Log-VOAs and 3-manifold invariants

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Based on:

3-Manifolds and VOA Characters

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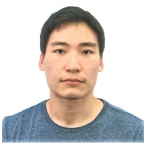

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Based also on (upcoming):

Quantum modularity of higher rank homological blocks

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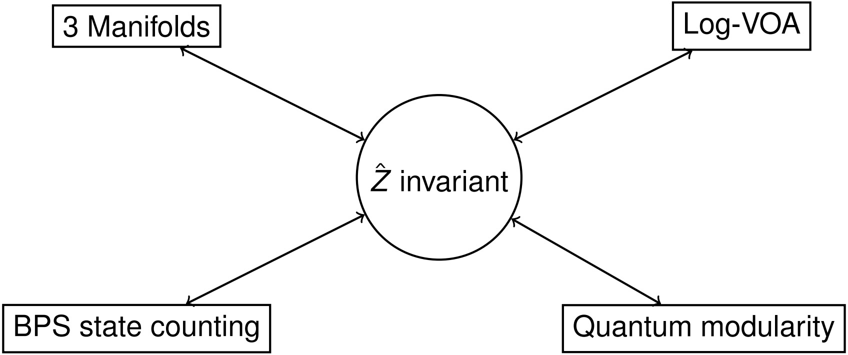
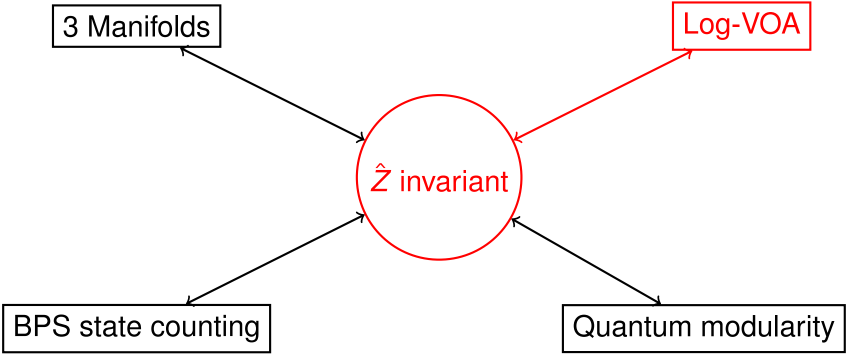

3 Manifolds
Log-VOA
$\hat{Z}$ invariant
BPS state counting
Quantum modularity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VOAs

History and applications

- Originally introduced (as vertex algebras) by Richard Borcherds to study the Leech lattice in the mid ’80s.
- Expanded to vertex operator algebras later by Frenkel, Lepowsky and Meurman with their work on the moonshine module
- Closely related to in two dimensional conformal field theory
- Mathematical applications to monstrous moonshine, representation theory of the Virasoro algebra, affine Kac-Moody algebras…

VA definition

- Data
  1. A vector space V, called the space of states.
  2. An identity element in the space of states $|0\rangle \in V$
  3. An endomorphism $T : V \to V$ called the “Translation.”
  4. A linear map, the vertex operator map:
     $Y(\bullet, x) : V \to (\operatorname{End} V)\left[\left[x, x^{-1}\right]\right]$
- Axioms
  1. **Vacuum property** $Y(|0\rangle, x) = 1$
  2. **Creation property** $Y(|a\rangle, x)|0\rangle = |a\rangle$
  3. **Translation covariance** $[T, Y(|a\rangle, z)] = \partial Y(|a\rangle, z)$ and $T|0\rangle = 0$
  4. **Locality**
     $\forall v_1,\ v_2 \in V,\ \exists N \in \mathbb{N} | \forall n > N\ (z - w)^n [Y(v_1, z), Y(v_2, w)] = 0$

VOA definition

- Data
  1. A vertex algebra $(V, |0\rangle, T, Y)$ with a $\mathbb{Z}$ graded vector space:
  $$V = \bigoplus_{n \in \mathbb{Z}} V_n$$
  2. A distinguished vector ω, conformal vector
- Axioms
  1. **Virasoro modes** $Y(\omega, x) = \sum_{n \in \mathbb{Z}} L_n x^{-n-2}$,
  $$[L_n, L_m] = (m-n)L_{n+m} + \frac{1}{12} m \left(m^2 - 1\right) \delta_{m+n,0} c_V$$
  2. **Conformal weight** For $v \in V_{h_v}$, $L_0 v = h_v v$
  3. **Translation generator** $T = L_{-1}$

Example: Heisenberg VOA

For $[\alpha_m, \alpha_n] = \delta_{m+n,0} c$, $[c, c] = [c, \alpha_n] = [\alpha_n, c] = 0$

- $V := \mathscr{U}(\mathfrak{h}) \otimes_{\mathscr{U}(\mathfrak{h}_{\geq 0})} (\mathbb{C}|0\rangle)$
- $|0\rangle$
- $T := \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_n \alpha_{-n-1}$
- $Y(\alpha_{j_1} \alpha_{j_2} \cdots \alpha_{j_n} |0\rangle, z) = \alpha(z)_{j_1} \left(\alpha(z)_{j_2} \left(\ldots \left(\alpha(z)_{j_n} I_V \right) \right) \right)$
- $\omega = \alpha_{-1}^2 |0\rangle$

VOA modules

Let V be a vertex algebra. A V-module is a vector space W and a map $Y_W : V \to \mathrm{End}(W)\left[\left[z, z^{-1}\right]\right]$

1. $Y_W(|0\rangle, z) = I_W$
2. The Jacobi identity holds:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_W(a, z_1)\, Y_W(b, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_W(b, z_2)\, Y_W(a, z_1) = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_W(Y(a, z_0)\, b, z_2)$$

Log-VOAs and Log-CFTs

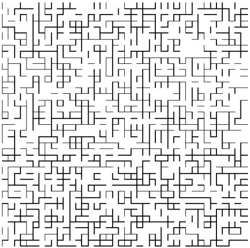
Log VOAs and Log CFTs

- CFTs with modules that are reducible but indecomposable
- First hints found by Rozansky and Saleur in 1992 while studying the WZW U(1|1) model.
- Link between logarithmic singularities $\leftrightarrow$ non-diagonalisability of the energy operator - Gurarie 1992
- Major advances in the computation of the fusion algebra - Nahm, Gaberdiel, Kausch 1994

Cardy's computation

Since then many applications have been found:

- Polymers
- Percolation (Limit of Q-Potts, $Q \to 1$)
- Self-avoiding walks (Limit of O(n) model, $n \to 1$)
- Dual to three dimensional chiral gravity models
- Transitions and incarnations of the quantum Hall effect



Lattice VOAs

- $\mathfrak{g}$ simply laced Lie algebra
- Λ root lattice, $\Lambda^\vee$ weight lattice
- $m \in \mathbb{N}$
- $\hat{\mathfrak{h}}$ rk(Λ) dimensional Heisenberg algebra
- $A \subseteq \frac{1}{\sqrt{m}}\Lambda^\vee$
- $V_A = \bigoplus_{\nu\in A} \mathscr{U}\left(\hat{\mathfrak{h}}\right) \otimes_{\mathscr{U}\left(\mathfrak{h}^{\geq 0}\right)} \mathbb{C}\,|\nu\rangle$
- Vertex operators:
  - $a \in \mathfrak{h}$, $a(z) = Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ as before
  - $\mu \in \frac{1}{\sqrt{m}}\Lambda^\vee$ $Y(|\mu\rangle, z) := e^\mu z^{\mu_0} E^-(-\mu, z) E^+(-\mu, z)$ where:
    - $e^\mu (a|\nu\rangle) = (\mu,\nu)\, a|\mu+\nu\rangle$
    - $z^{\mu_0} (a|\nu\rangle) = z^{(\mu,\nu)} a|\nu\rangle$
- $\omega = \sum_{i=1}^d \alpha^i_{-1}\alpha^i_{-1}\,|0\rangle$

Modules of Lattice VOAs

Let $\lambda \in \frac{1}{\sqrt{m}}\Lambda^\vee$

- $\bar{\lambda}$ is the representative of λ in $\frac{1}{\sqrt{m}}\Lambda^\vee / \sqrt{m}\Lambda^\vee$ - fractional part
- $\hat{\lambda}$ is the representative of $\frac{1}{\sqrt{m}}\left(\lambda - \bar{\lambda}\right)$ in $\Lambda^\vee/\Lambda$ - integral part

The irreducible modules $V_{\sqrt{m}Q+\lambda}$ are then labelled by:

$$\lambda \in \Lambda^* = \left\{\lambda = -\sqrt{m}\hat{\lambda} + \bar{\lambda} | \hat{\lambda} \in \hat{\Lambda},\ \bar{\lambda} \in \hat{\Lambda}_m\right\}$$

where:

$$\bar{\Lambda}_m = \left\{\sum_{i=1}^{l} \frac{s_i}{\sqrt{m}}\omega_i | 0 \leq s_i \leq m-1\right\}$$

$$\hat{\Lambda} = \{\lambda \in P_+ | (\lambda, \theta) = 1,\ \theta = \text{highest root}\}$$

$\mathscr{W}(m)_Q$ algebra

For $1 \leq i \leq l$ define:

$$F_{i,0} = \left| -\frac{1}{\sqrt{m}}\alpha_i \right\rangle_{(0)} \in \mathrm{Hom}_{\mathbb{C}}\left(V_{\sqrt{m}\Lambda^\vee}, V_{\sqrt{m}\Lambda^\vee - \frac{1}{\sqrt{m}}\alpha} \right)$$

With these we have the triplet model $\mathscr{W}(p)_Q$:

$$W(m)_Q = \bigcap_{i=1}^{l} \ker F_{i,0}|_{V_{\sqrt{m}Q}}$$

and the singlet model:

$$W(m)_Q = \bigcap_{i=1}^{l} \ker F_{i,0}|_{\mathscr{F}_0}$$

Modules and Characters

As for the modules are parametrized by elements in $\hat{\Lambda}_p$. Let

$$\Delta\left(\vec{\xi}\right) := \sum_{w \in W} (-1)^{\ell(w)}\, e^{\left\langle \vec{\xi}, w(\vec{\rho}) \right\rangle}$$

where $z = e^{\xi}$. The characters of the triplet algebra are given by:

$$\chi_{\vec{\lambda}'}(\tau, \vec{\xi}) = \frac{1}{\eta^{\mathrm{rank}\Lambda}(\tau)} \frac{1}{\Delta(\vec{\xi})} \sum_{\vec{\tilde{\lambda}} - \vec{\lambda} \in \Lambda} q^{\frac{1}{2}|\sqrt{m}\vec{\tilde{\lambda}} + \vec{\mu} + Q_0 \vec{\rho}|^2} \left(\sum_{w \in \mathscr{W}} (-1)^{l(w)} e^{\langle \vec{\xi}, w(\vec{\rho} + \vec{\tilde{\lambda}}) \rangle} \right)$$

For the singlet, take the z-constant terms.

$\hat{Z}$ invariants

M theory point of view

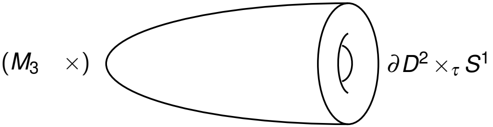
Spacetime	S^1	$\times$	$T^* M_3$	$\times$	TN
			$\cup$		$\cup$
N M5-branes	S^1	$\times$	M_3	$\times$	D^2

6d $\mathcal{N} = (2,0)$ theory on $M_3 \times D^2 \times_\tau S^1$

↙ ↘

$D^2 \times_\tau S^1$ — M_3

3d $\mathcal{N} = 2$, $T[M_3, \mathfrak{g}]$ — TQFT, 3d G Chern-Simons



$$\mathscr{H}(T[M_3,\mathfrak{g}]; b, D^2 \times_\tau S^1) = \bigoplus_{i,j} \mathscr{H}^{i,j}(M_3, b)$$

- i, j $U(1)$ charges of boundary of D^2 and compactified time
- τ complex structure on $T^2 = \partial D^2 \times S^1$
- $q := e^{2\pi i \tau}$
- Boundary conditions $\underline{\vec{b}}$ are labelled by $\mathrm{Tor}(H_1(M_3,\mathbb{Z}))/\mathbb{Z}_n$

$$\widehat{Z}^G_{\underline{\vec{b}}}(X,\tau) = \sum_{\substack{i \in \mathbb{Z}+\Delta_a \\ j \in \mathbb{Z}}} q^i (-1)^j \dim \mathscr{H}^{i,j}(M_3, b) \in \mathbb{Z}[q]$$

Relation to WRT invariant

$$\mathrm{WRT}: \mathbb{Z} \to \mathbb{C},\ k \mapsto \mathrm{WRT}(k)$$

Can we extend from $\mathbb{Z}$ to $\mathscr{H}$?

$$\text{q-series} \xrightarrow[q \to e^{2\pi i/k}]{\text{radial limit}} \mathrm{WRT}(k),\ \hat{Z}_b \xrightarrow[\text{summed over } b]{\text{radial limit}} \mathrm{WRT}$$

From the CS side, from resurgence:

$$Z^{CS}_{SU(2)}(M_3;k) = \sum_{a \in \mathscr{M}^{\mathrm{ab}}_{\mathrm{flat}}(M_3, SU(2))} e^{2\pi i k S^{CS}(a)} Z_a(k)$$

$$Z_a(k) \propto \sum_{b \in \mathrm{Tor} H_1(M_3)} S_{ab} \widehat{Z}_b(q) \bigg|_{q \to e^{\frac{2\pi i}{k}}} .$$

Expectations of modularity

We have a 2D-3D coupled system: we expect the 2D CFT at the boundary to have some modular properties.

$T[M_3]$	$\hat{Z}_b(M_3, \tau)$
trivial	modular
trivial but gapped	quantum modular
not gapped	???

The $\hat{Z}$ invariant

$$\hat{Z}^{G}_{\underline{\vec{b}}}(X,\tau) = q^{\Delta}(a_0 + a_1 q + a_2 q^2 + \dots)$$

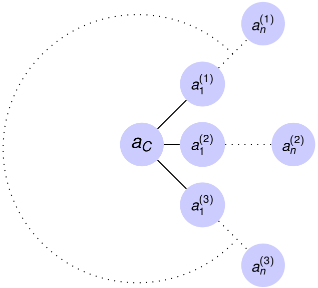
- Topological invariant for 3 manifolds
- Defined in physics from BPS state counting of M-theory compactifications
- Defined in mathematics for: plumbed manifolds and knot complements

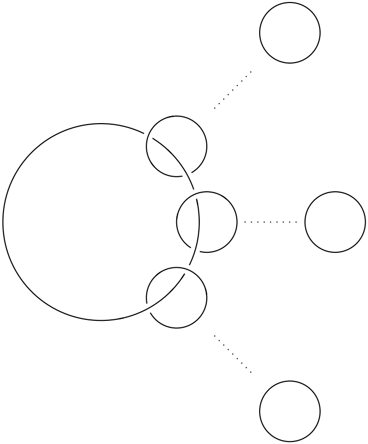
- G - Gauge group
- $\underline{\vec{b}}$ - Boundary condition
- X - 3 manifold
- τ - complex structure

Computation of $\hat{Z}$ invariants

Seifert manifolds and plumbed manifolds

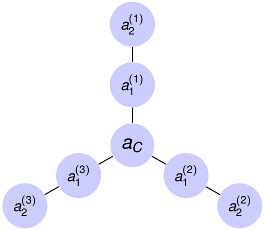
- Seifert manifolds are S^1 fibered 2d orbifolds.
- Can be expressed as plumbing manifolds
- N exceptional fibers $\rightarrow$ N legs of a star graph





What are Seifert manifolds

- We focus on 3 exceptional fibers
- The data is contained in plumbing/adjacency matrix



$$M = \begin{pmatrix} a_c & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & a_1^{(1)} & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & a_2^{(1)} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & a_1^{(2)} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & a_2^{(2)} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & a_1^{(3)} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & a_2^{(3)} \end{pmatrix}$$

$\hat{Z}$ invariants for plumbed manifolds

$$\hat{Z}^{G}_{\underline{\vec{b}}}(\tau) := C^{G}_{\Gamma}(q) \oint_{C} d\underline{\vec{\xi}} \left(\prod_{v \in V} \Delta(\vec{\xi}_v)^{(2-\deg(v))} \right) \times \sum_{w \in W} \sum_{\underline{\vec{\ell}} \in \Gamma_{M,G} + w(\underline{\vec{b}})} q^{-\frac{1}{2}||\ell||^2} \left(\prod_{v' \in V} e^{\left\langle \vec{\ell}_{v'}, \vec{\xi}_{v'} \right\rangle} \right)$$

Observations:

1. The contour integral means: pick out the $z \propto \log(\xi)$ constant terms
2. The sum is only interesting at the nodes where $\deg(v) \geq 3$

Quantum modular forms

Modular group

Definition (Modular group)

The modular group $\mathrm{SL}(2,\mathbb{Z})$is the group of 2×2 matrices on $\mathbb{Z}$ with unit determinant:

$$\mathrm{SL}(2,\mathbb{Z}) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a,b,c,d \in \mathbb{Z} \text{ and } ad-bc=1 \right\} \quad (1)$$

For $\tau \in \mathscr{H}$ the action is given by:

$$\gamma\tau = \frac{a\tau+b}{c\tau+d}. \quad (2)$$

Classical modular forms

Definition (Modular forms)

A holomorphic function $f : \mathscr{H} \to \mathbb{C}$ is a modular form if:

$$(f|_k\gamma)(z) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = f(z) \tag{3}$$

for all $z \in \mathscr{H}$ and $\gamma \in \mathrm{SL}(2,\mathbb{Z})$. k is a fixed integer called the weight of the quantum modular forms. For all γ, $f|_k\gamma$ must be bounded for $\mathrm{Im}(z) \to \infty$. We then write $\mathscr{M}_k(\mathrm{SL}(2,\mathbb{Z}))$ for the modular forms of weight k and

$$\mathscr{M}(\mathrm{SL}(2,\mathbb{Z})) = \bigoplus_{k\in\mathbb{Z}_{>0}} \mathscr{M}_k(\mathrm{SL}(2,\mathbb{Z}))$$

Examples

Definition (Modular forms)

For $q = e^{2\pi i\tau}$, $\tau \in \mathscr{H}$:

$$E_4(q) := 1 + 240\sum_{n=1}^{\infty}\frac{n^3 q^n}{1-q^n} = 1 + 240q + 2160q^2 + \dots \tag{4}$$

$$E_6(q) := 1 - 504\sum_{n=1}^{\infty}\frac{n^5 q^n}{1-q^n} = 1 - 504q - 16632q^2 + \dots \tag{5}$$

Classical modular forms are too easy

Theorem

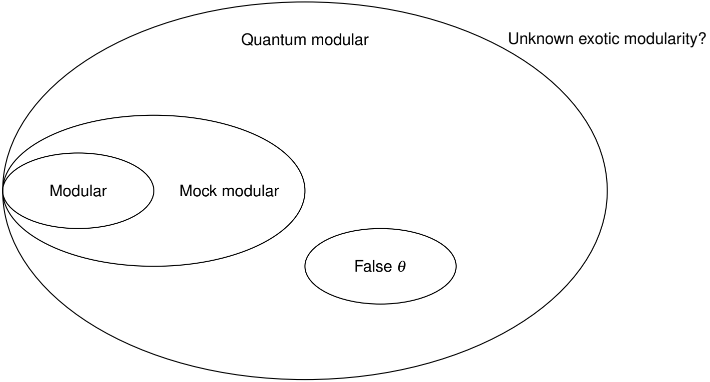
The ring of modular forms is freely generated by E_4 E_6:

$$\mathscr{M}(\mathrm{SL}(2,\mathbb{Z})) = \mathbb{C}[E_4, E_6].$$

Need for generalization:

- Vector valued functions
- Phase in the transformation
- Rational weight
- ...

A modular atlas



Where the generalizations lie

Some properties we relax:

- Domain: defined on $\mathbb{Q} \cup \{\infty\} \setminus S \subset \mathbb{P}^1(\mathbb{Q})$, for finite S.
- Functions defined asymptotically.
- Different transformation behavior.

What can we ask?

We cannot further ask for:

- Analycity
  - Topology of $\mathbb{P}^1(\mathbb{Q})$ is more naturally the discrete topology than that induced.
- Modular transformation properties: Γ-covariance
  - $SL(2,\mathbb{Z}))$ acts transitively on $\mathbb{P}(\mathbb{Q})$, trivial definition

Main idea

We do not require continuity/analycity or modularity but we require that the failure of one precisely offsets the failure of the other.

Definition of quantum modular forms

Definition (Weak/strong QMF)

A weak QMF is a function $f : \mathbb{Q} \to \mathbb{C}$ for which the "error in modularity" $h_\gamma(x)$ defined by:

$$h_\gamma(x) := f(x) - (f|_k\gamma)(x) \tag{6}$$

extends to a real-analytic function of $\mathbb{P}^1(\mathbb{R}) \setminus S_\gamma$, where S_γ is a finite set. A strong QMF is a function $f : \mathbb{Q} \to \mathbb{C}[[\varepsilon]]$ if h_γ extends holomorphically to a neighborhood of $\mathbb{P}^1(\mathbb{R}) \setminus S_\gamma \subset \mathbb{P}^1(\mathbb{C})$:

$$h_\gamma(x) := \lim_{t \to 0^+} (f - f|_k\gamma)(x + it)$$

Why quantum?

Quantum Modular Forms

Don Zagier

To Alain Connes on his 60th birthday, in friendship and admiration

A classical modular form is a holomorphic function f in the complex upper half-plane $\mathfrak{H}$ satisfying the transformation equation

In this note we want to discuss, in the simplest cases, another type of modular object which, because it has the "feel" of the objects occurring in perturbative quantum field theory and because several of the examples come from quantum invariants of knots and 3-manifolds, we call *quantum modular forms*. These are objects which live at the boundary of the space X, are defined only asymptotically, rather than exactly, and have a transformation behavior of a quite different type

Eichler integrals

Algorithmic procedure to construct quantum modular forms from regular modular forms.

$$f(\tau) = \sum_{m\geq 1} c_f(m) q^m, \; k \in \frac{1}{2}\mathbb{Z}, \; q = e^{2\pi i \tau}$$

$$\tilde{f}(\tau) := \sum_{m\geq 1} \frac{c_f(m)}{m^{k-1}} q^m = \int_{\tau}^{i\infty} f(w)(w-\tau)^{k-2} dw$$

$$\tilde{f}^*(\tau, \overline{\tau}) := \int_{-\overline{\tau}}^{i\infty} f(w)(w+\tau)^{k-2} dw$$

Example: False θ-function:

$$\theta^1_{m,r}(\tau) = \sum_{k\equiv r(2m)} k q^{\frac{k^2}{2m}}, \; w = 3/2$$

$$\tilde{\theta}^1_{m,r}(\tau) = \sum_{k\equiv r(2m)} \mathrm{sgn}(k) q^{\frac{k^2}{2m}}$$

Example - Kontsevitch's "strange function"

$$K(\tau) := \sum_{m\geq 0} \prod_{j=1}^{m} \left(1 - q^j\right) = \sum_{m\geq 0} (q;q)_m$$

Sum of tails identity

$$\sum_{m\geq 0} \left(\eta(\tau) - q^{1/24}(q;q)_m\right) = \eta(\tau) D(\tau) + \frac{1}{2}\tilde{\eta}(\tau).$$

$$\tau \to \frac{h}{k} \in \mathbb{Q},\ \eta(\tau), \eta(\tau) D(\tau) \to 0$$

$$K\left(e^{2\pi i\tau}\right)\Big|_{\tau\to\frac{h}{k}} = -\frac{q^{-1/24}}{2}\tilde{\eta}(\tau)\Bigg|_{\tau\to\frac{h}{k}}.$$

Definition (Higher depth quantum modular forms)

A function $f: \mathscr{Q} \to \mathbb{C} (\mathscr{Q} \subset \mathbb{Q})$ is called a *quantum modular form of depth* $N \in \mathbb{N}$, weight $k \in \frac{1}{2}\mathbb{Z}$, multiplier χ, and quantum set $\mathbb{Q}$ for Γ if for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$f(\tau) - \chi(M)^{-1}(c\tau+d)^{-k} f(M\tau) \in \bigoplus_j \mathscr{Q}_{k_j}^{N_j}(\Gamma, \chi_j \mathscr{O}(R)),\ N_j < N.$$

Eichler integrals $\to$ double Eichler integrals

$$\mathrm{DE}^{(m)}[f_1, f_2](\tau) = c_{w_2}(2\sqrt{m})^{2(w_2-1)} \times \int_{-\bar{\tau}}^{i\infty} \int_{\xi_1}^{i\infty} d\xi_1 d\xi_2 \frac{f_1(\xi_1)}{(-i(\xi_1+\tau))^{2-w_1}} \frac{f_2(\xi_2)}{(-i(\xi_2+\tau))^{2-w_2}}$$

Example

Let $f_1,\ f_2$ be quantum modular forms of depth 1: $f_i \in \mathscr{Q}^1_{k_i}(\Gamma_i, \chi_i)$.
Then: $f_1 f_2 \in \mathbb{Q}^2_{k_1+k_2}(\Gamma_1 \cap \Gamma_2, \chi_1\chi_2)$.

Proof.

$$f_i|_\gamma - f_i = \Phi_i.$$

$$(f_1 f_2)|_\gamma = f_1 f_2 + f_1\Phi_2 + f_2\Phi_1 + \Phi_1\Phi_2$$

$$(f_1 f_2)|_\gamma - f_1 f_2 = f_1\Phi_2 + f_2\Phi_1 + \Phi_1\Phi_2$$

□

$\hat{Z}$ invariants and characters of log-VOA

Proposition 4.2: a rewriting of the $\hat{Z}$ invariant

$$\widehat{Z}^G_{\underline{\vec{b}}}(X_\Gamma;\tau) = C^G_\Gamma(q)\int_{\mathscr{C}} d\underline{\vec{\xi}}\left(\prod_{v\in V}\Delta(\vec{\xi}_v)^{2-\deg v}\right)$$

$$\sum_{w\in W}\sum_{\underline{\vec{\ell}}\in\Gamma_{M,G}+w(\underline{\vec{b}})} q^{-\frac{1}{2}||\underline{\vec{\ell}}||^2}\left(\prod_{v'\in V} e^{\langle\vec{\ell}_{v'},\vec{\xi}_{v'}\rangle}\right)$$

$$\widehat{Z}^G_{\underline{\vec{b}}}(\tau) = C^G_\Gamma(q)\sum_{\hat{w}\in W^{\otimes N}}(-1)^{\ell(\hat{w})}\int_{\mathscr{C}} d\vec{\xi}\,\tilde{\chi}_{\hat{w};\underline{\vec{b}}}(\tau,\vec{\xi})$$

Either:

$$\chi_{\hat{w};\underline{\vec{b}}}(\tau,\vec{\xi}) = 0$$

or:

$$\exists\ \vec{\kappa}_{\hat{w};\underline{\vec{b}}}\in\Lambda/D\Lambda\ \text{ such that}$$

$$\vec{\ell} = \left(\vec{\kappa}_{\hat{w};\underline{\vec{b}}} + D\vec{\lambda} + \rho, -w_1(\rho),\dots,-w_N(\rho),0,\dots,0\right)\in\Gamma_{M_G}+w(\underline{\vec{b}})$$

Spherical case:
$|\det(M)| = 1 \implies D = 1$

Pseudo-spherical case:
$|\det(M)| \geq 1,\ D = 1$

Theorem 4.4:

$$\frac{q^{-\delta}}{\eta^{\mathrm{rk}(G)}}\tilde{\chi}_{\hat{w};\underline{\vec{b}}}(\tau,\vec{\xi}) = \chi_{\vec{\mu}_{\hat{w}}}(\tau,\vec{\xi})$$

Corollary 4.6: For $SU(2)$ and $SU(3)$:

$$C^G_\Gamma(q)^{-1}\,\frac{q^{-\delta}}{\eta^{\mathrm{rankG}}}\,\widehat{Z}^G_{\underline{\vec{b}}}(\tau;X_\Gamma) \in \left\{\sum_{\vec{\mu}} a_{\vec{\mu}}\chi^0_{\vec{\mu}} \,\middle|\, a_{\vec{\mu}} \in \mathbb{Z}\right\} + \text{finite polynomial in } q$$

Higher rank, higher depth

Theorem

For spherical Seifert manifolds with 3 singular fibers and $G = SU(3)$:

- $\hat{Z}_b^{SU(3)}(\tau)$ *sum of depth two quantum QMF*

$$M = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 0 & 0 & -9 \end{pmatrix}$$

$$\hat{Z}_b^{SU(2)} \to \left(\theta^1_{18,5}\left(\frac{1}{18}z_2\right) + \theta^1_{18,13}\left(\frac{1}{18}z_2\right)\right) = \theta_5^{18+9}\left(\frac{\tau}{18}\right)$$

$$\hat{Z}_b^{SU(3)}(\tau) \to \left(\theta_9^{18+9}\left(\frac{\tau}{6}\right) - \theta_3^{18+9}\left(\frac{\tau}{6}\right)\right)\theta_1^{18+9}\left(\frac{\tau}{18}\right) + \left(\theta_1^{18+9}\left(\frac{\tau}{6}\right) + \theta_5^{18+9}\left(\frac{\tau}{6}\right) - \theta_7^{18+9}\left(\frac{\tau}{6}\right)\right)\theta_5^{18+9}\left(\frac{\tau}{18}\right)$$

Open quesitons

- Z hat invariants for positive definite Seifert manifolds
- VOA[M_3] as a manifold invariant
- Recursion relations between higher depth QMF
- Physical insights from recursion relation and new VOAs

Thanks for listening.