CR Geometry, Mappings into Spheres, and Sums-Of-Squares
Lecture III

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Outline - Lecture III

CR submanifolds of complex manifolds (here $\mathbb{C}^N$).

Recall $\mathbb{C}^N \cong \mathbb{R}^{2N}$ via $z_j = x_j + iy_j$, or

$$x_j = \frac{1}{2}(z + \bar{z}), \quad y_j = \frac{1}{2}(z + \bar{z}).$$

Let $M \subset \mathbb{C}^N$ be submanifold of real codimension d; thus, locally near $p \in M$, defined by real equations:

$$\rho_1(z, \bar{z}) = \ldots = \rho_d(z, \bar{z}) = 0, \quad d\rho_1 \wedge \ldots d\rho_d \neq 0.$$

Recall $d = \partial + \bar{\partial}$; $\mathcal{V} = T_p^{0,1}M := \mathbb{C}T_pM \cap T_p^{0,1}\mathbb{C}^N$ (CR tangent space); i.e.

$$X = \sum_{j=1}^{N} X_j \frac{\partial}{\partial \bar{z}_j} \in T_p^{0,1}M \iff \langle \bar{\partial}\rho_k, X \rangle = 0, \quad j = 1, \ldots d.$$

CR submanifolds, cont.

- $\dim_{\mathbb{C}} T_p^{0,1}M = N - d_1$, where $d_1 = \operatorname{rank}_{\mathbb{C}}(\bar\partial\rho_1, \ldots, \bar\partial\rho_d)|_p$.
- Linear algebra $\implies N - d \leq \dim_{\mathbb{C}} T_p^{1,0}M \leq N - d/2$.
- M is CR if CR dim $M := \dim_{\mathbb{C}} T_p^{0,1}M$ is constant over M;
- M is generic if $\dim_{\mathbb{C}} T_p^{0,1}M = N - d$; i.e. when $\bar\partial\rho_1, \ldots, \bar\partial\rho_d$ linearly independent.

Example

- If M is a (real) hypersurface, i.e. $d = 1$, then it is generic.
- If $d = 2k$ and CR dim $M = N - k$, then M is complex (holomorphic).
- If CR dim $M = 0$, then M is totally real; if also generic, then maximally totally real. $\mathbb{R}^N \subset \mathbb{C}^N$ maximally totally real.

Basic facts about generic (mostly true also for CR) submanifolds

Let $M \subset \mathbb{C}^N$ be generic submanifold of codimension d. Then $N = n + d$ with $n = \text{CR dim}\, M$ $(= \dim_{\mathbb{C}} T_p^{0,1}M)$. Write $T_p^{1,0}M = \overline{T_p^{0,1}M}$.

- $T_p^{0,1}M \cap T_p^{1,0}M = \{0\} \implies \dim_{\mathbb{C}} T_p^{0,1}M \oplus T_p^{1,0}M = 2n$ $= \dim_{\mathbb{R}} M - d$.
- Locally, there are CR vector fields $\bar{L}_1, \ldots, \bar{L}_n$ forming a frame (basis) for the CR bundle $T^{0,1}M$; the CR bundle is formally integrable: $[\bar{L}_i, \bar{L}_j]$ is again a CR vector field.
- On the other hand, $[L_i, \bar{L}_j]$ is in general not in $T^{0,1}M \oplus T^{1,0}M$; constructions of commutators of this type is a source of invariants of M (Levi form, finite type, etc.).
- Levi form: $(L_i, L_j) \mapsto \pi([L_i, \bar{L}_j]) \in \mathbb{C}T_pM/(T_p^{0,1}M \oplus T_p^{1,0}M) \cong \mathbb{C}^d$.
- Finite type: $\text{span}_{\mathbb{C}}(\ldots, [L_i, \bar{L}_j], [[L_i, \bar{L}_j], L_k], \ldots) = \mathbb{C}T_pM$. See Kohn [6] Bloom-Graham [5].

CR functions and mappings

- A complex-valued function f on $M \subset \mathbb{C}^N$ is said to be CR if

$$\bar{L}_j f = 0, \quad j = 1, \ldots, n = \mathrm{CR} \dim M.$$

- A mapping $f\colon M \to M'$ is CR if $f_*(T^{0,1}M) \subset T^{0,1}M'$.
- Alternatively: If $M' \subset \mathbb{C}^{N'}$, $f\colon M \to \mathbb{C}^{N'}$, and $f(M) \subset M'$. Then f is a CR mapping $M \to M'$ if and only if each component if a CR function.
- Typical example: Restriction to M of a holomorphic function in a neighborhood of M; or boundary value of one with M in the boundary. **Basic question:** When is the converse true?

A wedge with edge on M.

Let Γ be an open convex cone in $\mathbb{R}^d$ and $U \subset \mathbb{C}^N$ an open neighborhood of $p \in M$.

Definition. A wedge $\mathcal{W} = \mathcal{W}_{\Gamma,U} \subset \mathbb{C}^N$ (in the direction Γ) with edge M centered at $p \in M$ is

$$\mathcal{W} = \{z \in U \colon \rho(z,\bar{z}) := (\rho_1(z,\bar{z}),\ldots,\rho_d(z,\bar{z})) \in \Gamma\}.$$

Remarks.

- $\rho \colon U \to \mathbb{R}^d$ is submersion; $\rho(M) = \{0\} \in \bar{\Gamma} \implies \mathcal{W} \subset \mathbb{C}^N$ is open subset with $M \subset \overline{\mathcal{W}}$.
- Wedges defined using different sets of defining functions are comparable. Given defining functions ρ, ρ' and Γ, U, there are $\Gamma' \subset \mathbb{R}^d$, $U' \subset U$ such that

$$\mathcal{W}'_{\Gamma',U'} \subset \mathcal{W}_{\Gamma,U}.$$

Wedge $\mathcal{W}$ with edge $M \subseteq \mathbb{C}^N$

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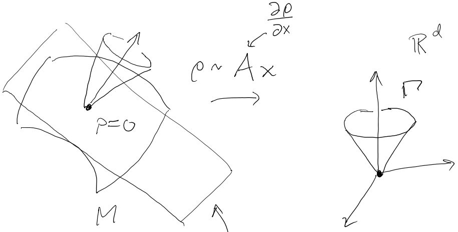

$x \in \mathbb{R}^{2N} \cong \mathbb{C}^N$; $T_pM \cong \{x : Ax = 0\}$

M hypersurface ; $d=1 \Rightarrow$

$\Gamma = \mathbb{R}_+$ or $\mathbb{R}_- \Rightarrow \mathcal{W} = \{\rho < 0\}$ or $\{\rho > 0\}$

"One side of M".

Extension of CR functions into wedges. Minimality.

Definition. A CR manifold M of CR dimension n is said to be non-minimal at $p \in M$ if there exists a CR manifold $S \subset M$ with $p \in S$, CR dim $S = n$, and $\dim S < \dim M$.

Theorem

Let $M \subset \mathbb{C}^N$ be a generic submanifold. If M minimal at $p \in M$, then every continuous CR function near p extends holomorphically to a wedge with edge M centered at p. If M is not minimal at p, then there is a continuous CR function that does not extend to any such wedge.

- When M is a real hypersurface, a "wedge with edge M" is simply (at least) "one side of M". In this case, the result is due to Trepreau [7].
- The sufficiency of minimality was proved by Tumanov [8]; the necessity was proved by Baouendi and Rothschild [2].
- Minimality is in general weaker than finite type.

Idea of proof (rough!) of sufficiency. See [4].

Two main ingredients:

- The Baouendi-Treves Approximation Theorem [3].
- Construction of analytic disks attached to M.

Baouendi-Treves Approximation Theorem (special case).

Theorem

Let $M \subset \mathbb{C}^N$ be a generic submanifold of codimension d and $p \in M$. There is $(p \in)U \subset M$ such that any continuous CR function f on M can be approximated uniformly in U by holomorphic polynomials.

Idea of Proof. The coordinate functions $Z_1, \ldots, Z_N$ restrict to CR functions on M. $N = n + d$. There are local coordinates $(x, y) \in \mathbb{R}^N \times \mathbb{R}^n$ vanishing at p such that $Z_j(x, y) = x_j + i\phi_j(x, y)$, $\phi_j(0) = 0$, $\phi_{j,x}(0) = 0$. Consider the N-forms

$$\alpha_\nu(z) := \left(\frac{\nu}{\pi}\right)^{N/2} \exp[-\nu(z - Z(x', y'))^2]\chi(x')f(x', y')dZ(x', y').$$

For fixed y, Stokes (Picture!) $\implies$

$$\int_{D_y} d\alpha_\nu(z) = \int_{\partial D_y} \alpha_\nu(z) = \int_{S_1} \alpha_\nu(z) - \int_{S_0} \alpha_\nu(z).$$

Integral over $D_y \subseteq \mathbb{R}^N_{x'} \times \mathbb{R}^n_{y'}$; $y \in \mathbb{R}^n$

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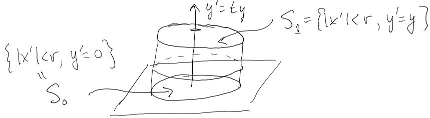
- $D_y := \{(x', y') : |x'| < r,\ y' = ty,\ t \in (0,1)\}$ (N-cycle)
- $\chi \in \mathcal{C}^\infty_0(\mathbb{R}^N)$, $\operatorname{supp} \chi \subset\subset \{|x'| < r\}$, $\chi \equiv 1$ if $|x'| \le r/2$



- $\int_{S_1} \alpha_\nu(z) = F_\nu(z; y)$; $\int_{S_0} \alpha_\nu(z) = H_\nu(z)$
- f CR $\Leftrightarrow$ $d(f\,dZ) = 0$

$\Rightarrow$ $d\alpha_\nu(z) = 0$ in $|x'| < r/2$ and

$|d\alpha_\nu(Z(x,y))| \le C e^{-\nu r^2/100}$ if $|x| < r/4$, $|x'| \ge r/2$, $|y| < r/50$.

Baouendi-Treves Approximation; conclusion of "proof".

Prove:

- $\int_{D_y} d\alpha_\nu(Z(x,y)) \to 0$ uniformly in $(x,y) \in U$.
- $\int_{S_1} \alpha_\nu(Z(x,y)) \to f(x,y)$ uniformly in $(x,y) \in U$.

Conclude:

$$H_\nu(z) := \int_{S_0} \alpha_\nu(z) =$$

$$\int_{\mathbb{R}^n} \left(\frac{\nu}{\pi}\right)^{N/2} \exp[-\nu(z - Z(x',0))^2]\chi(x')f(x',0)dZ(x',0)$$

satisfies:

- $H_\nu(Z(x,y)) \to f(x,y)$ uniformly in $(x,y) \in U$.
- $H_\nu(Z)$ entire. $\implies$ Can be approximated uniformly of compacts by holomorphic polynomials. $\implies$ QED.

Analytic disks attached to a generic submanifold $M \subset \mathbb{C}^N$.

- An analytic disk attached to $M \subset \mathbb{C}^N$ is a holomorphic mapping $A\colon \Delta \to \mathbb{C}^N$ such that $A(\partial\Delta) \subset M$; regularity of A is at $\partial\Delta$.

Theorem

If $M \subset \mathbb{C}^N$ is minimal at $p \in M$, then for any $(p \in)U \subset M$, there is a collection of small analytic disks of class $C^{1,\alpha}$ attached to U, with $p \in A(\partial\Delta)$, filling a wedge $\mathcal{W}$ with edge M centered at p.

Remarks:

- Originally due to Tumanov [8]. Careful analysis of the Bishop equation.
- A new approach based on Banach manifolds and the Implicit Function Theorem was introduced by Baouendi, Rothschild, and Trepreau [1]. See [4].

Sketch of proof of sufficiency of minimality in extension theorem.

By Baouendi-Treves Approximation, there is $U \subset M$ such that every continuous CR function on M can be uniformly approximated in U by polynomials. By Analytic Disk Theorem, you can find a collection of analytic disks, attached to U, filling a wedge $\mathcal{W}$ with edge U (after possibly shrinking U). Pick continuous CR function f on M and sequence of BT polynomials p_ν approximating f in U. For $z \in \mathcal{W}$, there is an analytic disk A, with $A(0) = z$, attached to U. By MMT, $p_\nu(z)$ is a Cauchy sequence. Indeed:

$$|p_\nu(z) - p_\mu(z)| \leq \sup_{q \in A(\partial\Delta)} |p_\nu(q) - p_\mu(q)| \leq \sup_{q \in U} |p_\nu(q) - p_\mu(q)|.$$

Since p_ν converges uniformly to f in U, we conclude that p_ν is a uniform Cauchy sequence in $\mathcal{W}$. $\implies$ there is continuous function F in $\mathcal{W} \cup M$ with $F|_U = f$ and F is holomorphic in the interior of $\mathcal{W}$. QED.

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