Chern-Simons Field Theory Invariants

Ramadevi Pichai

Deptartment of Physics, Indian Institute of Technology Bombay

28th Feb 2023, Learning Workshop on BPS states & 3-manifolds at ICTP

Outline

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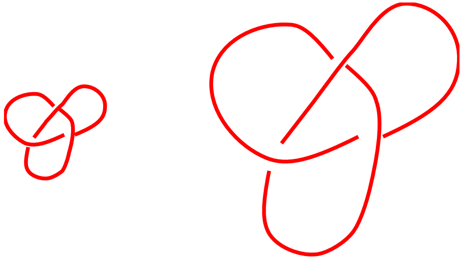
- Summary

Chern-Simons Theory



Topologically equivalent objects $\mathcal{K}$

(no notion of distance or size)

Hence the theory describing such objects must be metric independent

- Chern-Simons action S on a three-manifold M based on gauge group G:

$$S = \frac{k}{4\pi} \int_M \epsilon_{\mu\nu\lambda} \, d^3x \, Tr \left(A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \right)$$

k is the coupling constant, A_μ's are the gauge fields.

- Any Knot $\mathcal{K}$ carrying representation R is described by **expectation value of Wilson loop operators** $W_R(\mathcal{K}) = Tr[Pexp \oint_{\mathcal{K}} A_\mu dx^\mu]$:

$$P_R^G[\mathcal{K}] = \langle W_R^G(\mathcal{K}) \rangle = \frac{\int_M [\mathcal{D}A] \; W_R^G(\mathcal{K}) \; exp(iS)}{\mathcal{Z}[M]}$$

$$\text{where} \quad \mathcal{Z}[M] = \int_M [\mathcal{D}A] \; exp(iS) \quad \text{(partition function)}$$

$P_R^G[\mathcal{K}]$ are the **knot invariants**.

Hence the theory provides natural framework to study knots, links and three-manifolds

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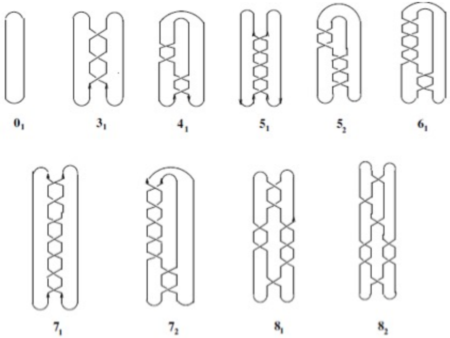
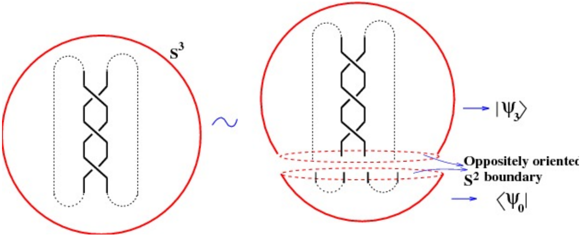
- Relation between Chern-Simons theory to G_k Wess-Zumino conformal field theory (WZNW) (Witten 1989)
- Any knot can be obtained as a **closure/plat/quasiplat of braid** **(Alexander, Birman)**



Example: Trefoil invariant

Basically, the trefoil T in S^3 is viewed as gluing two three-balls with oppositely oriented S^2 boundaries.



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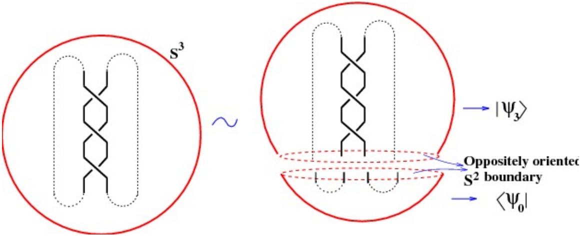

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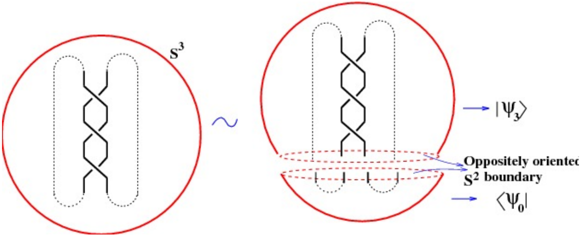

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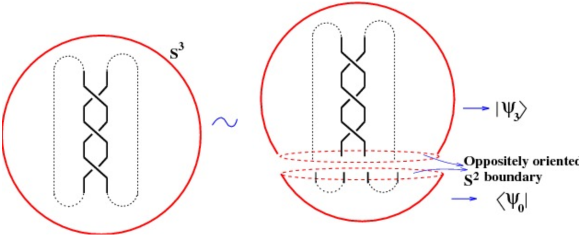

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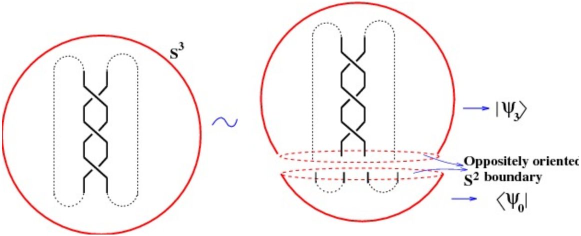

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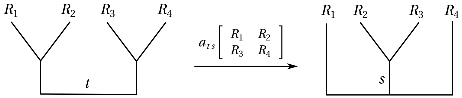
Braiding operator $\mathcal{B}$ eigenbasis will determine the polynomial form in variable q

Eigenbasis of Braiding operator $\mathcal{B}$

For the four-punctured S^2 boundary, the conformal block bases are:



where $t \in R_1 \otimes R_2 \cap \bar{R}_3 \otimes \bar{R}_4$ and $s \in R_2 \otimes R_3 \cap \bar{R}_1 \otimes \bar{R}_4$.

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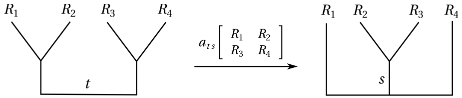

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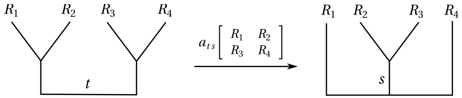

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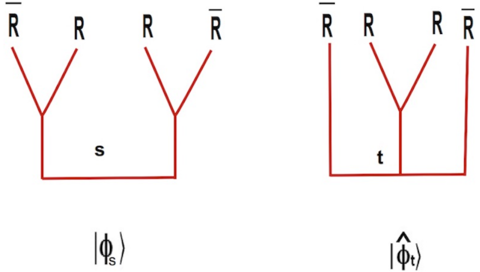
For knots, two of the R_i's will be R and the other two will be conjugate $\bar{R}$ depending on the orientation.

Polynomial invariant of trefoil



In the braid diagram for trefoil, middle two strands are parallely oriented and they are braided.

$$|\Psi_0\rangle \; = \; \sum_{s \in R \otimes R} \mu_s \, |\hat{\Phi}_t(\bar{R}, R, R, \bar{R})\rangle$$

where $\mu_s \; = \; \sqrt{S_{0s}/S_{00}} \equiv \sqrt{dim_q s}$ (**unknot** normalisation)

Trefoil evaluation continued

$$P_R^G[3_1] \;=\; \langle \Psi_0 | \mathcal{B}^3 | \Psi_0 \rangle \;=\; \sum_s dim_q s \; (\lambda_s(R,R))^3$$

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where braiding eigenvalue for parallelly oriented right-handed half-twists is

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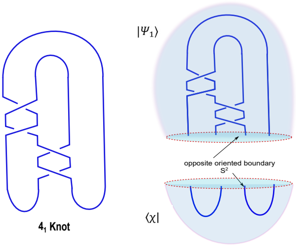
Take $R = \square$ representation of $SU(2)$, check that Jones polynomial is

$$J[3_1] = P_{\square}^{SU(2)}[3_1]/P_{\square}^{SU(2)}[U] = q + q^3 - q^4$$

Figure 8 knot invariant



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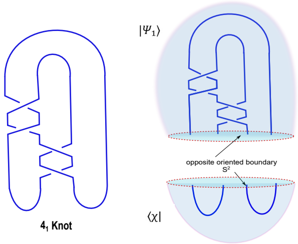

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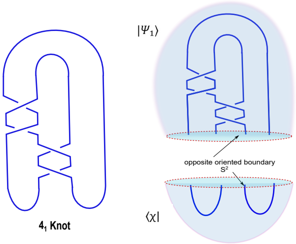

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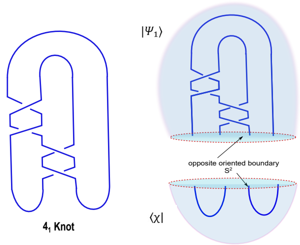

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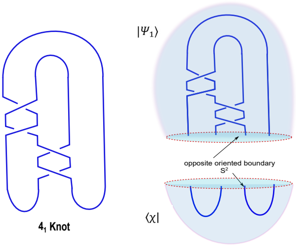

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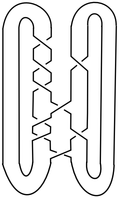
9_{42} chiral knot

Six-strand knot 9_{42} redrawn as quasi-plat:



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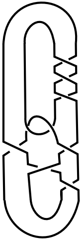

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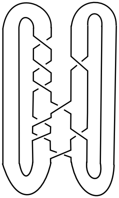

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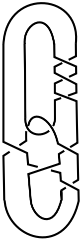

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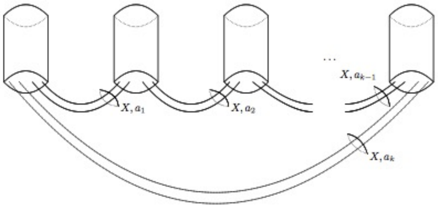
Arborescent Knots

- We had illustrated trefoil from 4- plat diagram and figure-eight from quasi-plat diagram.
- The knots with more than four-strands which can be drawn as



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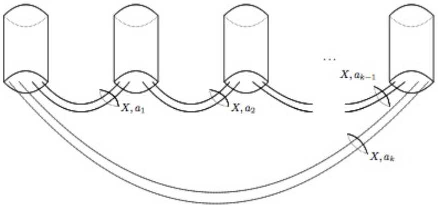

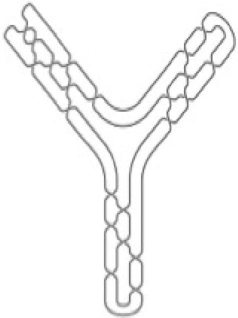

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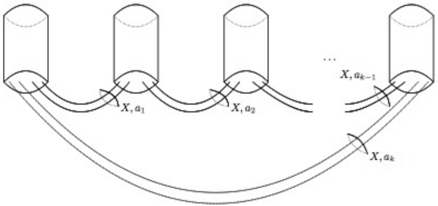
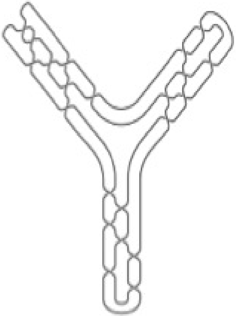
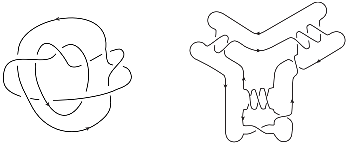


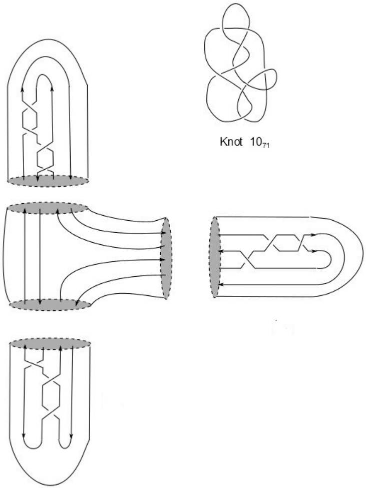


are called **Arborescent knots**.

• These knots in S^3 are obtained from gluing three-balls where some three-balls have two or more four-punctured S^2 boundaries

Knot 10_{152} and knot 10_{71}

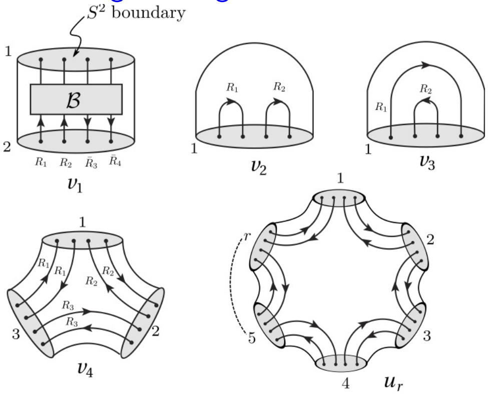




To write the polynomial form we need the states for the fundamental building blocks

Building blocks

The states for the following building blocks



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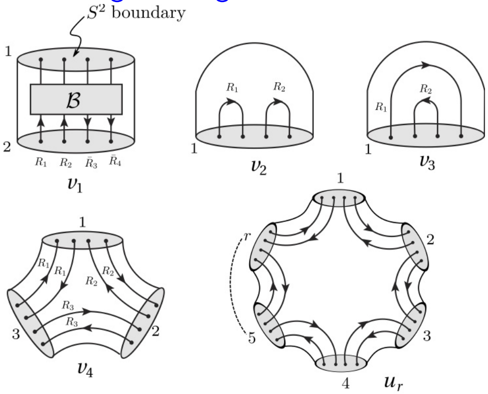

$$\nu_r = \sum_t (dim_q t)^{(1-r/2)} |\phi_t^{(1)}\rangle \ldots |\phi_t^{(r)}\rangle$$

Equivalent Building Blocks

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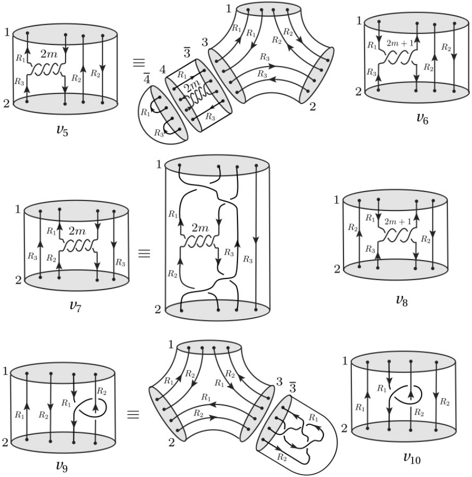
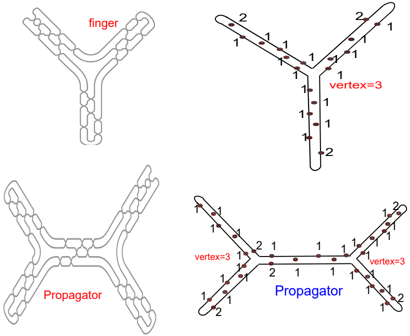


Arborescent knot- Feynman diagram analogy

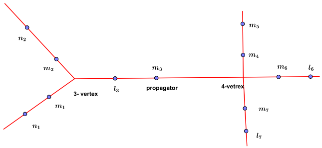


Arborescent knots (Feynman tree diagram)

Family Approach: Arborescent knots

one universal invariant as a function of parameters- choice of parameters gives different knot invariants!

Arborescent knot : drawn as Feynman tree diagram



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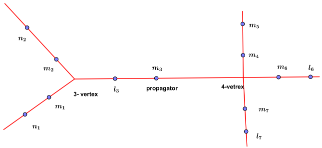

The best parametric family (for describing upto 10-crossing knots) in this class (of 4-point Feynman trees with up to 7 parameters) A.Mironov, A. Morozov, An. Morozov, V.Singh, A. Sleptsov, PR (2016)

$$d_R \sum_{X,\bar{Y}} F_{ap}^{m}(X) F_{pap}^{(m_1,n_1)}(X) T_X^n \bar{P}_{X\bar{Y}} F_{apa}^{(m_2,n_2)}(\bar{Y}) F_{aa}^{n_3}(\bar{Y})$$

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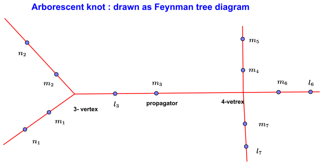
-

Family Approach: Arborescent knots

one universal invariant as a function of parameters- choice of parameters gives different knot invariants!



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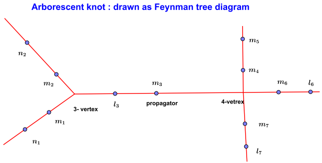

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- can we find quivers in terms of a parametric charge matrix for this family?-*Double twist knots*

Arborescent knot invariants

- arborescent knot invariants will involve braiding eigenvalues and two types of duality matrices $a_{ts}\begin{bmatrix}\bar{R} & R \\ \bar{R} & R\end{bmatrix}$ and or $a_{ts_1}\begin{bmatrix}\bar{R} & R \\ R & \bar{R}\end{bmatrix}$

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- However, other duality matrices are needed for **non-arborescent knot** invariants!

Do we know duality matrix elements

- **Duality matrices proportional to quantum Wigner 6j** (completely known for $SU(2)$ (Kirillov, Reshetikhin) and hence *we can write the polynomial form of any knot invariant (colored Jones' polynomials* $J_n(q)$*)*

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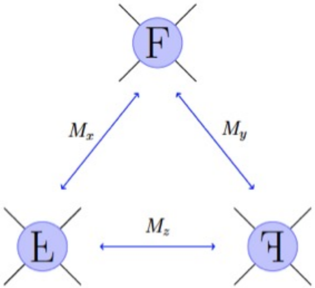
  Hence colored HOMFLY-PT for arborescent knot in polynomial form is possible for any symmetric color $\underbrace{\boxed{}\boxed{}\boxed{}\boxed{}}_{r}$.

Mutation operation on two tangles

- On any two tangle, mutation refers to π rotation about x or y axis (M_x, M_y)



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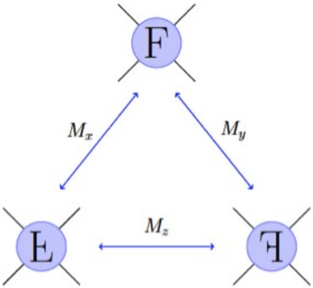

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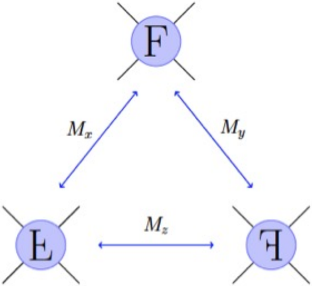

- Mutation is seen as **identity operation** by symmetric colors.
- need to go beyond symmetric representation.

[2,1] colored HOMFLY-PT

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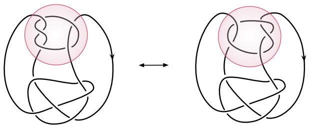

Kinoshita-Terasaka & Conway mutants

Additional information in mixed representation

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- Crucial input in the context of mixed representation: *multiplicity*

$$
\begin{aligned}
(21;0)\otimes(21;0) \quad = \quad & (42;0)_0 \oplus (2^3;0)_0 \oplus (31^3;0)_0 \oplus (321;0)_0 \\
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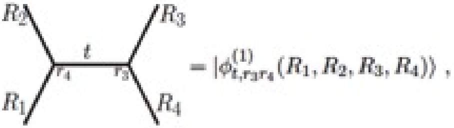
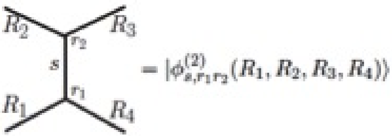
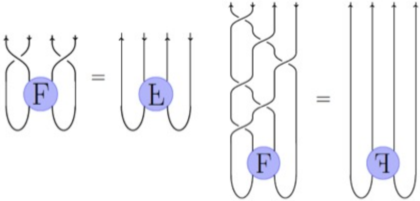
$$

- Hence the states in the four-point conformal blocks involve multiplicity index r_i : $|\phi_{s,r_1,r_2}\rangle$





Mutation operation on two-tangles



$$
\begin{aligned}
|\mathbf{Ⅎ}\rangle &= b_1^{(-)}[b_3^{(-)}]^{-1}|\mathbf{F}\rangle \\
&= \sum_{t,r_1,r_2} \{R,\bar{R},\bar{t},r_1\}\{R,\bar{R},\bar{t},r_2\}|\phi^{(1)}_{t,r_1,r_2}(R,\bar{R},R,\bar{R})\rangle\langle\phi^{(1)}_{t,r_1,r_2}(R,\bar{R},R,\bar{R})|\mathbf{F}\rangle
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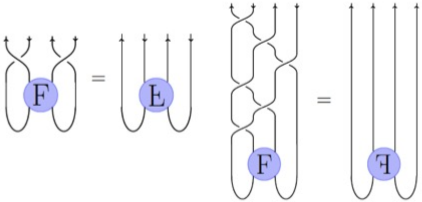

$$
\begin{aligned}
|\rotatebox[origin=c]{180}{\reflectbox{F}}\rangle &= b_1^{(-)}[b_3^{(-)}]^{-1}|\mathbf{F}\rangle \\
&= \sum_{t,r_1,r_2}\{R,\bar{R},\bar{t},r_1\}\{R,\bar{R},\bar{t},r_2\}|\phi^{(1)}_{t,r_1,r_2}(R,\bar{R},R,\bar{R})\rangle\langle\phi^{(1)}_{t,r_1,r_2}(R,\bar{R},R,\bar{R})|\mathbf{F}\rangle \\
|\reflectbox{F}\rangle &= \left([b_1^{(-)}]^{-1}b_2^{(+)}[b_1^{(-)}]^{-1}\right)b_1^{(-)}[b_3^{(-)}]^{-1}\left([b_1^{(-)}]^{-1}b_2^{(+)}[b_1^{(-)}]^{-1}\right)|\mathbf{F}\rangle \\
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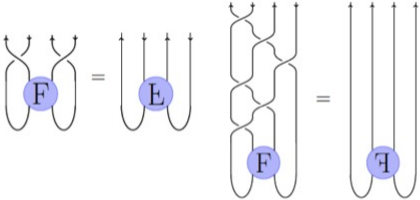

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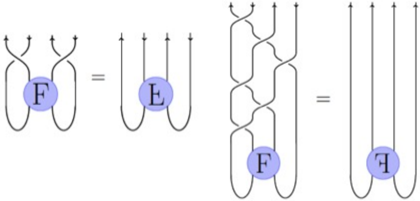
parenthesis denotes signs ± 1.

Mutation operation on two-tangles



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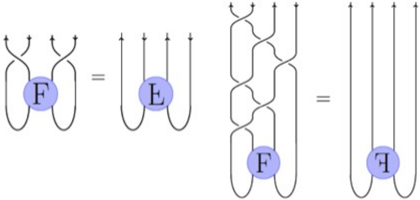
parenthesis denotes signs ± 1.Notice the amplitudes of mutant tangles are related by sign when $r_1 \neq r_2$

Mutation operation on two-tangles



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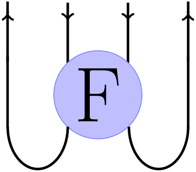
\end{aligned}
$$

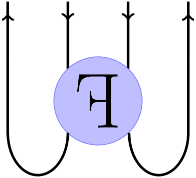
parenthesis denotes signs ± 1.Notice the amplitudes of mutant tangles are related by sign when $r_1 \neq r_2$ (occurs only for irreps with multiplicity)

Tangle and its M_y mutation

- The mutation operation (M_y) on $|\mathbf{F}\rangle$ which gives $|\text{Ⅎ}\rangle$ whose state can also be obtained.



$$= \sum_{s,r_1,r_2} f_{s,r_1,r_2} |\phi^{(1)}_{s,r_1,r_2}(R, \bar{R}, \bar{R}, R)\rangle \ ,$$

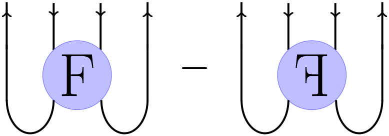


$$= \sum_{s,r_1,r_2} \tilde{f}_{s,r_1,r_2} |\phi^{(1)}_{s,r_1,r_2}(R, \bar{R}, \bar{R}, R)\rangle$$

- The coefficients are related by mutation operation :

$$\tilde{f}_{s,r_1,r_2} = (-1)^{r_1+r_2} f_{s,r_2,r_1} \ .$$

Difference between tangle F and mutant tangle of F



$$|\mathbf{F}\rangle - |\mathbf{Ⅎ}\rangle = (f_{(1;1),0,1} + f_{(1;1),1,0}) \sum_{r_1 \neq r_2} |\phi^{(1)}_{(1;1),r_1,r_2}(R, \bar{R}, \bar{R}, R)\rangle \ .$$

For some mutants, these coefficients could be zero(**for example, pretzel mutant knot pairs with odd antiparallel braidings**.)

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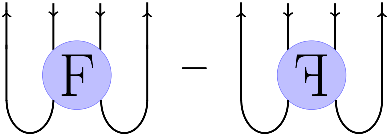

$$|\mathbf{F}\rangle - |\rotatebox{180}{$\mathbf{F}$}\rangle = (f_{(1;1),0,1} + f_{(1;1),1,0}) \sum_{r_1 \neq r_2} |\phi^{(1)}_{(1;1),r_1,r_2}(R, \bar{R}, \bar{R}, R)\rangle \ .$$

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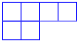
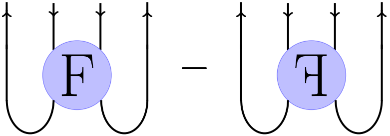
We require duality matrix for $R =$  (two-row representations) with multiplicity more than two to compute difference between such antiparallel pretzel mutants

Difference between tangle F and mutant tangle of F



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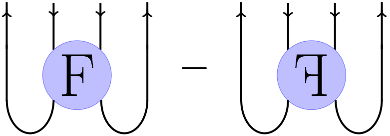
see our two papers with L. Bishler, Saswati Dhara, T. Grigoryev, A. Mironov, A. Morozov, An. Morozov, Vivek Kumar Singh, A. Sleptsov,PR (2020) -

Difference between tangle F and mutant tangle of F



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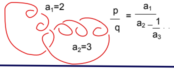
We require duality matrix for $R =$ [4,2] (two-row representations) with multiplicity more than two to compute difference between such antiparallel pretzel mutants

see our two papers with L. Bishler, Saswati Dhara, T. Grigoryev, A. Mironov, A. Morozov, An. Morozov, Vivek Kumar Singh, A. Sleptsov,PR (2020) - could get results for $SU(3)$ and $SU(4)$

Lickorish-Wallace Theorem

Surgery of framed knots and links in S^3 gives new three-manifolds

Frame link knot	3-Manifold
	$S^2 \times S^1$
	S^3
	RP^3
	$L(P,1)$
	$L(5,3)$
	P^3

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Frame link knot	3-Manifold
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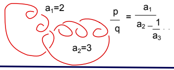
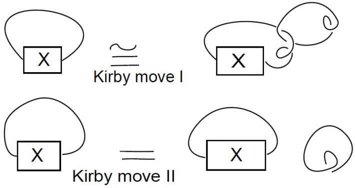
	L(5,3)
	P^3

Many to one Mapping

Kirby Moves

Framed links related by Kirby moves gives the same three-manifold



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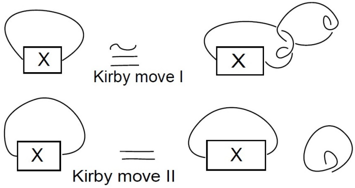

Hence an algebraic expression in terms of link invariants unchanged under Kirby moves will qualify as three manifold invariants.

Three-manifold invariants

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Lickorish's three-manifold invariant involves computation of bracket polynomials of several cables of the link (1991).

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We showed the Lickorish invariant is equivalent to $Z[M]$(upto overall normalisation) for $SU(2)$ group-Swatee Naik,PR (2000)

Three-manifold invariants

Lickorish's three-manifold invariant involves computation of bracket polynomials of several cables of the link (1991).

Three-manifold invariant proportional to Chern-Simons partition function $Z[M]$ respecting Kirby moves -Kaul,PR(2000)

$$Z[M] \propto \sum_{R} dim_q R \; P_R^{SU(N)}[\mathcal{K}]$$

We showed the Lickorish invariant is equivalent to $Z[M]$(upto overall normalisation) for $SU(2)$ group-Swatee Naik,PR (2000)

These three-manifold invariants are known as Witten-Reshetikhin-Turaev (WRT) invariants!

Homological Blocks

$$Z[M] \xrightarrow{\mathcal{S}} \hat{Z}[M, q] \in q^{\Delta}\mathbb{Z}[[q]]$$

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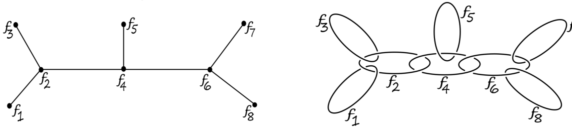
GPPV conjecture for negative definite plumbed three-manifolds



colored Jones related to colored $SO(3)$ knot polynomials & also to colored $OSp(1|2)$.

Zhat for $SO(3), OSp(1|2)$

For negative definite plumbed three-manifolds-Sachin Chauhan, PR(2022)

- $\hat{Z}^{SO(3)} = \hat{Z}^{SU(2)}$ - *probably Zhat depends on the Lie algebra*

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Proof for higher rank groups- *Ongoing work*

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Proof for higher rank groups- *Ongoing work*

- $\hat{Z}^{OSp(1|2)} = 2^{-c} q^{\Delta} \underbrace{\left(\sum_n a_n q^n \right)}_{q \to -q \text{ for } SU(2)}$

Summary

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Thank You