Lorentzian Geometry I

Todd A. Drumm (Howard University, USA)

ICTP

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Basic Definitions

- $E^{n,1}$ ($n \geq 2$) is the Lorentzian (flat) affine space with $n$ spatial directions
  - The tangent space: $\mathbb{R}^{n,1}$
  - Choose a point $o \in E^{n,1}$ as the origin
  - Identification of $E$ and its tangent space: $p \leftrightarrow v = p - o$
Basic Definitions

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- The tangent space $\mathbb{R}^{n,1}$
  - $v = [v_1, \ldots, v_n, v_{n+1}]^T$
  - The (standard, indefinite) inner product:
    \[ v \cdot w = v_1w_1 + \ldots + v_nw_n - v_{n+1}w_{n+1} \]

- $O(n, 1)$ is the group of matrices which preserve the inner product
  - In particular, for any $v, w \in \mathbb{R}^{n,1}$ and any $A \in O(n, 1)$
    \[ Av \cdot Aw = v \cdot w \]

- $SO(n, 1)$ is the subgroup whose members have determinant 1.
- $O^\circ(n, 1) = SO^\circ(n, 1)$ is the connected subgroup containing the identity
• $N = \{ v \in \mathbb{R}^{n,1} | v \cdot v = 0 \}$ is the light cone (or null cone) and vectors lying here are called lightlike
  - Inside cone: $v$ such that $v \cdot v < 0$, are called timelike
  - Outside cone: $v$ such that $v \cdot v > 0$, are called spacelike

• Time orientation
  - Choice of nappe, and timelike vectors upper nappe, is a choice of time orientation
  - Choose the upper nappe to be the future; vectors on or inside the upper nappe are future pointing
Models of Hyperbolic Spaces

• One sheet of hyperboloid
  • \( H^n \cong \{ v \in \mathbb{R}^{n,1} | v \cdot v = -1, \text{ and future pointing} \} \)
  • \( w \cdot w > 0 \) for \( w \) tangent to hyperbola.
  • Defined metric has constant curvature \(-1\).
  • Geodesics = \{Planes thru \( o\}\} \cap \{\text{hyperboloid}\}

• Projective model
  • \( v \sim w \) if \( v = kw \) for \( k \neq 0 \), written \((v) = (w)\)
  • \( H^n \cong \{ v \in \mathbb{R}^{n,1} | v \cdot v < 0 \} / \sim \)
  • Homogeneous coordinates
    \( (v) = [v_1 : v_2 : \ldots : v_n] \)

• Klein model
  • Project onto \( v_n = 1 \) plane.
  • Geodesics are straight lines.
  • Not conformal.
Isometries

- Linear Isometries
  - $O(n, 1)$ has four connected components.
  - Isometries of $H^n$
- Affine isometries: $\mathcal{A} = (A, a) \in \text{Isom}(E)$
  - $A \in O(n, 1)$ and $a \in \mathbb{R}^{n, 1}$
  - $\mathcal{A}(x) = A(x) + a$

**Proposition**

*For any affine isometry, $x \mapsto A(x) + a$, if $A$ does not have 1 as an eigenvalue, then the map has a fixed point.*

**Proof.**

If $A$ does not have 1 as an eigenvalue, you can always solve $A(x) + a = x$, or $(A - I)(x) = -a$
Three dimensions

- More on products
  - \( v^\perp = \{ w | w \cdot v = 0 \} \)
    - If \( v \) is spacelike, \( v^\perp \) defines a geodesic.
    - If \( v \) is lightlike, \( v^\perp \) is tangent to lightcone at \( v \).
  - **(Lorentzian) cross product**
    - \( v \times w \) is (Lorentzian) orthogonal to \( v \) and \( w \).
    - Defined by \( v \cdot (w \times u) = \text{Det}(v, w, u) \).

- Upper half plane model of the hyperbolic plane
  - \( U = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \} \) with boundary \( \mathbb{R} \cup \{ \infty \} \).
  - Geodesics are arcs of circles centered on \( \mathbb{R} \) or vertical rays.
  - \( \text{Isom}^+(\mathbb{H}^2) \cong \text{PSL}(2, \mathbb{R}) \)
\[ A \in \text{SO}^0(2, 1) \]

- All \( A \) have 1 eigenvalue.
- Classification: Nonidentity \( A \) is said to be ...
  - \textit{elliptic} if it has complex eigenvalues.
    - The 2 complex eigenvectors are conjugate.
    - The fixed eigenvector \( A^0 \) is timelike.
    - Acts like rotation about fixed axis.
  - \textit{parabolic} if 1 is the only eigenvalue.
    - The fixed eigenvector \( A^0 \) is \textit{lightlike}.
    - On \( \mathbb{H}^2 \), fixed point on boundary and orbits are \textit{horocycles}.
  - \textit{hyperbolic} if it has 3 distinct real eigenvalues \( \lambda < 1 < \lambda^{-1} \)
    - Fixed eigenvector \( A^0 \) is spacelike.
    - The \textit{contracting} eigenvector \( A^- \) and \textit{expanding} eigenvector \( A^+ \) are lightlike.
    - \( A^0 \cdot A^\pm = 0 \)
- \( A(x) = A(x) + a \) is called \textit{elliptic} /\textit{parabolic}/ \textit{hyperbolic} if \( A \) is elliptic /parabolic/ hyperbolic.
Hyperbolic affine transformations

- More on linear part
  - Choose $A^\pm$ are future pointing and have Euclidean length 1.
  - Choose so that $A^0 \cdot (A^- \times A^+) > 0$ and $A^0 \cdot A^0 = 1$.
  - $(A^0)^\perp$ determines the axis of $A$ on the hyperbolic plane.
- The Margulis invariant for a hyperbolic $A = (A, a)$
  - There exist a unique invariant line $C_A$ parallel to $A^0$.
  - The Margulis invariant: for any $x \in C_A$
    \[
    \alpha(A) = (A(x) - x) \cdot A^0
    \]
  - Signed Lorentzian length of unique closed geo in $E^{2,1}/\langle A \rangle$.
  - $\alpha(A) = 0$ iff $A$ has a fixed point.
  - Invariant given choice of $x \in E$.
  - Invariant under conjugation ($\alpha$ is a class function), and determines conjugation class for a fixed linear part.
  - $\alpha(A^n) = |n| \alpha(A)$
Proper actions

• For any discrete $G$ action on a locally compact Hausdorff $X$, if $G$ is proper then $X/G$ is Hausdorff.
  • Alternatively, $G$ is to act freely properly discontinuously on $X$.
  • (Bieberbach) For $X = \mathbb{R}^n$ and discrete $G \subset \text{Isom}(X)$, if $G$ acts properly on $X$ then $G$ has a finite index subgroup $\cong \mathbb{Z}^m$ for $m \leq n$.

• Cocompact affine actions

Conjecture (Auslander)

For $X = \mathbb{R}^n$ and discrete $G \subset \text{Aff}(\mathbb{R}^n)$, if $G$ acts properly and cocompactly on $X$ then $G$ is virtually solvable.

• No free groups of rank $\geq 2$ in virtually solvable gps.
• True up to dimension 6.
• (Milnor) Is Auslander Conj. true if “cocompact” is removed? NO.
Margulis Opposite Sign Lemma

Lemma (Margulis’ Opposite Sign)

If $\alpha(A)$ and $\alpha(B)$ have opposite signs then $\langle A, B \rangle$ does not act properly on $E^{2,1}$.

• The signs for elements of proper actions must be the same.
• Opposite Sign Lemma true in $E^{n,n-1}$
  • When $n$ is odd, $\alpha(A^{-1}) = -\alpha(A)$, so no groups with free groups (rank $\geq 2$) act properly.
  • Can find counterexamples to “noncompact Auslander” in $E^{2,1}, E^{4,3}, ...$
Margulis space-times

- First examples

**Theorem (Margulis)**

*There exist discrete free groups of \( \text{Aff}(\mathbb{E}^{2,1}) \) that act properly on \( \mathbb{E}^{2,1} \).*

- Next examples
  - Free discrete groups in \( \langle A_1, A_2, ..., A_n \rangle \subset \text{Isom}(\mathbb{H}^2) \).
  - Domain bounded by \( 2n \) nonintersecting geodesics \( \ell_n^\pm \) such that \( A_i(\ell^-_i) = \ell^+_i \).
Crooked Planes

• Problem: extend notion of lines in $H^2$ to $E^{2,1}$.

• A Crooked Plane
  • Stem is perpendicular to spacelike vector $v$ through vertex $p$ inside the lightcone at $p$.
    • Spine is the line through $p$ and parallel to $v$
  • Wings are half planes tangent to light cones at boundaries of stem, called the hinges.

• A Crooked half-space is one of the two regions in $E^{2,1}$ bounded by a crooked plane.
The Spaces Isometries Three dimensions Proper Actions Margulis space-times

Crooked domains

Theorem (D)

Given discrete $\Gamma = \langle A_1, A_2, ..., A_n \rangle \subset \text{Isom}(\mathbb{E}^{2,1})$. If there exist $2n$ mutually disjoint crooked half spaces $\mathcal{H}_n^\pm$ such that $A_i(\mathcal{H}_i^-) = \mathbb{E}^{2,1} \setminus \mathcal{H}_i^+$, then $\Gamma$ is proper.

- Example of a “ping-pong” theorem.
- Finding proper actions
  - Start with a free discrete linear group.
  - Find disjoint halfspaces whose complement is domain for a linear part.
  - Separate half planes, giving rise to proper affine group.
Crooked domains

- Two pair of *disjoint* halfspaces at the origin.

- Separated
Results

Theorem (D)

Given every free discrete group $G \subset \text{SO}(2,1)$ there exists a proper subgroup $\Gamma \subset \text{Isom}(\mathbb{E}^{2,1})$ whose underlying linear group is $G$.

Theorem (Danciger- Guéritaud - Kassel)

For every discrete $\Gamma \subset \text{Isom}(\mathbb{E}^{2,1})$ acting properly on $\mathbb{E}^{2,1}$, there exists a crooked fundamental domain for the action.

References

- (with V. Charette) *Complete Lorentz 3-manifolds*, Cont. Math. 630, (2015), pp. 43 72