Lecture 1: Survey

I will lecture on the connection between Minimal Surfaces in Euclidean Spaces and Complex Analysis.

- A brief history of minimal surfaces.
- Basics on minimal surfaces and holomorphic null curves.
- Isotopies of conformal minimal immersions.
- Desingularizing conformal minimal immersions in $\mathbb{R}^n$ ($n \geq 5$).
- Proper conformal minimal immersions to $\mathbb{R}^3$ and embeddings to $\mathbb{R}^5$
- On the Calabi-Yau problem for minimal surfaces.

Based on joint work with

- Barbara Drinovec Drnovšek and Franc Forstnerič, University of Ljubljana.
- Francisco J. López, University of Granada.
1744 Euler The only area minimizing surfaces of rotation in $\mathbb{R}^3$ are planes and catenoids.
Minimal Surfaces in $\mathbb{R}^3$

1760 **Lagrange** Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and $f : \overline{\Omega} \to \mathbb{R}$ be a smooth function. Then the graph

$$S = \{(x, y, f(x, y)) : (x, y) \in \overline{\Omega}\} \subset \mathbb{R}^3$$

is an **area minimizing surface** if and only if

$$(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy} = 0;$$

equivalently,

$$\text{div} \left( \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) = 0. \quad (1)$$

This is known as the **equation of minimal graphs**.
Minimal Surfaces in $\mathbb{R}^3$

A (smooth) surface $S \subset \mathbb{R}^3$ satisfies locally the above equation iff its mean curvature function vanishes identically.

Definition (Minimal Surface)

A smoothly immersed surface $M \rightarrow \mathbb{R}^3$ is said to be a **minimal surface** if its **mean curvature function** $H: M \rightarrow \mathbb{R}$ is identically zero: $H = 0$.

$$H = \frac{\kappa_1 + \kappa_2}{2}$$

$$K = \kappa_1 \kappa_2$$

where $\kappa_1, \kappa_2$ are the **principal curvatures**.

$H = 0 \Rightarrow K = \kappa_1 \kappa_2 \leq 0$ (**Gauss curvature**).
Minimal Surfaces in $\mathbb{R}^3$

1776 Meusnier The helicoid is a minimal surface.

1842 Catalan The helicoid and the plane are the only ruled minimal surfaces in $\mathbb{R}^3$ (unions of straight lines).
Local Theory: Plateau Problem

1873 Plateau Minimal surfaces can be physically obtained as soap films.

1932 Douglas, Radó Every continuous injective closed curve in $\mathbb{R}^n$ ($n \geq 3$) spans a minimal surface.
Examples by Riemann

1865 On the way to this solution, Riemann and others discovered new examples of minimal surfaces using the Weierstrass Representation.

2015 Meeks, Pérez, Ros Riemann’s minimal examples, catenoids, helicoids, and planes are the only properly embedded minimal planar domains in $\mathbb{R}^3$. 
Proposition

Let $D$ be a smoothly bounded, compact domain in $\mathbb{R}^2_{(x,y)}$, let $f : D \to \mathbb{R}$ be a smooth function, and let $S := \text{Graph}(f)$ be the graph of $f$.

If $S$ is minimal and $\tilde{S} := \text{Graph}(\tilde{f})$, where $\tilde{f} : D \to \mathbb{R}$ is a smooth function such that $\tilde{f}|_{bD} = f|_{bD}$ (equivalently, $b\tilde{S} = bS$), then the areas

$$A(S) \leq A(\tilde{S}).$$

Furthermore, $A(\tilde{S}) = A(S)$ if and only if $\tilde{f} = f$.

The same holds for minimal graphs in $\mathbb{R}^n$ for all $n \geq 3$. A graph $\{(u, f(u)) : u \in \overline{\Omega}\} \subset \mathbb{R}^n$ ($n \geq 3$) over a smoothly bounded compact domain $\overline{\Omega} \subset \mathbb{R}^{n-1}$ minimizes volume if and only if satisfies the minimal surface equation (1).
Minimal graphs minimize area. Proof

Let $N : D \to S^2$ be the normal map of the minimal surface

$$S = \{(x, y, f(x, y)) : (x, y) \in D\}.$$

Consider the unitary vector field $V : D \times \mathbb{R} \to \mathbb{R}^3$ given by

$$V(x, y, z) = N(x, y) = \frac{1}{\sqrt{1 + |\nabla f|^2}}(-f_x, -f_y, 1).$$

Computing,

$$\text{div}_{\mathbb{R}^3} V = \frac{\partial}{\partial x} \left( -\frac{f_x}{\sqrt{1 + |\nabla f|^2}} \right) + \frac{\partial}{\partial y} \left( -\frac{f_y}{\sqrt{1 + |\nabla f|^2}} \right)$$

$$= -\text{div}_{\mathbb{R}^2} \left( \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) = 0.$$
Minimal graphs minimize area. Proof

Assume \( S, \widetilde{S} \subset \{ z > 0 \} \). Let \( W \) and \( \widetilde{W} \) be the regions in \( D \times \mathbb{R} \) with boundaries \( bW = S \cup C \cup D \) and \( b\widetilde{W} = \widetilde{S} \cup C \cup D \), where \( C \subset bD \times \mathbb{R} \). By the divergence theorem,

\[
0 = \int_W \text{div}(V) \, dx dy dz = \int_{bW} \langle V, \nu_{bW} \rangle \, dA_{bW} \\
= \int_D \langle V, \nu_D \rangle \, dx dy + \int_C \langle V, \nu_C \rangle \, dA_C + \int_S \langle V, \nu_S \rangle \, dA_S
\]

(\( \nu_\bullet = \)outer normal, \( dA_\bullet = \)area element)

\[
0 = \int_{\widetilde{W}} \text{div}(V) \, dx dy dz = \int_{b\widetilde{W}} \langle V, \nu_{b\widetilde{W}} \rangle \, dA_{b\widetilde{W}} \\
= \int_D \langle V, \nu_D \rangle \, dx dy + \int_C \langle V, \nu_C \rangle \, dA_C + \int_{\widetilde{S}} \langle V, \nu_{\widetilde{S}} \rangle \, dA_{\widetilde{S}}.
\]

Hence,

\[
\int_S \langle V, \nu_S \rangle \, dA_S = \int_{\widetilde{S}} \langle V, \nu_{\widetilde{S}} \rangle \, dA_{\widetilde{S}}. \quad (2)
\]
Minimal graphs minimize area. Proof

We have $V = N = \nu_S$ on $S$, and so

$$\int_S \langle V, \nu_S \rangle d\mathcal{A}_S = \int_S d\mathcal{A}_S = \mathcal{A}(S).$$

On the other hand, by Schwarz, $|\langle V, \nu_{\tilde{S}} \rangle| \leq \|V\| \cdot \|\nu_{\tilde{S}}\| = 1$, hence

$$\int_{\tilde{S}} \langle V, \nu_{\tilde{S}} \rangle d\mathcal{A}_{\tilde{S}} \leq \int_{\tilde{S}} d\mathcal{A}_{\tilde{S}} = \mathcal{A}(\tilde{S}).$$

Therefore, $\mathcal{A}(S) \leq \mathcal{A}(\tilde{S})$ in view of (2).

If $\mathcal{A}(\tilde{S}) = \mathcal{A}(S)$, then $|\langle V, \nu_{\tilde{S}} \rangle| = \|V\| \cdot \|\nu_{\tilde{S}}\| = \|V\| \cdot \|\nu_S\|$, hence $\nu_{\tilde{S}} = \nu_S$ and so $\nabla \tilde{f} = \nabla f$ and $\tilde{f} - f$ is constant.

Since $\tilde{f}|_{bD} = f_{bD}$, $\tilde{f} = f$ and $\tilde{S} = S$. 
Curvature of surfaces in $\mathbb{R}^n$

Let $D$ be a domain in $\mathbb{R}^2_{(u_1,u_2)}$ and $X = (X_1, \ldots, X_n) : D \to \mathbb{R}^n$ be a $C^2$ embedding. Hence, $S = X(D) \subset \mathbb{R}^n$ is a parametrized surface in $\mathbb{R}^n$.

Every smooth embedded curve in $S$ is of the form

$$\lambda(t) = X(u_1(t), u_2(t)) \in S$$

where $t \mapsto (u_1(t), u_2(t))$ is a smooth embedded curve in $D$.

Let $s = s(t)$ denote the arc length on $\lambda$. The number

$$\kappa(T, N) := \frac{d^2 \lambda}{ds^2} \cdot N = \sum_{i,j=1}^{2} \left( X_{u_iu_j} \cdot N \right) \frac{du_i}{ds} \frac{du_j}{ds}$$

is the **normal curvature** of $S$ at $p = \lambda(t_0) \in S$ in the tangent direction $T = \lambda'(s_0) \in T_pS$ with respect to the normal vector $N \in N_pS$. 
Curvature in terms of fundamental forms

In terms of $t$-derivatives we get

$$
\kappa(T, N) = \frac{\sum_{i,j=1}^{2} \left( X_{u_i u_j} \cdot N \right) \dot{u}_i \dot{u}_j}{\sum_{i,j=1}^{2} g_{i,j} \dot{u}_i \dot{u}_j} = \frac{\text{second fundamental form}}{\text{first fundamental form}}.
$$

Fix a normal vector $N \in N_p S$ and vary the unit tangent vector $T \in T_p S$. The principal curvatures of $S$ at $p$ in direction $N$ are the numbers

$$
\kappa_1(N) = \max_T \kappa(T, N), \quad \kappa_2(N) = \min_T \kappa(T, N).
$$

Their average

$$
H(N) = \frac{\kappa_1(N) + \kappa_2(N)}{2} \in \mathbb{R}
$$

is the mean curvature of $S$ at $p$ in the normal direction $N \in N_p S$. 
The mean curvature vector

Let $G = (g_{i,j})$ and $b(N) = (b_{i,j}(N))$ be the matrix of the 1st and the 2nd fundamental form. The extremal values of $\kappa(T, N)$ are roots of

$$\det(b(N) - \mu G) = 0$$

$$(\det G)\mu^2 - (g_{2,2}b_{1,1}(N) + g_{1,1}b_{2,2}(N) - 2g_{1,2}b_{1,2}(N))\mu + \det b(N) = 0.$$ 

Note that $b_{i,j}(N) = X_{u_i u_j} \cdot N$. The Vieta formula gives

$$H(N) = \frac{g_{2,2}X_{u_1 u_1} + g_{1,1}X_{u_2 u_2} - 2g_{1,2}X_{u_1 u_2}}{2 \det G} \cdot N.$$ 

There is a unique normal vector $H \in N_p S$ such that

$$H(N) = H \cdot N$$

for all $N \in N_p S$.

This $H$ is the mean curvature vector of the surface $S$ at $p$. 
The mean curvature in isothermal coordinates

The formulas simplify drastically in isothermal coordinates:

\[(g_{i,j}) = \mu I, \quad \mu = \|X_{u_1u_1}\|^2 = \|X_{u_2u_2}\|^2.\]

\[H(N) = \frac{(X_{u_1u_1} + X_{u_2u_2})}{2\mu} \cdot N = \frac{\triangle X}{2\mu} \cdot N.\]

**Lemma**

Assume that \(D\) is a domain in \(\mathbb{R}^2_{(u_1,u_2)}\) and \(X: D \rightarrow \mathbb{R}^n\) is a conformal immersion of class \(C^2\) (i.e., \(u = (u_1,u_2)\) are isothermal for \(X\)). Then the Laplacian \(\triangle X = X_{u_1u_1} + X_{u_2u_2}\) is orthogonal to \(S = X(D)\) and satisfies

\[\triangle X = 2\mu H\]

where \(H\) is the mean curvature vector and

\[\mu = \|X_{u_1u_1}\|^2 = \|X_{u_2u_2}\|^2.\]
Proof of the lemma

It suffices to show that the vector

$$\triangle X(u)$$

is orthogonal to the surface $S$ at the point $X(u)$ for every $u \in D$. If this holds, it follows from the preceding formula that the normal vector

$$\frac{\triangle X(u)}{2\mu} \in \mathcal{N}_{X(u)}S$$

fits the definition of the mean curvature vector $H$, so it equals $H$.

Conformality of the immersion $X$ can be written as follows:

$$X_{u_1} \cdot X_{u_1} = X_{u_2} \cdot X_{u_2}, \quad X_{u_1} \cdot X_{u_2} = 0.$$ 

Differentiating the first identity on $u_1$ and the second one on $u_2$ yields

$$X_{u_1 u_1} \cdot X_{u_1} = X_{u_1 u_2} \cdot X_{u_2} = -X_{u_2 u_2} \cdot X_{u_1},$$

whence $\triangle X \cdot X_{u_1} = 0$. Similarly we get $\triangle X \cdot X_{u_2} = 0$ by differentiating the first identity on $u_2$ and the second one on $u_1$. 

Lagrange’s formula for the variation of area

The area of an immersed surface \( X: D \to \mathbb{R}^n \) with the 1st fundamental form \( G = (g_{i,j}) \) equals

\[
\mathcal{A}(X) = \int_D \sqrt{\det G} \cdot du_1 du_2.
\]

Let \( N: D \to \mathbb{R}^n \) be a normal vector field along \( X \) which vanishes in \( bD \). Consider the 1-parameter family of maps \( X^t: D \to \mathbb{R}^n \):

\[
X^t(u) = X(u) + tN(u), \quad u \in D, \ t \in \mathbb{R}.
\]

A calculation gives the formula for the first variation of area:

\[
\delta \mathcal{A}(X) N = \frac{d}{dt} \bigg|_{t=0} \mathcal{A}(X^t) = -2 \int_D H \cdot N \sqrt{\det G} \cdot du_1 du_2.
\]

It follows that \( \delta \mathcal{A}(X) = 0 \iff H = 0 \).
Conformal minimal surfaces are harmonic

In view of the formula $\Delta X = 2\mu H$ which holds for a conformal immersion $X$ we get the following corollary.

**Corollary**

The following are equivalent for a smooth conformal immersion $X : D \to \mathbb{R}^n$ from a domain $D \subset \mathbb{R}^2$:

- $X$ is minimal (a stationary point of the area functional)
- $X$ has vanishing mean curvature vector: $H = 0$
- $X$ is harmonic: $\Delta X = 0$
We wish to emphasize the difference between

- **minimal surfaces**: these are stationary (critical) points of the area functional, and are only locally area minimizing; and

- **area-minimizing surfaces**: these are surfaces which globally minimize the area among all nearby surfaces with the same boundary.

Minimal surfaces which are **graphs** are in fact (globally) area minimizing.
Definition

A **Riemann surface** $M$ is a complex manifold of complex dimension one.

So, a Riemann surface $M = (M, \mathcal{A})$ is nothing but a smooth (real) surface $M$ together with an **atlas** $\mathcal{A}$ such that the change of charts are **biholomorphic functions**.

**Holomorphic functions** $M \rightarrow \mathbb{C}$ and **harmonic functions** $M \rightarrow \mathbb{R}$ on a Riemann surface $M$ are well defined.
Riemann surfaces

Definition

A **compact bordered Riemann surface** $\overline{M}$ is a compact Riemann surface with nonempty boundary $bM \neq \emptyset$ consisting of finitely many pairwise disjoint smooth Jordan curves. The interior $M = \overline{M} \setminus bM$ is said a **bordered Riemann surface**.

Open Riemann surfaces $M$ (noncompact and with $bM = \emptyset$) are classified:

- **Hyperbolic**: there are non-constant negative subharmonic functions $M \to \mathbb{R}$; equivalently, the Brownian motion on $M$ is transient.
  The unit complex disc $\mathbb{D} \subset \mathbb{C}$ is hyperbolic. Every bordered Riemann surface is hyperbolic.

- **Parabolic**: every negative subharmonic function $M \to \mathbb{R}$, is constant; equivalently, the Brownian motion on $M$ is recurrent.
  The complex plane $\mathbb{C}$ is parabolic. Every compact Riemann surface with finitely many points removed is parabolic.
A Riemann surface $\mathcal{M} = (M, \mathcal{T})$ is the same thing as a smooth (real) oriented surface $M$ together with a conformal structure $\mathcal{T}$; i.e., a conformal class of Riemannian metrics on $M$. (Two metrics $g_1, g_2$ on $M$ are conformal if $g_1 = \lambda g_2$ for some smooth positive function $\lambda$ on $M$.)

**Angles** on a Riemann surface $\mathcal{M}$ are well defined.

**Definition**

An immersion $X: \mathcal{M} = (M, \mathcal{T}) \to \mathbb{R}^n$ ($n \geq 3$) is said to be conformal if it preserves angles; equivalently, if the Riemannian metric $X^* ds^2$ induced on $M$ by the Euclidean metric $ds^2$ in $\mathbb{R}^n$ via $X$ lies in $\mathcal{T}$. 
Conformal minimal surfaces are harmonic

Let \((M, g)\) be a Riemannian surface, let \(X : (M, g) \rightarrow \mathbb{R}^n (n \geq 3)\) be an isometric immersion, and let \(H\) be the mean curvature vector of \(X\). Then

\[ \triangle X = 2H. \]

**Corollary**

The following are equivalent for a conformal immersion \(X : M \rightarrow \mathbb{R}^n (n \geq 3)\):

- \(X\) is minimal (i.e., the mean curvature vector is identically zero).
- \(X\) is a stationary point of the area functional.
- \(X\) is a harmonic map (i.e., \(\triangle X = 0\) on \(M\)).

In the sequel we shall always assume that \(X : M \rightarrow \mathbb{R}^n\) is conformal and hence

\[ X \text{ is minimal } \iff \delta \mathcal{A}(X) \iff \triangle X = 0. \]
Weierstrass Representation (Osserman 1960s)

Let $\mathbf{M}$ be an open Riemann surface.
Let $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_n): \mathbf{M} \to \mathbb{R}^n$ be a conformal minimal immersion.
Let $z$ be a conformal parameter on $\mathbf{M}$ and denote by

$$\phi_j = \partial_z \mathbf{X}_j, \ j = 1, \ldots, n \quad \text{(holomorphic 1-forms in $\mathbf{M}$).}$$

- $\mathbf{X}^* ds^2 = \sum_{j=1}^{n} |\phi_j|^2 \neq 0 \quad \text{(conformal metric).}$
- $\Phi = (\phi_1, \ldots, \phi_n)$ satisfies $\sum_{j=1}^{n} \phi_j^2 = 0$.
- Hence $f = \frac{\Phi}{\vartheta}: \mathbf{M} \to \mathbb{C}^n$ assumes values in the Null Quadric:

$$\mathcal{A}^* = \{(\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n: \zeta_1^2 + \cdots + \zeta_n^2 = 0\} \setminus \{0\} \subset \mathbb{C}^n,$$
(here $\vartheta$ is any holomorphic 1-form on $\mathbf{M}$ with no zeros).
- $\Phi$ has no real periods: $\Re \int_{\gamma} \Phi = 0$ for any closed curve $\gamma \subset \mathbf{M}$.
- $\mathbf{X}(p) = \mathbf{X}(p_0) + \Re \int_{p_0}^{p} \Phi = \mathbf{X}(p_0) + \Re \int_{p_0}^{p} f \vartheta$. Conversely! (Period Problem)
- $\Phi \equiv \text{Weierstrass Representation of $\mathbf{X}$}$. 
Examples

• The plane: $M = \mathbb{C}$, $\Phi = (1, -i, 0) \, dz$.

• The catenoid: $M = \mathbb{C} \setminus \{0\}$, $\Phi = \left( \frac{1}{2} \left( \frac{1}{z^2} - 1 \right), \frac{i}{2} \left( \frac{1}{z^2} + 1 \right), \frac{1}{z} \right) \, dz$.

• The helicoid: $M = \mathbb{C}$, $\Phi = \left( \frac{i}{2} (e^{-z} - e^z), -\frac{1}{2} (e^{-z} + e^z), i \right) \, dz$. 
The Null Quadric is Oka

Let $\vartheta$ be a holomorphic 1-form on $M$ with no zeros. There is a bijective correspondence (up to translations)

$$\{X: M \to \mathbb{R}^n \text{ CMI}\} \leftrightarrow \{f: M \to \mathbb{A}^* \text{ holomorphic, } f\vartheta \text{ no real periods}\}$$

$$X(p) = X(p_0) + \Re \int_{p_0}^{p} f\vartheta, \quad f = \frac{\Phi}{\vartheta}.\$$

We construct CMIs $M \to \mathbb{R}^n$ by finding holomorphic maps $f: M \to \mathbb{A}^* \subset \mathbb{C}^n$ for which $f\vartheta$ has no real periods:

$$\Re \int_{\Gamma} f\vartheta \quad \text{for every closed curve } \Gamma \subset M.$$

The Null Quadric

$$\mathbb{A}^* = \{\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n: \zeta_1^2 + \cdots + \zeta_n^2 = 0\} \setminus \{0\} \quad (n \geq 3)$$

controlling minimal surfaces in $\mathbb{R}^n$ is an Oka Manifold.
Weierstrass Representation in $\mathbb{R}^3$: the Gauss map

Let $X = (X_1, X_2, X_3): M \to \mathbb{R}^3$ be a conformal minimal immersion with Weierstrass representation $\Phi = f \vartheta$, where $f = (\phi_1, \phi_2, \phi_3): M \to \mathfrak{a}^* \subset \mathbb{C}^3$. Then either $\phi_1 \equiv i \phi_2$ and $X$ is flat, or

$$f = \left( \frac{1}{2} \left( \frac{1}{g} - g \right), \frac{i}{2} \left( \frac{1}{g} + g \right), 1 \right) \phi_3$$

where

$$g = \frac{\phi_3}{\phi_1 - i \phi_2}$$

is the stereographic projection of the Gauss map $M \to S^2$ of $X$.

- $g: M \to \mathbb{C}$ is a meromorphic function, called the complex Gauss map of $X$.
- $X^* ds^2 = \sum_{j=1}^{n} |\phi_j|^2 = \frac{1}{4} \left( |g| + \frac{1}{|g|} \right)^2 |\phi_3|^2 |\vartheta|^2$. Hence, the zeros and poles of $g$ are zeros of $\phi_3$ with the same order.
- $(g, \phi_3 \vartheta) \equiv$ Weierstrass Representation of $X$. Conversely! (Period Problem)
Nonorientable minimal surfaces

Nonorientable surfaces present themselves quite naturally in the very origin of Minimal Surface Theory.
Nonorientable minimal surfaces

Assume that $M$ carries an antiholomorphic involution $\mathcal{J}: M \to M$ without fixed points. If $X: M \to \mathbb{R}^n$ is a conformal minimal immersion such that

$$X \circ \mathcal{J} = X,$$

(3)

then $X(M) \subset \mathbb{R}^n$ is a nonorientable minimal surface.

If $\Phi = f \vartheta$ are the Weierstrass data of $X$ and $\mathcal{J}^* \vartheta = \bar{\vartheta}$, then (3) is equivalent to

$$f \circ \mathcal{J} = \bar{f}.$$

If $n = 3$, $f = (\phi_1, \phi_2, \phi_3)$, and $g = \frac{\phi_3}{\phi_1 - i\phi_2}$, then (3) is equivalent to

$$\phi_3 \circ \mathcal{J} = \bar{\phi}_3, \quad g \circ \mathcal{J} = -\frac{1}{\bar{g}}.$$
Holomorphic null curves in $\mathbb{C}^n$

**Definition**

Let $\mathbf{M}$ be a Riemann surface. A holomorphic immersion

$$\mathbf{F} = (F_1, F_2, \ldots, F_n): \mathbf{M} \to \mathbb{C}^n$$

is a **null curve** if the derivative $\mathbf{F}' = (F'_1, F'_2, \ldots, F'_n)$ with respect to any local holomorphic coordinate $\zeta = x + iy$ on $\mathbf{M}$ satisfies

$$(F'_1)^2 + (F'_2)^2 + \ldots + (F'_n)^2 = 0,$$

The nullity condition is equivalent to

$$\mathbf{F}'(\zeta) \in \mathcal{A}^* = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} z_j^2 = 0 \},$$

where $\mathcal{A}^*$ is the **null quadric** in $\mathbb{C}^n$. 
Holomorphic null curves in $\mathbb{C}^n$

Let $\vartheta$ be a holomorphic 1-form on $\mathbf{M}$ with no zeros. There is a bijective correspondence (up to translations)

$$\{ F : \mathbf{M} \to \mathbb{C}^n \text{ null curves} \} \longleftrightarrow \{ f : \mathbf{M} \to \mathfrak{A}^* \text{ holomorphic, } f \vartheta \text{ exact} \}$$

$$F(p) = F(p_0) + \int_{p_0}^p f \vartheta, \quad f = \frac{\Phi}{\vartheta}.$$

We construct null curves $\mathbf{M} \to \mathbb{C}^n$ by finding holomorphic maps $f : \mathbf{M} \to \mathfrak{A}^* \subset \mathbb{C}^n$ for which $f \vartheta$ is an exact 1-form:

$$\int_{\Gamma} f \vartheta \quad \text{for every closed curve } \Gamma \subset \mathbf{M}.$$
If $F = X + iY : M \to \mathbb{C}^n$ is a holomorphic null curve, then

$$X = \Re F : M \to \mathbb{R}^n, \quad Y = \Im F : M \to \mathbb{R}^n$$

are conformal harmonic (hence minimal) immersions into $\mathbb{R}^n$.

Conversely, a conformal minimal immersion $X : D \to \mathbb{R}^n$ of the disc $D = \{z \in \mathbb{C} : |z| < 1\}$ is the real part of a holomorphic null curve $F : D \to \mathbb{C}^n$. (This fails on multiply connected Riemann surfaces.)

If $F = X + iY : M \to \mathbb{C}^n$ is a null curve then

$$F^* ds^2_{\mathbb{C}^n} = 2X^* ds^2_{\mathbb{R}^n} = 2Y^* ds^2_{\mathbb{R}^n}.$$
The calculation

Let $F = X + iY = (F^1, \ldots, F^n) : M \to \mathbb{C}^n$ be a holomorphic null curve and $\zeta = x + iy$ a local holomorphic coordinate on $M$. Then

$$0 = \sum_{j=1}^{n} (F^j_\zeta)^2 = \sum_{j=1}^{n} (F^j_x)^2 = \sum_{j=1}^{n} \left( X^j_x + iY^j_x \right)^2$$

$$= \sum_{j=1}^{n} \left( (X^j_x)^2 - (Y^j_x)^2 \right) + 2i \sum_{j=1}^{n} X^j_x Y^j_x.$$

Since $Y_x = -X_y$ by the Cauchy-Riemann equations,

$$0 = |X_x|^2 - |X_y|^2 - 2i X_x \cdot X_y \iff |X_x| = |X_y|, \ X_x \cdot X_y = 0.$$

It follows that $X$ is conformal harmonic and

$$F^* ds_{\mathbb{C}^n}^2 = |F_x|^2 (dx^2 + dy^2) = 2|X_x|^2 (dx^2 + dy^2) = 2X^* ds_{\mathbb{R}^n}^2 = 2Y^* ds_{\mathbb{R}^n}^2.$$
Example: catenoid and helicoid

The **catenoid** and the **helicoid** are **conjugate minimal surfaces**; i.e., the real and the imaginary part of the same null curve

\[ F(\zeta) = (\cos \zeta, \sin \zeta, -i\zeta), \quad \zeta = x + iy \in \mathbb{C}. \]

Consider the family of conformal minimal immersions \((t \in \mathbb{R})\):

\[
X_t(\zeta) = \Re \left( e^{it} F(\zeta) \right) \\
= \cos t \begin{pmatrix} \cos x \cdot \cosh y \\ \sin x \cdot \cosh y \\ y \end{pmatrix} + \sin t \begin{pmatrix} \sin x \cdot \sinh y \\ -\cos x \cdot \sinh y \\ x \end{pmatrix}
\]

At \( t = 0 \) we have a catenoid and at \( t = \pm \pi/2 \) a helicoid.

Two minimal surfaces are said to be **associated** to each other if they are obtained as projection of the same null curve.
The flux map of a conformal minimal immersion $X: M \to \mathbb{R}^n$ is the group homomorphism

$$\text{Flux}_X: H_1(M; \mathbb{Z}) \to \mathbb{R}^n$$

given on any closed curve $C$ in $M$ by

$$\text{Flux}_X(C) = \int_C \Im \Phi = \int_C d^c X$$

where $\Phi = \partial_Z X$ are the Weierstrass data of $X$ and

$$d^c X = i(\overline{\partial} X - \partial X)$$

is the conjugate differential. (Note that $d^c X$ is closed precisely when $X$ is harmonic: $dd^c X = 0$.)

A conformal minimal immersion $X: M \to \mathbb{R}^n$ is the real part of a holomorphic null curve $M \to \mathbb{C}^n$ if and only if $\text{Flux}_X = 0$. 
Let $X : M \to \mathbb{R}^n \ (n \geq 3)$ be a conformal minimal immersion.

**Definition**

$X$ (or $X(M)$) is said to be **complete** if the induced metric

$$X^* ds^2 = \sum_{j=1}^{n} |\phi_j|^2$$

is a complete Riemannian metric on $M$; equivalently if the Euclidean length of $X(\gamma) \subset \mathbb{R}^n$ is infinite for any divergent curve $\gamma \subset M$.

**Definition**

$X$ (or $X(M)$) is said to be **proper** if $X(\gamma) \subset \mathbb{R}^n$ diverges for any divergent curve $\gamma \subset M$.

Properness trivially implies completeness.
Global Theory

Properness \rightarrow \text{Completeness}

Topological Properties \quad \text{Geometric Properties} \quad \text{Conformal Properties}

Embedded \quad \text{Immersed}
Finite Total Curvature

\( \mathbf{X} : \mathcal{M} \to \mathbb{R}^3 \) is said to be of **finite total curvature** if

\[
\int_{\mathcal{M}} |-K| = -\int_{\mathcal{M}} K < +\infty,
\]

where \( K \leq 0 \) denotes the Gauss curvature function of \( \mathbf{X} \).

**1963 Osserman** A conformal complete minimal immersion \( \mathbf{X} : \mathcal{M} \to \mathbb{R}^3 \) is of FTC if and only if

- \( \mathcal{M} \cong \overline{\mathcal{M}} \setminus E \), where \( \overline{\mathcal{M}} \) is a compact Riemann surface and \( E \subset \overline{\mathcal{M}} \) is finite; that is, of **finite topology** and **parabolic conformal type**.
- The Weierstrass data \( \phi_j = \partial_z \mathbf{X}_j \) (and so \( g \)) extend meromorphically to \( \overline{\mathcal{M}} \).

The wealth of examples, discovered since about 1980, rely on this theorem.

Very well studied theory: General existence results, classification theorems, asymptotic behavior, behavior of the Gauss map, ...

**1983 Jorge-Meeks** Every complete minimal surface in \( \mathbb{R}^3 \) of FTC is **proper** in \( \mathbb{R}^3 \).
Completeness, Properness, and Conformal Structure

By the Maximum Principle for harmonic maps, there is no compact minimal surface in $\mathbb{R}^3$. All the complete minimal surfaces in $\mathbb{R}^3$ discovered up to the third quarter of the 19th century are proper and (except a few periodic ones) of parabolic conformal type.

1963 Calabi’s Conjecture Every complete minimal surface in $\mathbb{R}^3$ is unbounded. (Disproved by Nadirashvili in 1996.)

1980s Sullivan’s Conjecture Every proper minimal surface in $\mathbb{R}^3$ of finite topology is parabolic. (Disproved by Morales in 2003.)

1985 Schoen-Yau’s Conjecture Every minimal surface in $\mathbb{R}^3$ properly projecting into $\mathbb{R}^2$ is parabolic. (Disproved by AA-López in 2012.)
Question

How to construct complete/proper minimal surfaces in $\mathbb{R}^3$ with hyperbolic conformal type?
Even more, with given conformal structure?
Theorem

Let $\mathbb{M}$ be an open Riemann surface and let $K \subset \mathbb{M}$ be a Runge compact subset (consisting of finitely many pairwise disjoint, compact, smoothly bounded domains and smooth arcs).

Every smooth immersion $K \to \mathbb{R}^n$ ($n \geq 3$) being a conformal minimal immersion in $\bar{K}$ can be uniformly approximated on $K$ by conformal minimal immersions $\mathbb{M} \to \mathbb{R}^n$.


[A. Alarcón, F. Forstnerič, F.J. López: *Embedded minimal surfaces in $\mathbb{R}^n$.* Preprint 2014]
General position of minimal surfaces

Self-intersections of surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ are **stable** under deformations.

**Theorem**

Let $n \in \mathbb{N}$, $n \geq 5$, and let $M = \hat{M} \cup bM$ be a compact bordered Riemann surface. Every conformal minimal immersion $X: M \to \mathbb{R}^n$ can be uniformly approximated in the $C^0(M)$-topology by conformal minimal embeddings $Y: M \to \mathbb{R}^n$.

**Corollary**

Let $n \in \mathbb{N}$, $n \geq 5$, and let $M$ be an open Riemann surface. Every conformal minimal immersion $X: M \to \mathbb{R}^n$ can be uniformly approximated in compact subsets of $M$ by conformal minimal embeddings $Y: M \to \mathbb{R}^n$.

[A. Alarcón, F. Forstnerič, F.J. López: *Embedded minimal surfaces in $\mathbb{R}^n$.* Preprint 2014]
Schoen-Yau’s conjecture and the embedding theorem

Theorem

Let $n \in \mathbb{N}$, $n \geq 3$, and let $\mathbf{M}$ be an open Riemann surface. There is a conformal minimal immersion $(Y_1, Y_2, \ldots, Y_n): \mathbf{M} \to \mathbb{R}^n$, embedding if $n \geq 5$, such that $(Y_1, Y_2): \mathbf{M} \to \mathbb{R}^2$ is a proper map.

[A. Alarcón, F. Forstnerič, F.J. López: Embedded minimal surfaces in $\mathbb{R}^n$. Preprint 2014]
Corollary

Every open Riemann surface is the underlying conformal structure of a minimal surface in $\mathbb{R}^3$ properly projecting into $\mathbb{R}^2$.

1980s **Sullivan’s Conjecture** Every proper minimal surface in $\mathbb{R}^3$ of finite topology is parabolic. (Disproved by **Morales** in 2003.)

1985 **Schoen-Yau’s Conjecture** Every minimal surface in $\mathbb{R}^3$ properly projecting into $\mathbb{R}^2$ is parabolic.
The embedding theorem

1950s **Bishop, Narasimhan, Remmert** Every open Riemann surface properly embeds in $\mathbb{C}^3$ as a complex curve, hence in $\mathbb{R}^6$ as a conformal minimal surface.

1975 **Greene-Wu** Every open Riemann surface properly embeds in $\mathbb{R}^5$ by harmonic functions.

**Corollary**

*Every open Riemann surface is the underlying conformal structure of a properly embedded minimal surface in $\mathbb{R}^5$.*

False in $\mathbb{R}^3$.

Open in $\mathbb{R}^4$. (Examples with arbitrary orientable topology are known. Related to the **Bell-Narasimhan** conjecture.)
Null curves

We may prescribed the flux of the examples in all the previous results.

These results were inspired by similar ones for null holomorphic curves and, more generally, directed holomorphic immersions of open Riemann surfaces into $\mathbb{C}^n$ when the directing variety is Oka were proved in


In the holomorphic case, the general position is embedded in $\mathbb{C}^3$. 
Theorem

Let $n \geq 3$. Let $M$ be an open Riemann surface and let $K \subset M$ be a Runge compact subset (consisting of finitely many pairwise disjoint, compact, smoothly bounded domains and smooth arcs). Let $X_t : K \to \mathbb{R}^n \ (t \in [0, 1])$ be a smooth homotopy of smooth immersions such that $X_t$ is a conformal minimal immersion in $\overline{K}$ for all $t$ and $X_0$ is a conformal nonflat minimal immersion on $M$.

Then the homotopy $X_t$ can be uniformly approximated on $K$ by a smooth homotopy of conformal minimal immersions $Y_t : M \to \mathbb{R}^n$ such that

- $Y_0 = X_0$.
- $\text{Flux}_{Y_t} = \text{Flux}_{X_t}$ on $H_1(K, \mathbb{Z})$.
- $\text{Flux}_{Y_1}$ can be prescribed on $H_1(M, \mathbb{Z}) \setminus H_1(K, \mathbb{Z})$.

[A. Alarcón, F. Forstnerič: Every conformal minimal surface in $\mathbb{R}^3$ is isotopic to the real part of a holomorphic null curve. J. Reine Angew. Math. (Crelle), in press]
Theorem

Let $M$ be an open Riemann surface and $n \geq 3$.

- For every conformal minimal immersion $X_0 : M \to \mathbb{R}^n$ there exists a smooth homotopy $X_t : M \to \mathbb{R}^n$ ($t \in [0, 1]$) of conformal minimal immersions such that $X_t$ is the real part of a null curve $M \to \mathbb{C}^n$.

- If $X_0$ is nonflat and $p : H_1(M; \mathbb{Z}) \to \mathbb{R}^n$ is a group homomorphism then there exists a smooth homotopy $X_t : M \to \mathbb{R}^n$ ($t \in [0, 1]$) of conformal minimal immersions such that $X_1$ is complete and $\text{Flux}_{X_1} = p$.

- If $X_0$ is complete then we can choose $X_t$ (as above) to be complete for every $t \in [0, 1]$.

[A. Alarcón, F. Forstnerič: Every conformal minimal surface in $\mathbb{R}^3$ is isotopic to the real part of a holomorphic null curve. J. Reine Angew. Math. (Crelle), in press]
Theorem

Let $M$ be a compact bordered Riemann surface and let $X: M \rightarrow \mathbb{R}^n$ ($n \geq 3$) be a conformal minimal immersion (the central surface). Let $I$ be a compact subarc of $bM$ which is not a connected component of $bM$. Choose a small annular neighborhood $A \subset M$ of the component $C$ of $bM$ containing $I$ and a smooth retraction $\rho: A \rightarrow C$. Let $u, v \in \mathbb{R}^n$ be a couple of unitary orthogonal vectors (the direction vectors), let $\mu: C \rightarrow \mathbb{R}_+$ be a continuous function supported on $I$ (the size function), and consider the continuous map

$$\kappa: bM \times \overline{D} \rightarrow \mathbb{R}^n$$

$$\kappa(x, \zeta) = \begin{cases} X(p); & p \in bM \setminus I \\ X(p) + \mu(p) (\Re \zeta u + \Im \zeta v), & p \in I. \end{cases}$$
Theorem (Continued)

Then for any number $\epsilon > 0$ there exist an arbitrarily small open neighborhood $\Omega$ of $I$ in $A$ and a conformal minimal immersion $Y: M \rightarrow \mathbb{R}^n$ satisfying the following properties:

- $Y$ is $\epsilon$-close to $X$ in the $C^1$ topology on $M \setminus \Omega$.
- $\text{dist}(Y(p), \kappa(p, b\mathbb{D})) < \epsilon$ for all $p \in bM$.
- $\text{dist}(Y(p), \kappa(\rho(p), \overline{D})) < \epsilon$ for all $p \in \Omega$.


We do not change the conformal structure on $M$. 
1963 **Calabi’s Conjecture** Every complete minimal surface in $\mathbb{R}^3$ is unbounded. (Disproved by Nadirashvili in 1996.)

2000 **Yau** What is the geometry of complete bounded minimal surfaces?

2006 **Martín-Morales** There are complete proper minimal surfaces in any convex domain of $\mathbb{R}^3$.

**What about domains other than convex ones?**

2012 **Ferrer-Martín-Meeks** There are complete bounded minimal surfaces in $\mathbb{R}^3$ with arbitrary topology.

**What about the conformal structure?**
Calabi-Yau Problem

**Theorem**

Let \( n \in \mathbb{N}, \ n \geq 3, \) and let \( \mathcal{D} \) be a convex domain in \( \mathbb{R}^n. \) Let \( \mathcal{M} \) be a bordered Riemann surface and let \( X: \overline{\mathcal{M}} \to \mathbb{R}^n \) be a conformal minimal immersion with \( X(\overline{\mathcal{M}}) \subset \mathcal{D}. \) 

\( X \) may be approximated uniformly in compacta in \( \mathcal{M} \) by complete proper conformal minimal immersions (embeddings if \( n \geq 5 \)) \( \mathcal{M} \to \mathcal{D}. \)

**Theorem**

Every compact bordered Riemann surface \( \mathcal{M} \) is the underlying conformal structure of complete minimal surface in \( \mathbb{R}^3 \) bounded by Jordan curves.


[A. Alarcón, B. Drinovec, F. Forstnerič, F.J. López: *Every bordered Riemann surface is a complete conformal minimal surface bounded by Jordan curves.* Preprint 2015]
Calabi-Yau Problem

Theorem

Let $\mathcal{D}$ be a mean-convex domain in $\mathbb{R}^3$. Every bordered Riemann surface $\mathbf{M}$ carries a complete proper conformal minimal immersion $\mathbf{M} \to \mathcal{D}$.


These results contribute to the conformal and asymptotic Calabi-Yau problems.