\infty$. A slight rephrasing of Theorem 3.5.10, as proved in [3], ensures that the crooked surfaces are displaced within the original crooked halfspaces, away from $p_\infty$. Thus we obtain $2n$ pairwise disjoint crooked surfaces:

$$\mathcal{S}_1^-, \mathcal{S}_1^+, \ldots, \mathcal{S}_n^-, \mathcal{S}_n^+.$$

Finally, we may find suitable maps $\gamma_1, \ldots, \gamma_n$ such that, for $i = 1, \ldots, n$:

$$\gamma_i(\mathcal{S}_i^-) = \mathcal{S}_i^+.$$

In some sense, this will be an "Einstein deformation" of a Schottky subgroup of $\mathrm{SO}(2,1)$! The disjoint crooked surfaces bound pairwise disjoint regions. In other words, we have built a Schottky-type fundamental domain for the group $\langle\gamma_1, \ldots, \gamma_n\rangle$.

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