Rozansky-Witten theory and KZ-equations

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Outline

Tensor Categories

Definition. A *monoidal category* $(\mathcal{C}, \otimes, \mathbb{I})$ is a category in which we have tensor product $\otimes$ and identity object $\mathbb{I}$, such that

- $\otimes$ is associative up to a family of natural isomorphism maps, that is there exists a natural isomorphism
$$a_{X,Y,Z} : X \otimes (Y \otimes Z) \tilde{\to} (X \otimes Y) \otimes Z.$$
- $\forall X \in \mathcal{C}$, we have natural isomorphisms for identity object
$$l_X : \mathbb{I} \otimes X \tilde{\to} X, \; r_X : X \otimes \mathbb{I} \tilde{\to} X.$$

Fix k to be a field. Then a monoidal category $\mathcal{C}$ is a *tensor category* over k if

- its sets of morphisms are k-vector spaces
- $\mathcal{C}$ has finite direct sum decomposition
- compositions are k-linear

Fusion Categories

A tensor category is *semi-simple* if there exists a subset of *simple objects* $\mathcal{I} \subset \mathcal{C}$ such that any object in $\mathcal{C}$ is a direct sum of simple objects, and $\forall X, Y \in \mathcal{I}$,

$$\mathrm{Hom}(X, Y) = \begin{cases} k\mathrm{Id}_X, & X = Y \\ 0, & X \neq Y \end{cases}$$

A *fusion category* is a semi-simple finite tensor category. For simple objects $X_i \in \mathcal{I}$, define the coefficient

$$N_k^{i,j} := \dim\left(\mathrm{Hom}(X_k, X_i \otimes X_j)\right)$$

the multiplicity of decomposition of $X_i \otimes X_j$ into single X_k, then the date $\left\{N_k^{i,j}\right\}$ is the *fusion rule* of $\mathcal{C}$, with the corresponding *fusion algebra* $\mathbb{Z}[\mathcal{I}]/\sim$ as the free $\mathbb{Z}$-module generated by $\mathcal{I}$ quotiented by the relation

$$X_i \cdot X_j \sim \sum_k N_k^{i,j} X_k$$

A $\mathcal{C}$ is *rigid* if any object $X \in \mathcal{C}$ has "duality" $X^* \in \mathcal{C}$, with the following evaluation and coevaluation map

$$\mathrm{ev}_X : \; X^* \otimes X \to \mathbb{I},$$
$$\mathrm{ev}_X{}^* : \; \mathbb{I} \to X^* \otimes X.$$

Example. A fusion category above is equivalent to an abelian category Rep$A$ for a finite dimensional k-algebra A. Thus it's not surprising that most examples of fusion category are representation categories for algebraic objects, for instance, the representation category $\mathrm{Rep}_{\mathbb{C}} SL_2$, which consists complex vector spaces for a representation of SL_2 to act on. The simple objects are highest weight representations $V_j, \; j \in \mathbb{Z}$, with fusion rule as Clebsh-Gordan rule

$$V_i \otimes V_j = \sum_{k=|i-j|}^{i+j} V_k \;\; (k = i + j \mod 2).$$

Remark. The difference between fusion category and modular tensor category is sufficiently the "modularity" data, which comes from the following additional braiding structures of $\mathcal{C}$:

Braiding

Definition. A *braiding* in a tensor category is a family of natural isomorphisms:

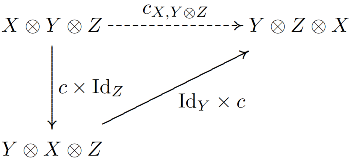
$$c_{X,Y}: \; X \otimes Y \xrightarrow{\sim} Y \otimes X$$

satisfying the following compatibility with tensor:

$$c_{X,Y\otimes Z} = (\mathrm{Id}_Y \otimes c_{X,Z})\,(c_{X,Y} \otimes \mathrm{Id}_Z)\,,$$

$$c_{X\otimes Y,Z} = (c_{X,Z} \otimes \mathrm{Id}_Y)\,(\mathrm{Id}_X \otimes c_{Y,Z})\,,$$

that is the diagram(e.g., for $c_{X,Y\otimes Z}$)



A *twist* of braided tensor category is a family of natural isomorphisms:

$$\theta_X : \ X \xrightarrow{\sim} X$$

such that it's compatible with tensor as

$$\theta_{X\otimes Y} = c_{Y,X} \circ c_{X,Y} \left(\theta_X \otimes \theta_Y\right),$$

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{\theta_{X\otimes Y}} & X \otimes Y \\ {\scriptstyle \theta_X \otimes \theta_Y}\downarrow & & \uparrow{\scriptstyle c} \\ X \otimes Y & \xrightarrow[c]{} & Y \otimes X \end{array}$$

in additional with the compatibility of duality

$$\theta_{X^*} = \theta_X{}^*.$$

Ribbon Category

A rigid tensor category with the compatible braiding and twist is called *ribbon category*. For a morphism $f \in \mathrm{Hom}(X, X) = \mathrm{End}(X)$, define its *trace* in the following manner:

$$\mathrm{Tr}(f) := \mathrm{ev}_X \circ C_{X,X^*} \circ (\theta_X f \otimes \mathrm{Id}_{X^*}) \circ \mathrm{ev}_X{}^*$$

which is in $\mathrm{End}(\mathbb{I})$, hence a k-number.

$$\begin{array}{ccccc} \mathbb{I} & & \xrightarrow{\quad \mathrm{Tr}\, f \quad} & & \mathbb{I} \\ \downarrow \mathrm{ev}_X{}^* & & & & \uparrow \mathrm{ev}_X \\ X \otimes X^* & \xrightarrow[\theta_X \circ f]{} & X \otimes X^* & \xrightarrow{c} & X^* \otimes X \end{array}$$

In particular,

$$\dim(X) = \mathrm{Tr}(\mathrm{Id}_X).$$

Modular Tensor Category

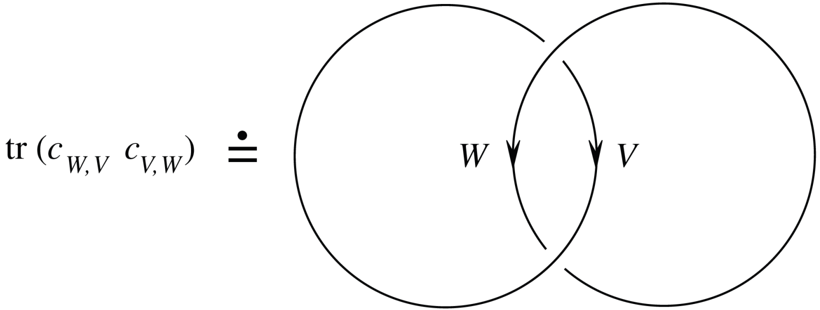
Finally, a *modular tensor category* is a semi-simple rigid ribbon category with an invertible S-matrix:

$$S_{i,j} = \mathrm{Tr}(c_{X_j,X_i} \circ c_{X_i,X_j}),$$

where $X_i, X_j \in \mathcal{I}$. Let diagonal T-matrix as (note that $\theta_{X_i} \in \mathrm{End}(X_i)$ is a k-number for simple X_i)

$$T_{i,j} = \delta_{i,j}\theta_{X_i},$$

its easy to varify that matrices $\{S, T\}$ form a projective representation of modular group $SL_2(\mathbb{Z})$, hence the modularity.



Outline

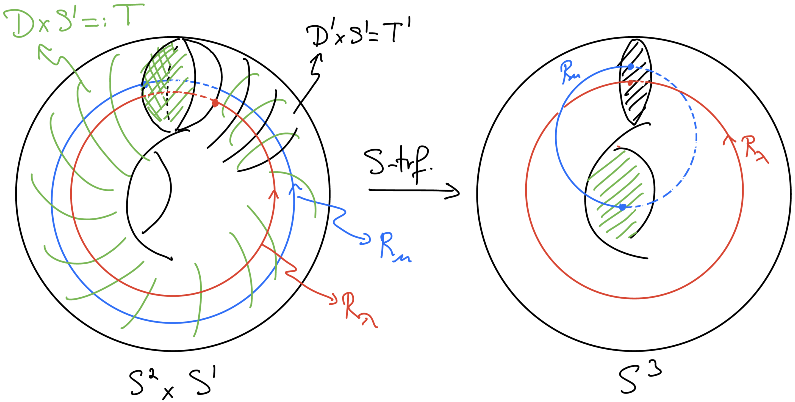
TQFT

A topological quantum field theory (TQFT) in $2+1$ dimensions is a functor Z satisfying the following conditions:

- To each compact oriented 2-dimensional smooth manifold without boundary Σ one associates a finite dimensional complex vector space Z_Σ.
- A compact oriented $(2+1)$-dimensional smooth manifold Y with $\partial Y = \Sigma$ determines a vector $Z(Y) \in Z_\Sigma$.

Furthermore, Z (known as the *partition function*) has to satisfy the following properties:

1. Denote by $-\Sigma$ the manifold Σ with the orientation reversed. Then, we have $Z_{-\Sigma} = Z_\Sigma^*$ where Z_Σ^* is the dual of $Z_{-\Sigma}$ as a complex vector space
2. For a disjoint union $\Sigma_1 \cup \Sigma_2$ we have $Z_{\Sigma_1 \cup \Sigma_2} = Z_{\Sigma_1} \otimes Z_{\Sigma_2}$.
3. For the composition of *cobordisms* $\partial Y_1 = (-\Sigma_1) \cup \Sigma_2$ and $\partial Y_2 = (-\Sigma_2) \cup \Sigma_2$, we have $Z(Y_1 \cup Y_2) = Z(Y_2) \circ Z(Y_1)$
4. For an empty set $\emptyset$ we have $Z(\emptyset) = \mathbb{C}$.
5. Let I denote the closed nit interval. Then, $Z(\Sigma \times I)$ is the identity map as a linear transformation of Z_Σ.



$$Z(S^3; L(R_\mu, R_\lambda)) = \sum_\nu S_\mu{}^\nu \underbrace{Z(S^2 \times S^1; R_\nu, R_\lambda)}_{= \delta_{\nu\lambda}} = S_{\mu\lambda}$$

Outline

The Rozansky-Witten TQFT

Let X be a hyper-Kähler manifold of real dimension $4n$. The complexification of the tangent bundle admits a decomposition

$$TX \otimes_{\mathbb{R}} \mathbb{C} = V \otimes S$$

where V is a rank $2n$ complex vector bundle with structure group $Sp(n)$, and S is a trivial rank two bundle. Denote local coordinates on 3-manifold M as x^μ, $\mu = 1, 2, 3$. Define TQFT with fields:

- bosons $\Phi : M \to X, \quad \phi^i(x^\mu),\ i = 1, \ldots, 4n$
- fermions are scalar η^I and a one-form χ^I_μ with values in V

and action (Ω is completely symmetric tensor)

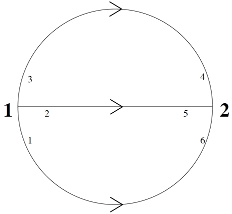
$$\begin{aligned} S &= \int_M (L_1 + L_2)\sqrt{h} d^3x, \\ L_1 &= \frac{1}{2} g_{ij} \partial_\mu \phi^i \partial^\mu \phi^j + \epsilon_{IJ} \chi^I_\mu \nabla^\mu \eta^J \\ L_2 &= \frac{1}{2} \frac{1}{\sqrt{h}} \epsilon^{\mu\nu\rho} \left(\epsilon_{IJ} \chi^I_\mu \nabla^J_\rho + \frac{1}{3} \Omega_{IJKL} \chi^I_\mu \chi^J_\nu \chi^K_\rho \eta^L \right) \end{aligned}$$

Partition Function

The partition function of this theory can be evaluated via a Feynman diagram expansion which takes the following general form

$$Z_X(M) = \sum_{\Gamma} b_\Gamma(X) I_\Gamma^{\mathrm{RW}}(M)$$

where Γ denotes trivalent graphs. The quantities $b_\Gamma(X)$ are known as *weights* and vanish except when Γ has $2n$ (= dimension of X) vertices. The invariants $I_\Gamma^{\mathrm{RW}}(M)$ solely depend on the 3-manifold M.



To evaluate diagrams Γ, assign structure constants c_{ijk} to inner vertices and a symmetric tensor σ^{ij} to propagators.

In case of Rozansky-Witten theory, we have the following correspondence:

$$\Phi \in \Omega^{0,1}(X, \mathrm{Sym}^3 T^*) \quad \leftrightarrow \quad c_{ijk}$$
$$\tilde{\omega} \in H^0(X, \Lambda^2 T) \quad \leftrightarrow \quad \sigma^{ij}$$

One can also equip the 3-manifold M with a link $\mathcal{L}$ composed of *Wilson lines* L_i labeled by elements of the category $\mathcal{C}$. To each element $a \in \mathcal{C}$ one assigns a representation V_a which label outer circles of Γ. Each *outer* vertex carries then a tensor $(B_a)_{iJ_a}{}^{K_a}$ where $J_a, K_a = 1, \ldots, \dim V_a$. For the chord diagram of the half-circle one then gets for example

$$\sum_{i,j,K,L} B_{iK}{}^{L} B_{jL}{}^{K} \sigma^{ij}$$

For an m-component link, one chooses holomorphic vector bundles $E_1, \ldots, E_m$ over X, of ranks $r_1, \ldots, r_m$. Let the curvatures of the corresponding connections be

$$R_a \in \Omega^{1,1}(\mathrm{End} E_a),$$

One then has the correspondence

$$R_a \quad \leftrightarrow \quad B_a.$$

Conformal Blocks

The "space of conformal blocks" on $\Sigma_{g,m}$ is the Hilbert space in RW theory on $\Sigma_{g,m}$, where $\Sigma_{g,m}$ denotes a genus-g surface with m punctures labeled by holomorphic vector bundles $E_1, \ldots, E_m$ over X.
We are mostly interested in the cases with $g = 0$. For $m = 0$, the Hilbert space is

$$\mathcal{H}_{RW[X]}(S^2) = H^{0,\bullet}(X) = \oplus_n H^n(\mathcal{O}_X)$$

The corresponding Verlinde formula that gives the dimension of this space is obtained by taking the (super-)trace, *i.e.* evaluating the invariant on $M_3 = S^1 \times S^2$:

$$\mathrm{sdim}\mathcal{H}_{RW[X]}(S^2) = \sum_i (-1)^i \dim H^{0,i}(X)$$

A generalization of this Verlinde formula to $m \neq 0$ looks like :

$$\mathrm{sdim}\mathcal{H}_{RW[X]}(S^2; E_1, \ldots, E_m) = \chi(E_1 \otimes \ldots \otimes E_m)$$

and the corresponding Hilbert spaces are

$$\mathcal{H}_{RW[X]}(S^2; E_1, \ldots, E_m) = \bigoplus_{n=0}^{\dim_{\mathbb{C}} X} H^n_{\bar\partial}(X, E_1 \otimes \ldots \otimes E_m)$$

Outline

Braiding

We have seen that line operators correspond to sheaves on the manifold X. These can be viewed as objects within the *derived category of coherent sheaves* on X, denoted by $D(X)$. Given two objects, A, B of $D(X)$, let

$H_{A,B} \in \mathrm{Ext}^2(A \otimes B, A \otimes B)$ and $C_A \in \mathrm{Ext}^2(A, A)$ be two morphisms:

In terms of these, the braiding morphism $\tau_{A,B}$ may be described as

$$\tau_{A,B} = \exp\left(H_{A,B}/2\right) \in \mathrm{Ext}^*(A \otimes B, B \otimes A) = \mathrm{Hom}_{D(X)}(A \otimes B, B \otimes A).$$

The associator $\Phi_{A,B,C}$ is written as a polynomial in the non-commuting variables

$$X \equiv H_{A,B} \otimes \mathrm{id}_C, \quad Y \equiv \mathrm{id}_A \otimes H_{B,C}.$$

KZ-equation

In order to obtain it, one solves the KZ-equation

$$\frac{dG}{dz} = \hbar\left(\frac{X}{z} + \frac{Y}{z-1}\right) G.$$

This ODE is Fuchsian and there are two solutions of the form

$$G_0(z) = P(z) z^{\hbar X}, \quad G_1(z) = Q(1-z)(1-z)^{\hbar Y},$$

where P, Q are power series in z with value 1 in $z = 0$. This can be seen by inserting the above ansatz into the equation and considering the resulting equations for each power of z. Since G_0, G_1 are nonzero solutions of our ODE and this is homogeneous, their ratio is independent of z,

$$G_0(z) =: G_1(z)\Phi(X, Y).$$

The ratio $\Phi \in \mathbb{C}[[\hbar]]\langle X, Y\rangle$ is the associator we are interested in. If G_a, $a \in [0, 1]$, is the unique solution of our ODE with $G_a(a) = 1$, then one has

$$\Phi(X, Y) = \lim_{a\to 0} a^{-\hbar Y} G_a(1-a) a^{\hbar X}.$$

$\Phi(X,Y)$ can then be computed and is given by

$$\Phi(X,Y) = 1 - \frac{\zeta(2)}{(2\pi\sqrt{-1})^2}[X,Y]\hbar^2 + \frac{\zeta(3)}{(2\pi\sqrt{-1})^3}\left([[X,Y],Y] - [X,[X,Y]]\right)\hbar^3 + \mathcal{O}(\hbar^4)$$

From $G_0(z) \sim z^{\hbar X}$ we see that it changes under a whole circle around zero by a factor $e^{\hbar X}$. Hence the half-monodromy (braiding) around zero amounts to

$$B_0 = e^{\hbar X/2}.$$

Similarly, one gets from $G_1(1-z) \sim (1-z)^{\hbar Y}$ for the monodromy around the point $z = 1$

$$B_1 = \Phi(X,Y)^{-1} e^{\hbar Y/2} \Phi(X,Y).$$

Computation for K3 surface

Let now S be a K3 surface. We next want to compute $\mathrm{Ext}^2(A \otimes B, A \otimes B)$ for A and B two line bundles of the form $A = \mathcal{O}_S(C_1)$ and $B = \mathcal{O}_S(C_2)$ with C_1 and C_2 two curves in the K3. Thus, the task is to compute

$$\mathrm{Ext}^2_X(\mathcal{O}_S(C_1) \otimes \mathcal{O}_S(C_2), \mathcal{O}_S(C_1) \otimes \mathcal{O}_S(C_2))$$

Using

$$\mathrm{Ext}^n_S(\mathcal{E}, \mathcal{F}) = H^n(S, \mathcal{E}^\vee \otimes \mathcal{F}),$$

we thus have to compute

$$H^2(S, \mathcal{E}^\vee \otimes \mathcal{E}),$$

where $\mathcal{E} = \mathcal{O}_S(C_1) \otimes \mathcal{O}_S(C_2)$. But we know

$$\mathcal{E}^\vee = \mathcal{O}_S(-C_1) \otimes \mathcal{O}_S(-C_2),$$

and hence we get

$$H^2(S, \mathcal{E}^\vee \otimes \mathcal{E}) = H^2(S, \mathcal{O}_S) = \mathbb{C},$$

generated by the holomorphic two-form of the K3 surface.

Let us next do the computation for a general vector bundle E of rank $r+1$. That is, we want to compute

$$\mathrm{Ext}^2_S(E,E) = H^2(S, E\otimes E^\vee).$$

The rank $E\otimes E^\vee$ is simply $\mathrm{rk}(E)^2=(r+1)^2$. Using that $c_1(E)=c_1(\mathrm{det}E)=c_1(L)$ for $L=\mathrm{det}E$ and $c_1(E^\vee)=c_1((\mathrm{det}E)^\vee)=c_1(L^\vee)=-c_1(L)$, we see that from the properties of the Chern character we get

$$\begin{aligned}
\mathrm{ch}(E\otimes E^\vee) &= \mathrm{ch}(E)\mathrm{ch}(E^\vee)\\
&= (r+1+c_1(E)+\mathrm{ch}_2(E)+\ldots)(r+1-c_1(E)+\mathrm{ch}_2(E^\vee)+\ldots)\\
&= (r+1)^2-c_1(E)^2+2(r+1)\mathrm{ch}_2(E)+\ldots
\end{aligned}$$

Comparing this with

$$\mathrm{ch}(E \otimes E^\vee) = (r+1)^2 + c_1(E \otimes E^\vee) + \frac{c_1(E \otimes E^\vee) - 2c_2(E \otimes E^\vee)}{2} + \dots$$

and noting that $c_1(E \otimes E^\vee) = 0$, wee see that

$$c_2(E \otimes E^\vee) = -2(r+1)\mathrm{ch}_2(E) + c_1(E)^2.$$

Using $c_1(E)^2 = 2g-2$ and $c_2(E) = \mathrm{deg} Z = d$, this gives

$$c_2(E \otimes E^\vee) = -r(2g-2) + 2(r+1)d.$$

Together with the Hirzebruch-Riemann-Roch formula for the Euler number of a coherent sheaf $\mathcal{F}$ on a K3 surface,

$$\chi(\mathcal{F}) = \frac{c_1(\mathcal{F})^2 - 2c_2(\mathcal{F})}{2} + 2\mathrm{rk}(\mathcal{F}),$$

we finally obtain

$$\chi(E \otimes E^\vee) = 2h^0(S, E \otimes E^\vee) - h^1(S, E \otimes E^\vee) = 2 - 2(g - (r+1)(r-d+g)).$$

Example sheaves

As a test of this formula, let us assume that E is a line bundle L. Then $r = 0$ and $d = 0$ as the second Chern class vanishes. In this case our formula gives

$$\chi(L \otimes L^\vee) = \chi(\mathcal{O}_S) = 2 - 2(g - (0 - 0 + g)) = 2,$$

which agrees with the fact that on a K3 surface $h^0(S, \mathcal{O}_S) = h^2(S, \mathcal{O}_S) = 1$ and $h^1(S, \mathcal{O}_S) = 0$ giving $\chi(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S) - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S) = 2$. As a further example, let us consider the tangent bundle $E = \mathcal{T}_S$. Then $E^\vee = \mathcal{T}_S^\vee \equiv \mathcal{T}_S$ and moreover $H^0(S, S^m\mathcal{T}_S) = 0$. We compute

$$E \otimes E^\vee = \mathcal{T}_S \otimes \mathcal{T}_S \equiv S^2\mathcal{T}_S \oplus \mathcal{O}_S,$$

and hence $h^0(S, \mathcal{T}_S \otimes \mathcal{T}_S) = h^2(S, \mathcal{T}_S \otimes \mathcal{T}_S) = 1$. As for $E = \mathcal{T}_S$, $r = 1$, $g = 1$ (as $c_1(\mathcal{T}_S) = 0$) and $d = c_2(\mathcal{T}_S) = 24$, thus

$$\chi(\mathcal{T}_S \otimes \mathcal{T}_S) = 2 - 2(1 - 2(1 - 24 + 1)) = -88.$$

This finally gives $h^1(S, \mathcal{T}_S \otimes \mathcal{T}_S) = 90$.

Outline

Physics Interpretation: 6d LSTs

- Consider N small, coincident $SO(32)$ instantons in the heterotic string

  $\longrightarrow$ 6d Little String Theory (LST) with $\mathcal{N}=(1,0)$ SUSY and gauge group $Sp(N)$

- Taking $N=1$ gives gauge group $SU(2)$

  $\longrightarrow$ Compactification along T^3 leads to classical Coulomb branch metric

$$ds^2 = \frac{\sqrt{\det h}}{g_6^2}(h^{-1})^{ab}d\phi_a d\phi_b + \frac{g_6^2}{\sqrt{\det h}}(d\phi_4 - \theta^a d\phi_a)^2, \quad a = 1,2,3,$$

  where g_6 : coupling constant, ϕ_a : Wilson lines of gauge field along cycles of T^3, ϕ_4 : dualized 3d $U(1)$ gauge field.

  $\longrightarrow$ metric on T^4

- Modding out by $\mathbb{Z}_2$ Weyl action of $SU(2)$ leads to K3 surface as low-energy Coulomb branch geometry!

More generally, compactification of any LST on T^3 leads to a sigma model with a Hyperkähler target space X of dimension

$$\dim_{\mathbb{R}} X = 4(r_V + n_T),$$

where r_V is the total rank of the 6d gauge group while n_T is the number of tensor multiplets. $\longrightarrow$ Topological twisting leads to RW-theory with target X: ϕ, $\bar{\phi}$ are complex coordinates of X and we have the map

$$\phi : M_3 \to X.$$

The fermionic fields χ are section of $\Omega^1_M \otimes \phi^* T^{1,0}X$ and moreover $\bar{v} \in \phi^* T^{0,1}X$.

Wilson lines in the *adjoint* representation are given by ($C \subset M_3$)

$$W_C(T^{1,0}X) = \mathrm{Tr}\mathbf{P}\exp\int_C \mathcal{A}, \quad \text{where } \mathcal{A}^i_j = -d\phi^k \Gamma^i_{kj} + \bar{v}^{\bar{i}} \chi^k R^i_{\bar{i}kj}.$$

Wilson lines in the *fundamental* representation arise from holomorphic vector bundles $E \to X$ on X. Their explicit form is

$$W_C(E) = \mathrm{Tr}\mathbf{P}\exp\int_C \mathcal{A}, \quad \text{where } \mathcal{A} = -d\phi^k A_k + \bar{v}^{\bar{i}} \chi^k F_{\bar{i}k}.$$

Wilson lines from 1-form symmetry of LST

So we are looking for line defects with a K3 moduli space. Indeed there are such objects in our 6d LST!
6d gauge theories give rise to 1-form continuous global $U(1)$ symmetries associated to the 2-form instanton current,

$$J^{(2)} \sim \star \mathrm{Tr}\left(f^{(2)} \wedge f^{(2)}\right).$$

$\longrightarrow$ Integration over a 4-manifold Σ_4 defines a conserved charge operator,

$$Q(\Sigma_4) = -i \int_{\Sigma_4} \star J^{(2)}.$$

Acting with such a codimension 2 operator on the vacuum gives rise to a line defect carrying charge $Q \in \mathbb{Z}$ and linked with the surface Σ_4. Coupling to a 2-form background gauge field $B^{(2)}$ gives rise to an additional term in the action,

$$\int B^{(2)} \wedge \star J^{(2)},$$

and conservation of the current requires invariance under background gauge transformations $B^{(2)} \to B^{(2)} + d\Lambda^{(1)}$.

The degrees of freedom (moduli) of our line defect $\mathcal{L}$ are moduli spaces of instantons on Σ_4. Since in our case 3 directions of Σ_4 are already compactified on T^3 and the defect is extended along a line within the perpendicular 3-manifold M_3, the only 4-cycles which links this line is composed of a circle linking $\mathcal{L}$ within M_3 together with T^3. Thus we see $\Sigma_4 \equiv T^4$.

$\longrightarrow$ moduli space $\mathcal{M}_{\mathcal{L}}$ is the moduli space of $SU(2)$-instantons on T^4.

$\longrightarrow$ For a single $SU(2)$ instanton we get $\mathcal{M}_{\mathcal{L}} = \mathbf{K3}$.

Outline

Summary

- *Modular Tensor Categories* describe the structure of a $2+1$d TQFT in terms of braiding morphisms and modular S- and T-matrices.
- The Rozansky-Witten TQFT assigns a modular tensor category to a Hyperkähler manifold X from its derived category of coherent sheaves.
- Anyons (simple objects) are given in terms of sheaves on X and the TQFT Hilbert space is the space of sections of tensor products of sheaves.
- Braiding is defined in terms of the KZ-equations which encodes the τ-morphisms exchanging the order in the tensor product of two sheaves.
- The *associator* is the *connection matrix* of the KZ-equation connecting solutions at two different points in moduli space.
- In the case of $X = K3$, we find that the Associator is trivial and the τ-morphism satisfies the Yang-Baxter equation.
- Physically, we can identify our RW-theories with compactifications of little string theories along T^3 together with a topological twist.

Thank you!