Homological blocks and R-matrices

Sunghyuk Park

Mar 3, 2023

Learning workshop on BPS states and 3-manifolds @ ICTP

Homological block $\hat{Z}$, is a machinery that produces q-series out of 3-manifolds.

$$Y \dashrightarrow \hat{Z}_Y(q) = q^{\Delta} \sum_{n \geq 0} a_n q^n$$

I will discuss what's known about its mathematical construction, from the perspective of quantum topology.

This talk is based on

- **"Large color R-matrix for knot complements and strange identities"**. *J. Knot Theory Ramifications* 29 (2020), no. 14, 2050097. [arXiv:2004.02087]
- **"Inverted state sums, inverted Habiro series, and indefinite theta functions"**. [arXiv:2106.03942]

Plan:

1. Brief review of what we've seen in previous talks
2. Large-color R-matrix and inverted state sum
3. Final remarks

1. Brief review

Homological blocks $\hat{Z}$

Let Y be a 3-manifold with $b_1(Y) = 0$, and set $\mathfrak{g} = \mathfrak{sl}_2$ for simplicity.

It was conjectured in [Gukov, Pei, Putrov, Vafa '17] that the WRT invariant of Y can be decomposed into a certain linear combination of $q \to e^{\frac{2\pi i}{k}}$ limit of $\hat{Z}_{Y,b}(q)$:

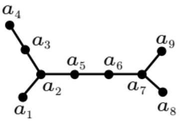
$$WRT_Y(e^{\frac{2\pi i}{k}}) = \sum_{a \in H_1(Y;\mathbb{Z})/\mathbb{Z}_2} e^{2\pi i k CS(a)} \sum_{b \in \mathrm{Spin}^c(Y)/\mathbb{Z}_2} S_{ab} \lim_{q \to e^{\frac{2\pi i}{k}}} \frac{\hat{Z}_{Y,b}(q)}{2(q^{\frac{1}{2}} - q^{-\frac{1}{2}})},$$

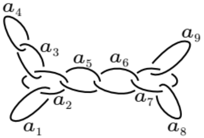
where S_{ab} is a matrix determined by the linking pairing of Y that does not depend on k.

Homological blocks $\hat{Z}$

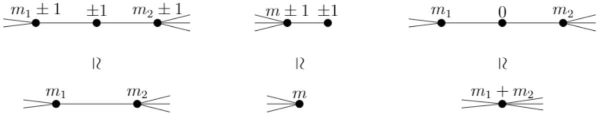
For a class of 3-manifolds known as negative-definite plumbed 3-manifolds, a mathematical definition of $\hat{Z}$ was given by [Gukov, Pei, Putrov, Vafa '17].

Plumbed 3-manifolds are 3-manifolds that can be naturally associated to trees whose vertices are labelled by integers.





Two plumbing graphs represent the same 3-manifold iff they are related via a sequence of Neumann moves.



Homological blocks $\hat{Z}$

When the linking matrix B of a plumbing graph Γ is negative-definite, the resulting plumbed 3-manifold Y_Γ is called negative-definite.

For a negative-definite plumbed 3-manifold Y_Γ, define

$$\hat{Z}_{Y_\Gamma,b}(q) := \oint \prod_{v\in V} \frac{dx_v}{2\pi i x_v} \left(\prod_{v\in V} (x_v^{\frac{1}{2}} - x_v^{-\frac{1}{2}})^{2-\delta_v} \sum_{\ell \in 2B\mathbb{Z}^V + b} q^{-\frac{1}{4}(\ell, B^{-1}\ell)} x^{\frac{\ell}{2}} \right),$$

where δ_v is the degree of v, and

$$b \in (2\mathbb{Z}^V + \delta)/2B\mathbb{Z}^V \cong \mathrm{Spin}^c(Y_\Gamma).$$

This is invariant under Neumann moves (and hence well-defined) [Gukov, Manolescu '19].

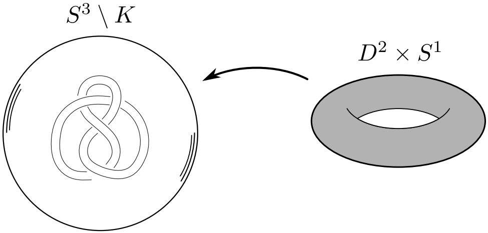
Homological blocks $\hat{Z}$

- GPPV conjecture (relating $\hat{Z}$ with WRT invariants) has been proven in many cases [Andersen, Mistegard '18], [Fuji, Iwaki, H. Murakami, Terashima '20], [Mori, Y. Murakami '21], and a proof for all negative-definite plumbed 3-manifolds appeared on arXiv just a few days ago [Y. Murakami '23].
- They provide examples of quantum modular forms (of possibly higher depth) [Zagier '10], [Cheng, Chun, Ferrari, Harrison, Gukov '18], [Bringmann, Mahlburg, Milas '18, '19], [Bringmann, Kaszian, Milas, Nazaroglu '21].
- This definition has been generalized to more general gauge groups [Chung '18], [S.P. '19], [Ferrari, Putrov '20], [Chauhan, Ramadevi '22].
- Connection to Heegaard-Floer homology [Akhmechet, Johnson, Krushkal '21], [Akhmechet, Johnson, S.P. to appear].

An approach toward $\hat{Z}$ via surgery

Physics predicts $\hat{Z}$ can be extended to an invariant defined for **all** 3-manifolds.

An approach toward $\hat{Z}$ for general 3-manifolds:



1. study $\hat{Z}$ for link complements, and then
2. study the Dehn surgery formula.

This is the approach initiated in [Gukov, Manolescu '19].

Melvin-Morton-Rozansky (MMR) expansion

The following asymptotic large-color behavior of colored Jones polynomials was conjectured by [Melvin, Morton '95], [Rozansky '96].

Theorem ([Bar-Natan, Garoufalidis '96], [Rozansky '96])

Set $q = e^{\hbar}$. Consider the limit where $\hbar \to 0$ and $n \to \infty$ while $n\hbar$ is fixed. In this large-color limit, the colored Jones polynomial has the following expansion:

$$J_{K,n}(q) = \frac{1}{\Delta_K(x)} + \frac{P_1(x)}{\Delta_K(x)^3}\hbar + \frac{P_2(x)}{\Delta_K(x)^5}\frac{\hbar^2}{2!} + \cdots ,$$

where $x = q^n = e^{n\hbar}$, $\Delta_K(x)$ is the Alexander polynomial, and $P_j(x) \in \mathbb{Z}[x, x^{-1}]$.

Gukov-Manolescu conjecture

Conjecture ([Gukov, Manolescu '19])

The MMR expansion of the colored Jones polynomials can be resummed into a two-variable series $F_K(x,q)$:

$$(x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \sum_{j\geq 0} \frac{P_j(x)}{\Delta_K(x)^{2j+1}} \frac{\hbar^j}{j!} = F_K(x,q).$$

That is, there is a formal power series $F_K(x,q)$ in x with coefficients in $\mathbb{Z}((q))$ whose $\hbar$-expansion agrees with the MMR expansion.

Moreover,

$$\hat{A}_K(\hat{x},\hat{y},q)F_K(x,q) = 0,$$

where $\hat{A}_K$ is the quantum A-polynomial for the unreduced colored Jones polynomials of K.

The two-variable series $F_K(x,q)$ should really be thought of as the $\hat{Z}$ for the knot complement $S^3 \setminus K$.

Dehn surgery

Conjecture ([Gukov, Manolescu '19])

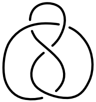
$$\hat{Z}_{S^3_{p/r}(K),b}(q) = \oint \frac{dx}{2\pi i x}\left((x^{\frac{1}{2r}} - x^{-\frac{1}{2r}}) F_K(x,q) \sum_{u \in \frac{p}{r}\mathbb{Z} + \frac{b}{r}} q^{-\frac{r}{p}u^2} x^u \right),$$

provided that the right-hand side converges (i.e. when $-\frac{r}{p}$ *is big enough).*

This is a theorem for surgeries on torus knots that result in negative-definite plumbed 3-manifolds.

Dehn surgery: an example

Consider $Y = -\Sigma(2,3,7) = S^3_{-1}(\mathbf{4}_1) = S^3_{+1}(\mathbf{3}_1)$.



$\hat{Z}$ for the figure-eight knot complement is given by [Gukov, Manolescu '19], [S.P. '21]

$$F_{\mathbf{4}_1}(x,q) = -(x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \sum_{n\geq 0} \frac{1}{\prod_{0\leq j\leq n}(x + x^{-1} - q^j - q^{-j})}$$

$$= x^{\frac{1}{2}} + 2x^{\frac{3}{2}} + (q^{-1}+3+q)x^{\frac{5}{2}} + (2q^{-2}+2q^{-1}+5+2q+2q^2)x^{\frac{7}{2}} + O(x^{\frac{9}{2}}).$$

Dehn surgery: an example

Applying the surgery formula, we get

$$\begin{aligned}\hat{Z}_{S^3_{-1}(\mathbf{4}_1)}(q) &= \left[(x^{-\frac{1}{2}} - x^{\frac{1}{2}})F_{\mathbf{4}_1}(x,q)\right]_{x^m \mapsto q^{m^2}} \\ &= \left[1+x+(q^{-1}+1+q)x^2+(2q^{-2}+q^{-1}+2+q+2q^2)x^3+O(x^4)\right]_{x^m \mapsto q^{m^2}} \\ &= 1+q+q^3+q^4+q^5+2q^7+\cdots \\ &= \sum_{n\geq 0} \frac{q^{n^2}}{\prod_{1\leq k\leq n}(1-q^{n+k})},\end{aligned}$$

which is a Ramanujan's mock theta function of order 7.

Other descriptions of $Y = -\Sigma(2,3,7)$ give the same result.

2. Large-color R-matrix and inverted state sum

R-matrices

Colored Jones polynomials can be computed using the R-matrices for finite-dimensional representations V_n of the quantum group $U_q(\mathfrak{sl}_2)$ [Kirillov, Reshetikhin '88].

Hence, in order to tackle Gukov-Manolescu conjecture, it is natural to study the large-color limit of these R-matrices.

$U_q(\mathfrak{sl}_2)$

Quantum $\mathfrak{sl}_2$, $U_q(\mathfrak{sl}_2)$, is the associative algebra over $\mathbb{C}(q^{\frac{1}{2}})$ generated by $E, F, K^{\pm 1}(= q^{\pm\frac{H}{2}})$ with relations

$$KE = qEK, \quad KF = q^{-1}FK, \quad [E, F] = \frac{K - K^{-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

For each $n \geq 1$, let V_n be the n-dimensional $U_q(\mathfrak{sl}_2)$-module with basis $\{v^0, \cdots, v^{n-1}\}$ on which the generators act by

$$Ev^j = [j]v^{j-1}, \quad Fv^j = [n-1-j]v^{j+1}, \quad Kv^j = q^{\frac{n-1-2j}{2}}v^j,$$

where $[m] := \frac{q^{\frac{m}{2}} - q^{-\frac{m}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$ for any $m \in \mathbb{Z}$.

$U_q(\mathfrak{sl}_2)$

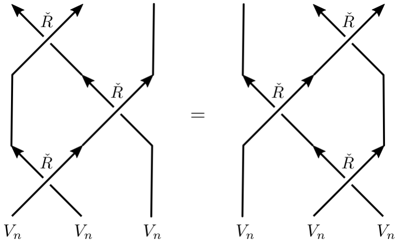
$U_q(\mathfrak{sl}_2)$ admits a universal R-matrix

$$R \in U_q(\mathfrak{sl}_2) \hat{\otimes} U_q(\mathfrak{sl}_2).$$

Applied to V_n, we obtain an automorphism $\check{R} \in \mathrm{Aut}(V_n \otimes V_n)$ given by

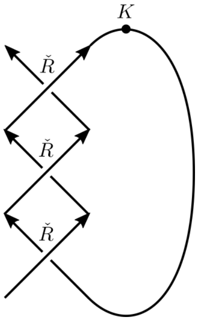
$$\check{R}(v^i \otimes v^j) = \sum_{k \geq 0} q^{\frac{n^2-1}{4} - \frac{(i-k+j+1)(n-1)}{2} + (i-k)j} \begin{bmatrix} i \\ k \end{bmatrix}_q \prod_{1 \leq l \leq k} (1 - q^{-n+j+l}) v^{j+k} \otimes v^{i-k}$$

that satisfies the Yang-Baxter equation.



Colored Jones polynomials

Let L be a link which can be presented as the closure of a braid β. Then the *n*-colored Jones polynomial $J_{L,n}(q)$ is the quantum trace of the automorphism given by the product of R-matrices.



Large-color R-matrix

In the large-color limit, the representation V_n becomes a Verma module V_∞ with generic highest (or lowest) weight.

The highest weight Verma module with highest weight $\lambda = \log_q x - 1$ has a basis $\{v^j\}_{j\geq 0}$ on which the generators of $U_q(\mathfrak{sl}_2)$ act by

$$Ev^j = [j]v^{j-1}, \quad Fv^j = [\lambda - j]v^{j+1}, \quad Kv^j = q^{\frac{\lambda-2j}{2}}v^j.$$

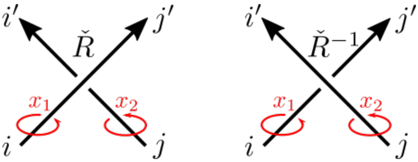
Let's call the R-matrix for these Verma modules the large-color R-matrix.

$$\check{R}(x_1,x_2)_{i,j}^{i',j'} = \delta_{i+j,i'+j'}\, q^{(j+\frac{1}{2})(j'+\frac{1}{2})} x_1^{\frac{-i'-j-1}{4}} x_2^{\frac{i-3j'-1}{4}} \begin{bmatrix} i \\ j' \end{bmatrix}_q \prod_{1\leq l\leq i-j'} (1 - q^{j+l}x_2^{-1})$$

Large-color R-matrix

$$\check{R}(x_1,x_2)_{i,j}^{i',j'} = \delta_{i+j,i'+j'} q^{(j+\frac{1}{2})(j'+\frac{1}{2})} x_1^{\frac{-i'-j-1}{4}} x_2^{\frac{i-3j'-1}{4}} \begin{bmatrix} i \\ j' \end{bmatrix}_q \prod_{1\le l\le i-j'} (1-q^{j+l}x_2^{-1})$$

Geometrically, x_1 and x_2 are the holonomy eigenvalues around the meridians of the two strands, in $SL_2(\mathbb{C})$ Chern-Simons theory at the abelian branch.



Extending the R-matrix

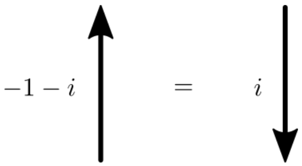
While the indices $i, j, i', j' \in \mathbb{Z}_{\geq 0}$ represent the basis vectors $\{v^i\}_{i \geq 0}$ of the highest weight Verma module V_∞, the R-matrix element $\check{R}(x_1, x_2)_{i,j}^{i',j'}$ makes sense for all $i, j, i', j' \in \mathbb{Z}$.

$$\check{R}(x,y)_{i,j}^{i',j'} =$$

$$\begin{cases} \delta_{i+j,i'+j'} q^{(j+\frac{1}{2})(j'+\frac{1}{2})} x^{-\frac{i'+j+1}{4}} y^{-\frac{3j'-i+1}{4}} \begin{bmatrix} i \\ i-j' \end{bmatrix}_q \prod_{1 \leq l \leq i-j'} (1-q^{j+l} y^{-1}) & \text{if } \begin{matrix} i \geq j' \geq 0 \\ \text{or} \\ 0 > i \geq j' \end{matrix} \\ \delta_{i+j,i'+j'} q^{(j+\frac{1}{2})(j'+\frac{1}{2})} x^{-\frac{i'+j+1}{4}} y^{-\frac{3j'-i+1}{4}} \begin{bmatrix} i \\ j' \end{bmatrix}_q \frac{1}{\prod_{0 \leq l \leq j'-i-1} (1-q^{j-l} y^{-1})} & \text{if } j' \geq 0 > i \\ 0 & \text{otherwise} \end{cases}$$

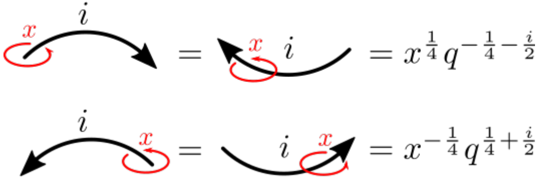
Meaning of the extension

The "inversion" of the domain of an index from $\mathbb{Z}_{\geq 0}$ to $\mathbb{Z}_{<0}$ corresponds to passing from the highest weight Verma module to the lowest weight Verma module, which can be thought of diagrammatically as changing the orientation of the arc of the link.



Meaning of the extension

There are natural cups and caps, depicted in the following figure:

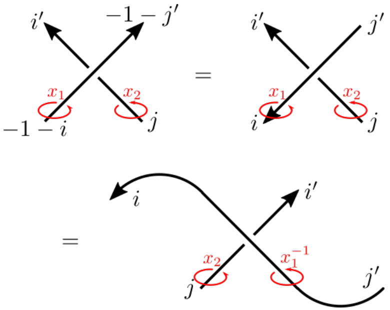


Meaning of the extension

The large-color R-matrix enjoys various symmetries that are natural from the diagrammatic point of view. For instance,

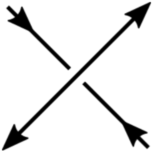
$$\check{R}(x_1, x_2)^{i', -1-j'}_{-1-i, j} = q^{\frac{i-j'}{2}} \check{R}^{-1}(x_2, x_1^{-1})^{i,i'}_{j,j'},$$

which can be diagrammatically understood as



New type of crossings

Note, by inverting some arcs, we also get some non-standard orientations such as



It is important to allow these non-standard orientations for better convergence of the state sum.

Inverted state sum

Verma modules are infinite-dimensional, so the state sum becomes an infinite sum, and one needs to be careful about its convergence.

Let's say a link L is "nice" if it admits a link diagram with an orientation datum such that the inverted state sum is convergent in $\mathbb{Z}[q, q^{-1}][[x^{-1}]]$.

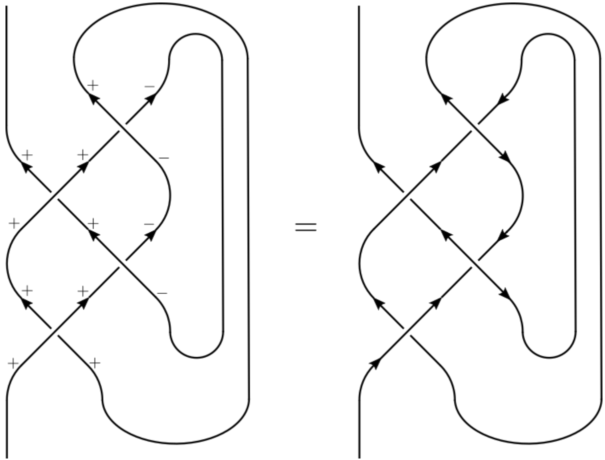
Theorem ([S.P. '21])

Gukov-Manolescu conjecture is true for any nice link.

- Homogeneous braid links are nice.
- All fibered knots up to 10 crossings are nice.

Inverted state sum - a detailed example: $\mathbf{4}_1$

For the figure-eight knot,



Inverted state sum - a detailed example: $\mathbf{4}_1$

$$
\begin{aligned}
F_{\mathbf{4}_1}(x,q) &= -(x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \sum_{\substack{m\geq 0\\ k<0}} \check{R}(x)^{m,0}_{0,m}\, \check{R}^{-1}(x)^{0,k}_{0,k}\, \check{R}(x)^{0,m}_{m,0}\, \check{R}^{-1}(x)^{m,k}_{m,k} \cdot x^{\frac{1}{2}} q^{-\frac{1}{2}-m} \cdot x^{\frac{1}{2}} q^{-\frac{1}{2}-k} \\
&= -x^{-\frac{1}{2}} - 2x^{-\frac{3}{2}} - (\tfrac{1}{q}+3+q)x^{-\frac{5}{2}} - (\tfrac{2}{q^2}+\tfrac{2}{q}+5+2q+2q^2)x^{-\frac{7}{2}} + O(x^{-\frac{9}{2}}) \\
&= -(x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \sum_{n\geq 0} \frac{1}{\prod_{0\leq j\leq n}(x + x^{-1} - q^j - q^{-j})}.
\end{aligned}
$$

Inverted state sum - an example of consistency check via surgery

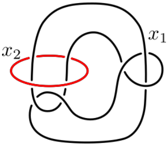


Figure: The link **L7a1**

$$F_{\mathbf{L7a1}}(x_1, x_2, q) = q^{\frac{1}{2}} \sum_{n,m \geq 0} f^{\mathbf{L7a1}}_{n,m} x_1^{-n-\frac{1}{2}} x_2^{-m-\frac{1}{2}},$$

where

$$f^{\mathbf{L7a1}} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \\ 2 & 2 & 2 & 2 & \cdots \\ 3-q & -\frac{1}{q^2}+4-q & -\frac{1}{q^3}-\frac{1}{q^2}+4 & -\frac{1}{q^4}-\frac{1}{q^3}-\frac{1}{q^2}+4+q^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

-1 (resp. $+1$) surgery gives $\mathbf{6}_3$ (resp. $\mathbf{6}_2$), and it can be checked that surgery formula works!

Inverted state sum

It is well known [Stallings '78] that for any link L, we can add an unknot component K (with any desired linking number with each component of L) so that $L' = L \cup K$ is a homogeneous braid link.

Corollary

For any 3-manifold Y, there is a link $L \subset Y$ for which $Y \setminus L$ is homeomorphic to the complement of a homogeneous braid link in S^3, so that we can compute $\hat{Z}_{Y \setminus L}(q)$.

Therefore, in order to mathematically construct $\hat{Z}$ for general 3-manifolds, it suffices to find a general Dehn surgery formula.

This problem is still open, but there are many hints, such as inverted Habiro series [S.P. '21] and the use of indefinite theta functions [Cheng, Ferrari, Sgroi '19]

Finiteness conjecture for fibered knots

Motivated by experimental evidences, we conjecture that:

Conjecture ([S.P. '21])

For any fibered knot K, the coefficients of $F_K(x, q)$ are in $\mathbb{Z}[q, q^{-1}]$.

Note, this is a well-defined conjecture, as any Laurent polynomial in $q = e^{\hbar}$ is uniquely determined by its power series expansion in $\hbar$.

This conjecture is also motivated from the enumerative geometry perspective: When the knot is fibered, we can shift the knot complement Lagrangian off from the zero section S^3 of T^*S^3, just like the conormal Lagrangian.

3. Final remarks

Final remarks

- These R-matrix expressions were also used to prove knots-quivers correspondence of F_K for some knots [Ekholm, Gruen, Gukov, Kucharski, S.P., Stosic, Sulkowski '21].
- Open questions:
  - Topological characterization of "nice links"?
  - How to go beyond fibered knots, e.g. $\overline{\mathbf{5}_2}$?
  - What is the appropriate category of line operators? It should contain Verma modules and their duals, as well as all the finite-dimensional modules.
  - $F_K(x, q)$ for fibered knots as traces of monodromy maps? In particular, does the symplectic representation of the mapping class group

$$\mathrm{MCG}(\Sigma_{g,1}) \circlearrowright \mathrm{Sym}(\mathbb{C}[H_1(\Sigma_{g,1})])$$

    admit a q-deformation?

4. Appendix: Proof sketch

Proof sketch

The idea is to use Foata-Zeilberger formula and combine it with Rozansky's proof of MMR conjecture.

The main ingredients are bosonic and fermionic random walks on the knot diagram.

Foata-Zeilberger formula

A cycle on an oriented graph is called primitive if it is not a power of any other cycle. A multi-cycle is an unordered tuple of cycles. A multi-cycle is called primitive if each of the cycles are primitive.

Theorem ([Foata, Zeilberger (1998)])

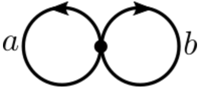
Let G be an oriented graph with weighted edges. Then the weighted sum of all primitive multi-cycles on G is

$$\frac{1}{\det(I - \mathcal{B})},$$

where $\mathcal{B}$ denotes the transition matrix.

Foata-Zeilberger formula: an example

Example:



$$
\begin{aligned}
&1\\
&+(a)+(b)\\
&+(ab)+(a)(b)+(a)^2+(b)^2\\
&+(a^2b)+(ab^2)+(a)(ab)+(b)(ab)+(a)^3+(a)^2(b)+(a)(b)^2+(b)^3\\
&+\cdots\\
&=\frac{1}{1-(a+b)}.
\end{aligned}
$$

Proof of the classical limit

In the classical limit $q = 1$, the (inverted) state sum becomes the weighted count of all primitive multi-cycles on the oriented graph whose vertices are the set of internal segments. Therefore,

$$Z^{\mathrm{inv}}(D) = \frac{1}{\det(I - \mathcal{B}_{\mathrm{inv}})},$$

which is the partition function of a random walk of free bosons on the knot diagram.

On the other hand, the denominator can be interpreted as the partition function of a random walk of free fermions on the knot diagram, which is the signed weighted count of all simple multi-cycles. (A multi-cycle is called simple if it uses each edge at most once.)

Proof of the classical limit

The main lemma of the proof is to show that there is a nice one-to-one correspondence between the simple multi-cycles in the weighted oriented graphs before the inversion and after the inversion, that preserves the weight.

It follows that, up to a simple factor,

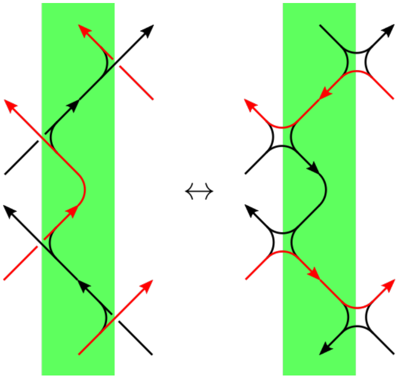
$$\det(I - \mathcal{B}_{\mathrm{inv}}) = \det(I - \mathcal{B}) = \Delta_K(x).$$

This establishes

$$\frac{1}{\det(I - \mathcal{B}_{\mathrm{inv}})} = \frac{1}{\det(I - \mathcal{B})} = \frac{1}{\Delta_K(x)}.$$

Note, the left-hand side is a power series in x^{-1}, whereas the right-hand side is a power series in $(1 - x)$.

Proof of the classical limit



Proof of the higher perturbative terms

To show that all the higher perturbative terms agree with the MMR expansion, we use the idea that Rozansky used in his proof of MMR conjecture.

That is, we use the parametrized R-matrices:

$$\mathrm{R}(\alpha,\beta,\gamma)_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \binom{i}{j'} \alpha^{j} \beta^{j'} \gamma^{i-j'},$$
$$\mathrm{R}^{-1}(\alpha,\beta,\gamma)_{i,j}^{i',j'} = \delta_{i+j,i'+j'} \binom{j}{i'} \alpha^{i} \beta^{i'} \gamma^{j-i'}.$$

They are *not* inverses to each other, and they do *not* satisfy the Yang-Baxter equation. They just provide a generic form of a state sum that can be computed by a weighted count of primitive multi-cycles.

Proof of the higher perturbative terms

The usual R-matrices in the classical limit are specializations of these parametrized R-matrices:

$$\check{R}(x)_{i,j}^{i',j'}\Big|_{q=1} = \mathrm{R}(\alpha,\beta,\gamma)_{i,j}^{i',j'}\Big|_{\alpha=x^{-\frac{1}{2}},\beta=x^{-\frac{1}{2}},\gamma=1-x^{-1}},$$

$$\check{R}^{-1}(x)_{i,j}^{i',j'}\Big|_{q=1} = \mathrm{R}^{-1}(\alpha,\beta,\gamma)_{i,j}^{i',j'}\Big|_{\alpha=x^{\frac{1}{2}},\beta=x^{\frac{1}{2}},\gamma=1-x}.$$

Higher perturbative terms of the R-matrix can be obtained by applying some differential operators to the parametrized R-matrix and then specializing the parameters. That is, there is a sequence of differential operators D_n in the parameters $\{\alpha_c,\beta_c,\gamma_c\}_{c\in\{\text{crossings}\}}$ such that the $\hbar^n$-coefficient of the MMR expansion is

$$D_n\,\mathrm{Z}(\{\alpha\},\{\beta\},\{\gamma\})\Big|_{\text{specialize as above}}$$

Proof of the higher perturbative terms

The differential operators don't change upon extension of R-matrix, so it follows that the higher perturbative terms of the MMR expansion remain the same after inversion. This proves the theorem.