Composite-boson formalism for one-dimensional models

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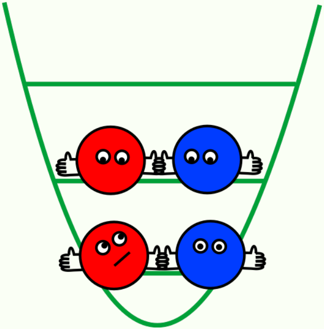
Warnings

- I'm not a condensed matter physicist
- I'm rather a quantum optics/information person
- This talk is about a field that is quite new to me
- I'm happy to get feedback!

The usual bosons are not “true” bosons

- Take composite particles with an even number of fermions. Exchange two of these particles, and the wave function must be symmetric.
- Is this it? How can you have Bose-Einstein condensation if particles can’t really occupy the same state?

How can bosons made of fermions condense?



Watch out!
Very misleading
pictorial representation

- Condensation refers to degrees of freedom of the composite particle as a whole.
- Internal degrees of freedom must be playing a role.
- ...

Towards a rigorous treatment of composite bosons

For a composite particle,
"boson-like behaviour" is a state-dependent property

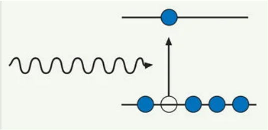
Towards a rigorous treatment of composite bosons

Monique Combescot

Coboson formalism

[Combescot, Betbeder-Matibet & Dubin, Phys. Rep. 463, 215 (2008)]

- Motivation: semiconductor physics - excitons made of holes and electrons.
- A “bosonic” exciton has creation operator $c^\dagger$, so that $|\psi\rangle = c^\dagger|0\rangle$ is the single-exciton state with $|0\rangle$ the vacuum.



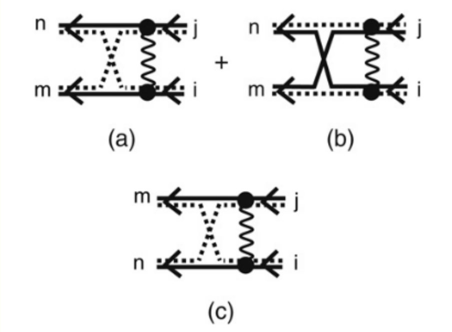
[Image from photonics.com, article by David Snoke]

How bosonic is $c^\dagger$?

How can we properly describe compositeness?

Towards a rigorous treatment of composite bosons

[Combescot, Betbeder-Matibet & Dubin, Phys. Rep. 463, 215 (2008)]



“Shiva-diagrams” represent interaction scatterings and fermion exchanges.

Composite bosons as entangled systems

PHYSICAL REVIEW A **71**, 034306 (2005)

Quantum entanglement as an interpretation of bosonic character in composite two-particle systems

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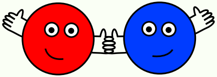
(Received 31 October 2004; published 21 March 2005)

We consider a composite particle formed by two fermions or two bosons. We discover that composite behavior is closely related to the quantum entanglement between the constituent particles. By analyzing the properties of creation and annihilation operators, we show that bosonic character emerges if the constituent particles become strongly entangled. Such a connection is demonstrated explicitly in a class of two-particle wave functions.

Composite bosons as entangled systems

We focus on cobosons
made of two different fermions:



$$|1\rangle = \sum_{n=1}^{S} \sqrt{\lambda_n}\, |\psi_n\rangle \otimes |\phi_n\rangle \quad \Rightarrow \quad c^\dagger = \sum_{n=1}^{S} \sqrt{\lambda_n}\, a_n^\dagger b_n^\dagger$$

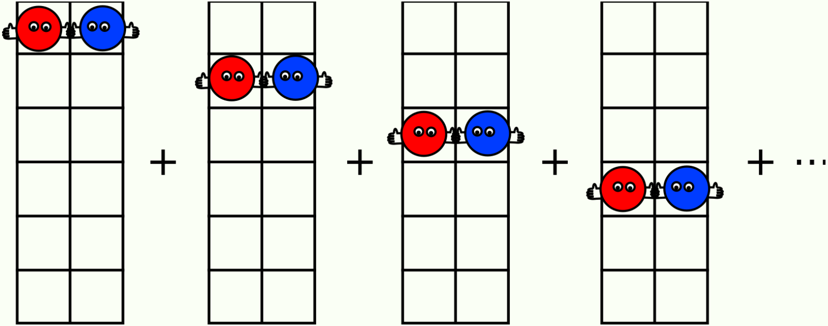
$$|N\rangle = \frac{\left(c^\dagger\right)^N}{\sqrt{N!\,(1-\delta_N)}}|0\rangle.$$

$\delta_2 = \mathcal{P}$, $\mathcal{P}$: one-fermion purity.

More entanglement gives “better bosons”.

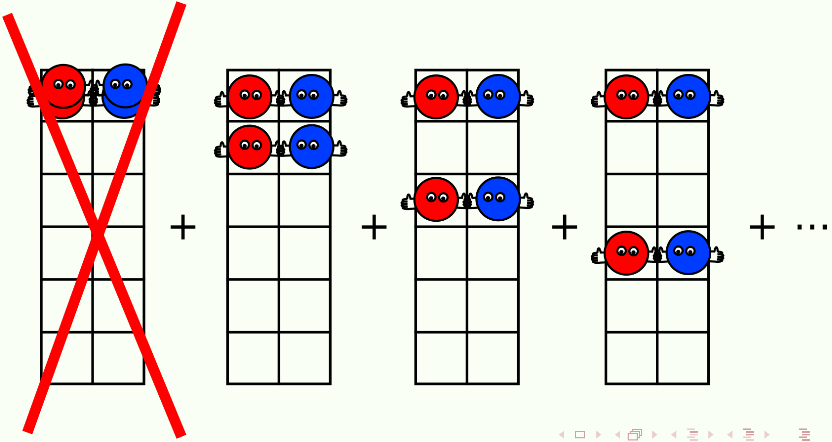
Intuitive picture - one pair

$$|1\rangle = c^\dagger|0\rangle, \quad c^\dagger = \sum_{n=1}^{S} \frac{1}{\sqrt{S}} a_n^\dagger b_n^\dagger, \quad \mathcal{K} = S, \quad \mathcal{P} = 1/S$$

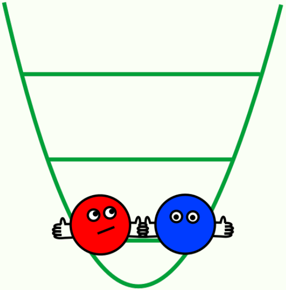


Intuitive picture - two pairs

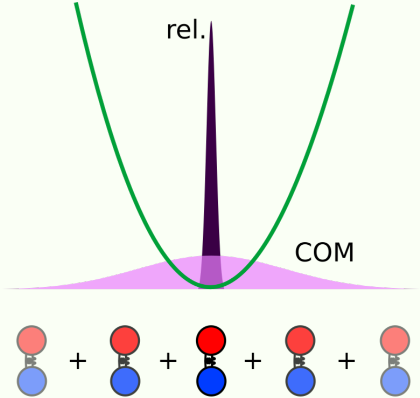
$$|2\rangle \propto (c^\dagger)^2|0\rangle, \quad c^\dagger = \sum_{n=1}^{S} \frac{1}{\sqrt{S}} a_n^\dagger b_n^\dagger$$



How can bosons made of fermions condense?



Watch out!
Very misleading
pictorial representation



Careful!

"To behave like a good boson" can mean two different things:

- That creation and annihilation operators are boson-like (this depends on the entanglement between constituents).
- That the ground state is condensate-like:

  If $c^\dagger|0\rangle$ is the 1-coboson ground state, then $|N\rangle \propto (c^\dagger)^N|0\rangle$ is a good approximation of the N-coboson ground state.

  This is a different issue! We call it "condensate ansatz"

When do composite bosons “behave well”?

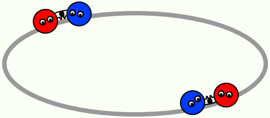
Cobosons made of two fermions are expected to have boson-like operators **and** a condensate-like ground state when:

- Constituents are highly entangled
- The system is dilute
- Interactions are not too long-ranged

This makes the low-T many-body description **way** simpler... and coboson theory in its most compact form can be used.

One-dimensional systems are different

- Two fermionic species on a ring, with δ-type attractive interactions.
- (solvable with Bethe ansatz)



UN SYSTEME A UNE DIMENSION DE FERMIONS EN INTERACTION

M. GAUDIN
Service de Physique Théorique, Centre d'Etudes Nucléaires de Saclay, Gif-sur-Yvette

Reçu le 24 Novembre 1966

PHYSICAL REVIEW VOLUME 161, NUMBER 1 5 SEPTEMBER 1967

Delta-Function Fermi Gas with Two-Spin Deviates*

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AND
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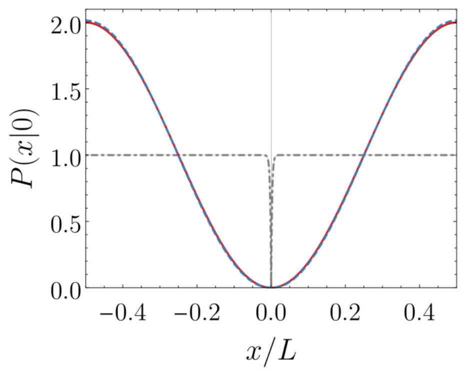
Physics Department, Northeastern University, Boston, Massachusetts

The infinite attraction limit

- For infinite attraction fermion pairs are point-like hard-core bosons.
- The ground state then becomes much easier.
- For two bound pairs on the ring:

$$\psi(X_1, X_2) \propto |\sin[\pi(X_1 - X_2)/L]|$$

Spatial correlations for two strongly bound fermion pairs



Conditional probability of finding a fermion at position x if an identical fermion was found at $x = 0$ (strongly attractive regime).

Clearly the ground state is not condensate-like in this case...

So when do composite bosons “behave well”?

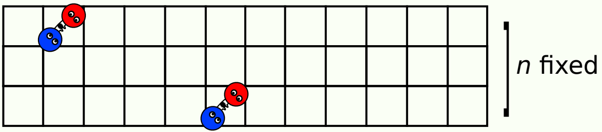
Cobosons made of two fermions can be expected to have boson-like operators and a condensate-like ground state when:

- Constituents are highly entangled
- The system is dilute
- Interactions are not too long-ranged

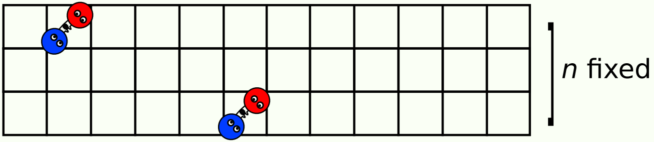
  and
- The system is not 1D

Why this is not entirely trivial

- Some people approach coboson theory from quantum information or quantum optics areas.
- The role of dimensionality is not clear in the coboson literature.
- And there are simple lattice models where the ansatz fails even though they are not strictly 1D:



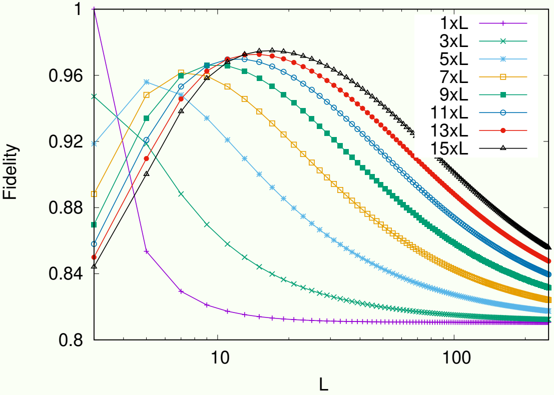
Let’s have a better look at that case



$$H = -U \sum_j a_j^\dagger a_j b_j^\dagger b_j + \frac{J}{2} \sum_{<j,k>} (a_j^\dagger a_k + b_j^\dagger b_k + H.c.)$$

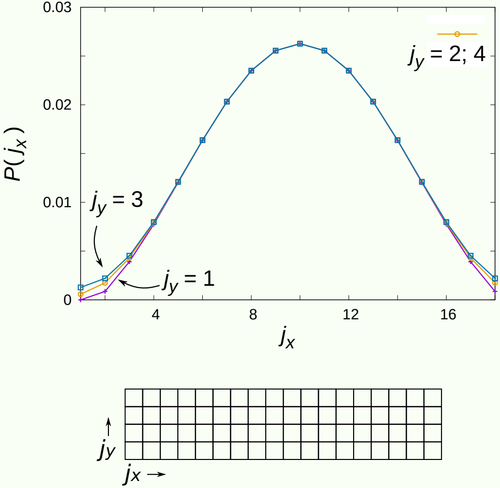
Periodic boundary conditions; $U \gg J$ so pairs hop as a whole.

Fidelity between ground state and condensate ansatz for two pairs



Fidelity for $n \times L$ torus.

Spatial correlations in “ladder” models



Two-pair correlations in the ground state of the 4×18 lattice: conditional probability to find one pair in each site given one pair at site (1,1).

Who cares about two pairs? Since when is that many-body?

- Many results from the condensate ansatz $|N\rangle$ can be formulated as a power expansion in density.
- For low density, the dominant terms are calculated from one- and two-pair expectation values.
- If the two-pair description is not reliable, things don't get better with more particles.

The coboson formalism can be useful even if the condensate ansatz fails

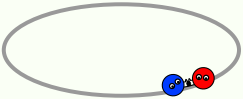
as has already been shown for the case of Cooper pairs:

- M. Combescot and S.-Y. Shiau, Excitons and Cooper pairs: two composite bosons in many-body physics (Oxford Univ. Press, 2015).
- M. Combescot and G. Zhu, EPJ B 79, 263 (2011).
- M. Combescot, W. Pogosov, and O. Betbeder-Matibet, Phys. C: Supercond. 485, 47–57 (2013).

The coboson formalism can be useful also in 1D

- Let's go back to the continuous model on the ring...
  The Bethe ansatz leads to states that are rather ugly to write down.
- The "coboson basis" can be very convenient for strong attraction - but you need more than one creation operator!
- The basics are in [Combescot et al., Phys. Rep. (2008)].
  But sticking to the recipe is not always the best!

Building the coboson basis (I)



First look at the case of one pair.
Solve in terms of center-of-mass and relative coordinates:

$$\psi_{CM}(x_{CM})^{(K)} = \frac{1}{\sqrt{L}} e^{iKx_{CM}}, \quad K = 2\pi k/L\,,$$

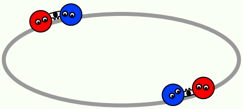
$$\psi_r(x_r) \simeq \sqrt{\lambda}\; e^{-\lambda|x_r|}\,.$$

Coboson creation operator

for one pair with momentum K and energy $E_K = \frac{\hbar^2 K^2}{4m} - \frac{\hbar^2\lambda^2}{m}$:

$$\boxed{B_K^\dagger = \sqrt{\frac{\lambda}{L}} \int dx_{CM}\, dx_r\; e^{iKx_{CM}}\, e^{-\lambda|x_r|}\, \Psi_a^\dagger(x_{CM} + x_r/2)\, \Psi_b^\dagger(x_{CM} - x_r/2)}$$

Building the coboson basis (II)



- Then we can form states of two pairs: $B^{\dagger}_{K_1} B^{\dagger}_{K_2} |v\rangle$
  (with total momentum $K_1 + K_2$)

- These states are neither normalized nor orthogonal!
  We need an overlap matrix:

  $$S_{K_1,K_2;K_3,K_4} = \langle v| B_{K_1} B_{K_2} B^{\dagger}_{K_3} B^{\dagger}_{K_4} |v\rangle$$

  The overlaps are spatial integrals.

- For the ground state we only need $K_1 + K_2 = 0$.

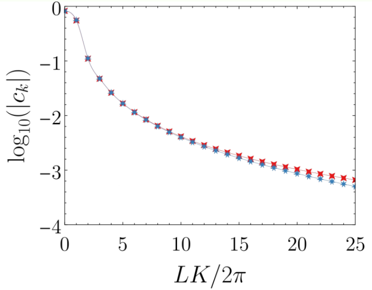
Using the coboson basis

- Also the Hamiltonian matrix elements are integrals.
- For the ring, these integrals can be solved exactly.
- The ground state is found from a generalized eigenvalue equation: $Hc = ESc$
  with c the coefficients of the state in the coboson basis.
- Careful! We have an energy cutoff... the basis needs to be truncated consistently (the states we chose only make sense for strong coupling).

An analytical approximate solution

- We know the ground state for infinite attraction.
- We can calculate the coefficients for that limit exactly.
- Then we can use the same values for strong but finite coupling: $c_K(\lambda) \simeq c_K(\infty)$

  (but $B_K^\dagger(\lambda) \neq B_K^\dagger(\infty)$)



Blue: numerical calculation for $\lambda L = 200$; red: analytical result for infinite attraction.

Another example: harmonic 1D trap

- In terms of center-of-mass and relative coordinates:

$$\psi_{CM}(x_{CM})^{(n)} = \varphi_n(x_{CM}) \,,$$

(the harmonic oscillator eigenfunctions),

$$\psi_r(x_r) \simeq \sqrt{\lambda}\; e^{-\lambda|x_r|} \,.$$

- **Coboson creation operator**:

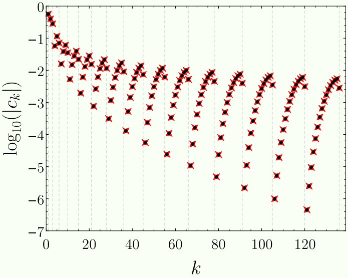
$$\boxed{B_n^\dagger = \sqrt{\lambda} \int dx_{CM}\, dx_r\, \varphi_n(x_{CM}) e^{-\lambda|x_r|} \Psi_a^\dagger(x_{CM} + x_r/2)\, \Psi_b^\dagger(x_{CM} - x_r/2)}$$

The rest is as before, with Taylor expansions for ugly integrals.

Again the infinite attraction limit works well

- Again we can calculate the coefficients for infinite attraction.
- Then we can use the same coefficients for strong but finite coupling:

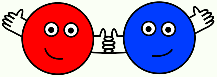
$$|GS\rangle_{N=2} \simeq \sum_{m,n} c_{mn}(\infty) B_m^\dagger(\lambda) B_n^\dagger(\lambda)|0\rangle$$



Black: numerical calculation for $\lambda x_\omega = 30$; red: analytical result for infinite attraction.

Summing up:

- Fermion pairs ressemble non-interacting bosons for:
  - Highly entangled constituents
  - Dilute systems
  - Short-ranged interactions
  - Dimension $d > 1$



- The coboson formalism can be useful in the strong attraction regime even if the condensate ansatz fails!

  (The same ideas can be used for more than 2 pairs.)

More on this line of work:

SciPost Phys. Core 6, 012 (2023)
PRA 105, 013302 (2022)
PRA 100, 012309 (2019)

E. Cuestas

M. Jiménez

P. Céspedes

E. Rufeil-Fiori

A. Bouvrie

A. Majtey

Thank you for your attention!