CR Geometry, Mappings into Spheres, and Sums-Of-Squares
Lecture III

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Outline - Lecture III

1. CR Submanifolds of Complex Manifolds
2. Wedges with Edges on CR submanifolds
3. Minimal CR Submanifolds and Extension of CR Functions
4. Baouendi-Treves Approximation Theorem
5. Analytic Disks Attached to a CR Submanifold
6. References
7. End
Recall $\mathbb{C}^N \cong \mathbb{R}^{2N}$ via $z_j = x_j + iy_j$, or

$$x_j = \frac{1}{2}(z + \bar{z}), \quad y_j = \frac{1}{2}(z + \bar{z}).$$

Let $M \subset \mathbb{C}^N$ be submanifold of real codimension $d$; thus, locally near $p \in M$, defined by real equations:

$$\rho_1(z, \bar{z}) = \ldots = \rho_d(z, \bar{z}) = 0, \quad d\rho_1 \wedge \ldots d\rho_d \neq 0.$$
\[ \dim_{\mathbb{C}} T^{0,1}_p M = N - d_1, \text{ where } d_1 = \text{rank}_{\mathbb{C}}(\bar{\partial}\rho_1, \ldots, \bar{\partial}\rho_d)|_p. \]

Linear algebra \( \implies N - d \leq \dim_{\mathbb{C}} T^{1,0}_p M \leq N - d/2. \)

- \( M \) is CR if CR dim \( M := \dim_{\mathbb{C}} T^{0,1}_p M \) is constant over \( M \);
- \( M \) is generic if \( \dim_{\mathbb{C}} T^{0,1}_p M = N - d \); i.e. when \( \bar{\partial}\rho_1, \ldots, \bar{\partial}\rho_d \) linearly independent.

**Example**

- If \( M \) is a (real) hypersurface, i.e. \( d = 1 \), then it is generic.
- If \( d = 2k \) and CR dim \( M = N - k \), then \( M \) is complex (holomorphic).
- If CR dim \( M = 0 \), then \( M \) is totally real; if also generic, then maximally totally real. \( \mathbb{R}^N \subset \mathbb{C}^N \) maximally totally real.
Basic facts about generic (mostly true also for CR) submanifolds

Let $M \subset \mathbb{C}^N$ be generic submanifold of codimension $d$. Then $N = n + d$ with $n = \text{CR dim} \ M (\equiv \dim_{\mathbb{C}} T^{0,1}_p M)$. Write $T^{1,0}_p M = \overline{T^{0,1}_p M}$.

- $T^{0,1}_p M \cap T^{1,0}_p M = \{0\} \implies \dim_{\mathbb{C}} T^{0,1}_p M \oplus T^{1,0}_p M = 2n = \dim_{\mathbb{R}} M - d$.

- Locally, there are CR vector fields $\overline{L}_1, \ldots, \overline{L}_n$ forming a frame (basis) for the CR bundle $T^{0,1} M$; the CR bundle is formally integrable: $[\overline{L}_i, \overline{L}_j]$ is again a CR vector field.

- On the other hand, $[L_i, \overline{L}_j]$ is in general not in $T^{0,1}_p M \oplus T^{1,0}_p M$; constructions of commutators of this type is a source of invariants of $M$ (Levi form, finite type, etc.).

- Levi form: $(L_i, L_j) \mapsto \pi([L_i, \overline{L}_j]) \in \mathbb{C} T_p M/(T^{0,1}_p M \oplus T^{1,0}_p M) \cong \mathbb{C}^d$.

- Finite type: $\text{span}_{\mathbb{C}}(\ldots, [L_i, \overline{L}_j], [[L_i, \overline{L}_j], L_k], \ldots) = \mathbb{C} T_p M$. See Kohn [6] Bloom-Graham [5].
A complex-valued function \( f \) on \( M \subset \mathbb{C}^N \) is said to be CR if
\[
\bar{L}_j f = 0, \quad j = 1, \ldots, n = \text{CR dim } M.
\]

A mapping \( f: M \to M' \) is CR if \( f_\ast(T^{0,1} M) \subset T^{0,1} M' \).

Alternatively: If \( M' \subset \mathbb{C}^{N'} \), \( f: M \to \mathbb{C}^{N'} \), and \( f(M) \subset M' \). Then \( f \) is a CR mapping \( M \to M' \) if and only if each component if a CR function.

Typical example: Restriction to \( M \) of a holomorphic function in a neighborhood of \( M \); or boundary value of one with \( M \) in the boundary. **Basic question:** When is the converse true?
A wedge with edge on $M$.

Let $\Gamma$ be an open convex cone in $\mathbb{R}^d$ and $U \subset \mathbb{C}^N$ an open neighborhood of $p \in M$.

**Definition.** A wedge $\mathcal{W} = \mathcal{W}_{\Gamma, U} \subset \mathbb{C}^N$ (in the direction $\Gamma$) with edge $M$ centered at $p \in M$ is

$$\mathcal{W} = \{z \in U : \rho(z, \bar{z}) := (\rho_1(z, \bar{z}), \ldots, \rho_d(z, \bar{z})) \in \Gamma\}.$$

**Remarks.**

- $\rho : U \to \mathbb{R}^d$ is submersion; $\rho(M) = \{0\} \in \overline{\Gamma} \implies \mathcal{W} \subset \mathbb{C}^N$ is open subset with $M \subset \overline{\mathcal{W}}$.

- Wedges defined using different sets of defining functions are comparable. Given defining functions $\rho$, $\rho'$ and $\Gamma$, $U$, there are $\Gamma' \subset \mathbb{R}^d$, $U' \subset U$ such that

$$\mathcal{W}_{\Gamma', U'} \subset \mathcal{W}_{\Gamma, U}.$$
Wedge $W$ with edge $M \subseteq \mathbb{C}^N$

$x \in \mathbb{R}^{2N} \subseteq \mathbb{C}^N$; $T_p M := \{x : A x = 0\}$

$M$ hypersurface; $d = 1 \Rightarrow T = \mathbb{R}_+ \text{ or } \mathbb{R}_- \Rightarrow W = \{\rho < 0\} \text{ or } \{\rho > 0\}$

"One side of $M".$
Definition. A CR manifold \( M \) of CR dimension \( n \) is said to be non-minimal at \( p \in M \) if there exists a CR manifold \( S \subset M \) with \( p \in S \), \( \text{CR dim } S = n \), and \( \dim S < \dim M \).

Theorem

Let \( M \subset \mathbb{C}^N \) be a generic submanifold. If \( M \) minimal at \( p \in M \), then every continuous CR function near \( p \) extends holomorphically to a wedge with edge \( M \) centered at \( p \). If \( M \) is not minimal at \( p \), then there is a continuous CR function that does not extend to any such wedge.

- When \( M \) is a real hypersurface, a ”wedge with edge \( M \)” is simply (at least) ”one side of \( M \”). In this case, the result is due to Trepreau [7].
- The sufficiency of minimality was proved by Tumanov [8]; the necessity was proved by Baouendi and Rothschild [2].
- Minimality is in general weaker than finite type.
Idea of proof (rough!) of sufficiency. See [4].

Two main ingredients:

- The Baouendi-Treves Approximation Theorem [3].
- Construction of analytic disks attached to $M$. 
Baouendi-Treves Approximation Theorem (special case).

**Theorem**

Let $M \subset \mathbb{C}^N$ be a generic submanifold of codimension $d$ and $p \in M$. There is $(p \in) U \subset M$ such that any continuous CR function $f$ on $M$ can be approximated uniformly in $U$ by holomorphic polynomials.

**Idea of Proof.** The coordinate functions $Z_1, \ldots, Z_N$ restrict to CR functions on $M$. $N = n + d$. There are local coordinates $(x, y) \in \mathbb{R}^N \times \mathbb{R}^n$ vanishing at $p$ such that $Z_j(x, y) = x_j + i\phi_j(x, y)$, $\phi_j(0) = 0$, $\phi_{j,x}(0) = 0$. Consider the $N$-forms

$$\alpha_\nu(z) := \left(\frac{\nu}{\pi}\right)^{N/2} \exp[-\nu(z - Z(x', y'))^2] \chi(x') f(x', y') dZ(x', y').$$

For fixed $y$, Stokes (Picture!) \implies

$$\int_{D_y} d\alpha_\nu(z) = \int_{\partial D_y} \alpha_\nu(z) = \int_{S_1} \alpha_\nu(z) - \int_{S_0} \alpha_\nu(z).$$
Integral over $D_y \subseteq \mathbb{R}_x^N \times \mathbb{R}_y^N$; $y \in \mathbb{R}^n$

- $D_y := \{(x',y'): |x'| < r, y' = ty, t \in (0,1]\}$ (N-cycle)
- $\chi \in C_0^\infty (\mathbb{R}^n), \supp \chi \subseteq \{|x'| < \frac{r}{2}\}$ $\chi \equiv 1$ if $|x'| \leq \frac{r}{2}$

$S_0 = \{|x'| < r, y' = y\}$

$S_1 = \{|x'| < r, y' = y\}$

$S_0 = \{\chi = 0\}$

$\chi \equiv 0$ in $|x'| < \frac{r}{2}$ and

$|dw_\nu (Z(x,y))| \leq C e^{-u r^2/100}$ if $|x| < \frac{r}{4}$, $|x'| \geq \frac{r}{2}$, $|y| < \frac{r}{50}$.
Prove:

- $\int_{D_y} d\alpha_\nu(Z(x, y)) \to 0$ uniformly in $(x, y) \in U$.
- $\int_{S_1} \alpha_\nu(Z(x, y)) \to f(x, y)$ uniformly in $(x, y) \in U$.

Conclude:

$$H_\nu(z) := \int_{S_0} \alpha_\nu(z) =$$

$$\int_{\mathbb{R}^n} \left(\frac{\nu}{\pi}\right)^{N/2} \exp[-\nu(z - Z(x', 0))^2] \chi(x') f(x', 0) dZ(x', 0)$$

satisfies:

- $H_\nu(Z(x, y)) \to f(x, y)$ uniformly in $(x, y) \in U$.
- $H_\nu(Z)$ entire. $\implies$ Can be approximated uniformly of compacts by holomorphic polynomials. $\implies$ QED.
Analytic disks attached to a generic submanifold $M \subset \mathbb{C}^N$.

- An **analytic disk** attached to $M \subset \mathbb{C}^N$ is a holomorphic mapping $A: \Delta \to \mathbb{C}^N$ such that $A(\partial \Delta) \subset M$; regularity of $A$ is at $\partial \Delta$.

**Theorem**

If $M \subset \mathbb{C}^N$ is minimal at $p \in M$, then for any $(p \in) U \subset M$, there is a collection of small analytic disks of class $C^{1,\alpha}$ attached to $U$, with $p \in A(\partial \Delta)$, filling a wedge $\mathcal{W}$ with edge $M$ centered at $p$.

**Remarks:**

- Originally due to Tumanov [8]. Careful analysis of the Bishop equation.
- A new approach based on Banach manifolds and the Implicit Function Theorem was introduced by Baouendi, Rothschild, and Trepreau [1]. See [4].
Sketch of proof of sufficiency of minimality in extension theorem.

By Baouendi-Treves Approximation, there is $U \subset M$ such that every continuous CR function on $M$ can be uniformly approximated in $U$ by polynomials. By Analytic Disk Theorem, you can find a collection of analytic disks, attached to $U$, filling a wedge $\mathcal{W}$ with edge $U$ (after possibly shrinking $U$). Pick continuous CR function $f$ on $M$ and sequence of BT polynomials $p_\nu$ approximating $f$ in $U$. For $z \in \mathcal{W}$, there is an analytic disk $A$, with $A(0) = z$, attached to $U$. By MMT, $p_\nu(z)$ is a Cauchy sequence. Indeed:

$$|p_\nu(z) - p_\mu(z)| \leq \sup_{q \in A(\partial \Delta)} |p_\nu(q) - p_\mu(q)| \leq \sup_{q \in U} |p_\nu(q) - p_\mu(q)|.$$

Since $p_\nu$ converges uniformly to $f$ in $U$, we conclude that $p_\nu$ is a uniform Cauchy sequence in $\mathcal{W}$. $\implies$ there is continuous function $F$ in $\mathcal{W} \cup M$ with $F|_U = f$ and $F$ is holomorphic in the interior of $\mathcal{W}$. QED.


T. Bloom and I. Graham. On “type” conditions for generic real submanifolds of $c^n$. 
J. J. Kohn.
Boundary behavior of $\delta$ on weakly pseudo-convex manifolds of dimension two.


Collection of articles dedicated to S. S. Chern and D. C. Spencer on their sixtieth birthdays.

J.-M. Trépreau.
Sur le prolongement holomorphe des fonctions C-R définies sur une hypersurface réelle de classe $C^2$ dans $\mathbb{C}^n$.


A. E. Tumanov.
Extension of CR-functions into a wedge from a manifold of finite.
(Russian).