Karlsruhe Institute of Technology

Disorder and quantum coherence:
From ergodicity to Anderson and many-body localization

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Plan

PART 1 (Lectures 1-3)

- **I. Disorder and localization**
  - disorder: diagrammatics, quantum interference, localization
  - field theory: non-linear σ-model; quasi-1D geometry: exact solution
- **II. Criticality and multifractality**
  - RG, metal-insulator transition, criticality
  - Multifractality of wave functions
- **III. Symmetries and topologies**
  - symmetry classification of disordered electronic systems
  - topological insulators and superconductors; disordered Dirac fermions

PART 2 (Lectures 4-5)

- **IV. Interaction**
  - electron-electron-interaction: dephasing and renormalization
  - Interplay of disorder and interaction; superconductor-insulator transition
- **V. Localization on tree-like graphs (Random Regular Graphs)**
- **VI. Many-body localization**

PART 2

Electron-electron interaction effects

- **Renormalization**

  Virtual processes, energy transfer $\gtrsim T$,
  become stronger when T is lowered

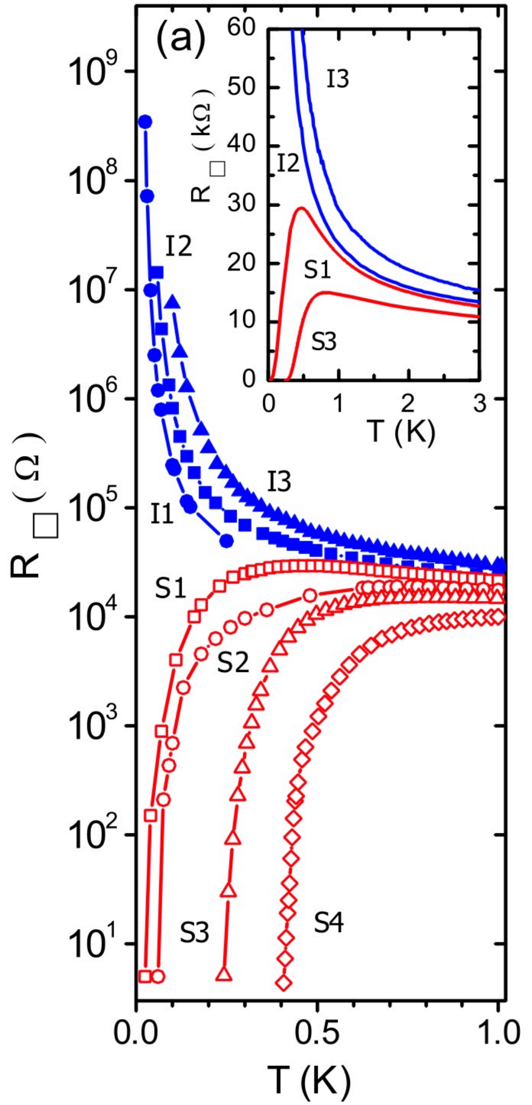
  - **mutual renormalization of resistivity and interaction,**
  - **zero-T phase diagram and quantum phase transitions**
  - **effect of disorder on superconducting and magnetic instabilities**

- **Dephasing / Decay**

  Real inelastic scattering processes, energy transfer $\lesssim T$,
  become weaker when T is lowered

  - **dephasing of quantum interference**
  - **decay of single-particle excitations**
  - **finite-T broadening of localization quantum phase transitions**
  - **$T > 0$ many-body delocalization**

Superconductor-insulator transition (SIT) in 2D disordered films



experiment: TiN films

Baturina et al, PRL'07

Superconductivity vs Anderson localization

related talk on Tuesday by E. Andriyakhina

SIT in disordered 2D system with short-range interaction

Burmistrov, Gornyi, ADM, 2012 ... 2015

σ model with interaction — Finkelstein 1983 (Coulomb interaction)

short range interaction $\longrightarrow$ RG for 4 coupling constants:
resistance t, interactions γ_s (singlet), γ_t (triplet), γ_c (Cooper)

weak interactions

$$\frac{d}{dy}\begin{pmatrix}\gamma_s\\ \gamma_t\\ \gamma_c\end{pmatrix} = -\frac{t}{2}\begin{pmatrix}1 & 3 & 2\\ 1 & -1 & -2\\ 1 & -3 & 0\end{pmatrix}\begin{pmatrix}\gamma_s\\ \gamma_t\\ \gamma_c\end{pmatrix} - \begin{pmatrix}0\\ 0\\ 2\gamma_c^2\end{pmatrix}; \qquad \frac{dt}{dy} = t^2$$

Eigenvalues of the linear problem (without BCS term γ_c^2):
$\lambda = 2t, \quad \lambda' = -t$

2D system is "weakly critical" (on scales shorter than ξ)

The eigenvalues λ, λ' are exactly multifractal exponents:

$\lambda \equiv -\Delta_2 > 0$ (RG relevant), $\quad \lambda' = -\Delta_{(1,1)} < 0$ (RG irrelevant)

$\longrightarrow$ enhnacement of interaction, and consequently of superconductivity, by multifractality

SIT in disordered 2D system

$$T_c \sim \exp\{-1/|\gamma_{c,0}|\} \quad \text{(BCS)}\ , \qquad G_0 \gtrsim |\gamma_0|^{-1}$$

$$T_c \sim \exp\{-2G_0\}\ , \qquad |\gamma_0|^{-1/2} \lesssim G_0 \lesssim |\gamma_0|^{-1}$$

$$\text{insulator}\ , \qquad G_0 \lesssim |\gamma_0|^{-1/2}$$

$$\gamma_0 = \tfrac{1}{6}(-\gamma_{s,0} + 3\gamma_{t,0} + 2\gamma_{c,0}) < 0$$ – bare interaction

G_0 – bare conductivity

Non-monotonic dependence

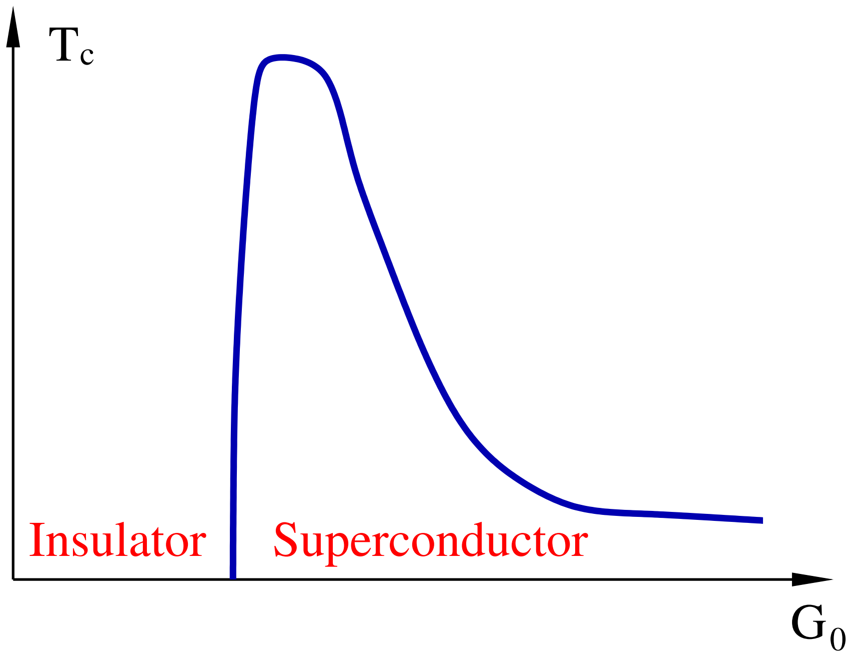
of T_c on disorder (G_0)

Exponentially strong enhancement

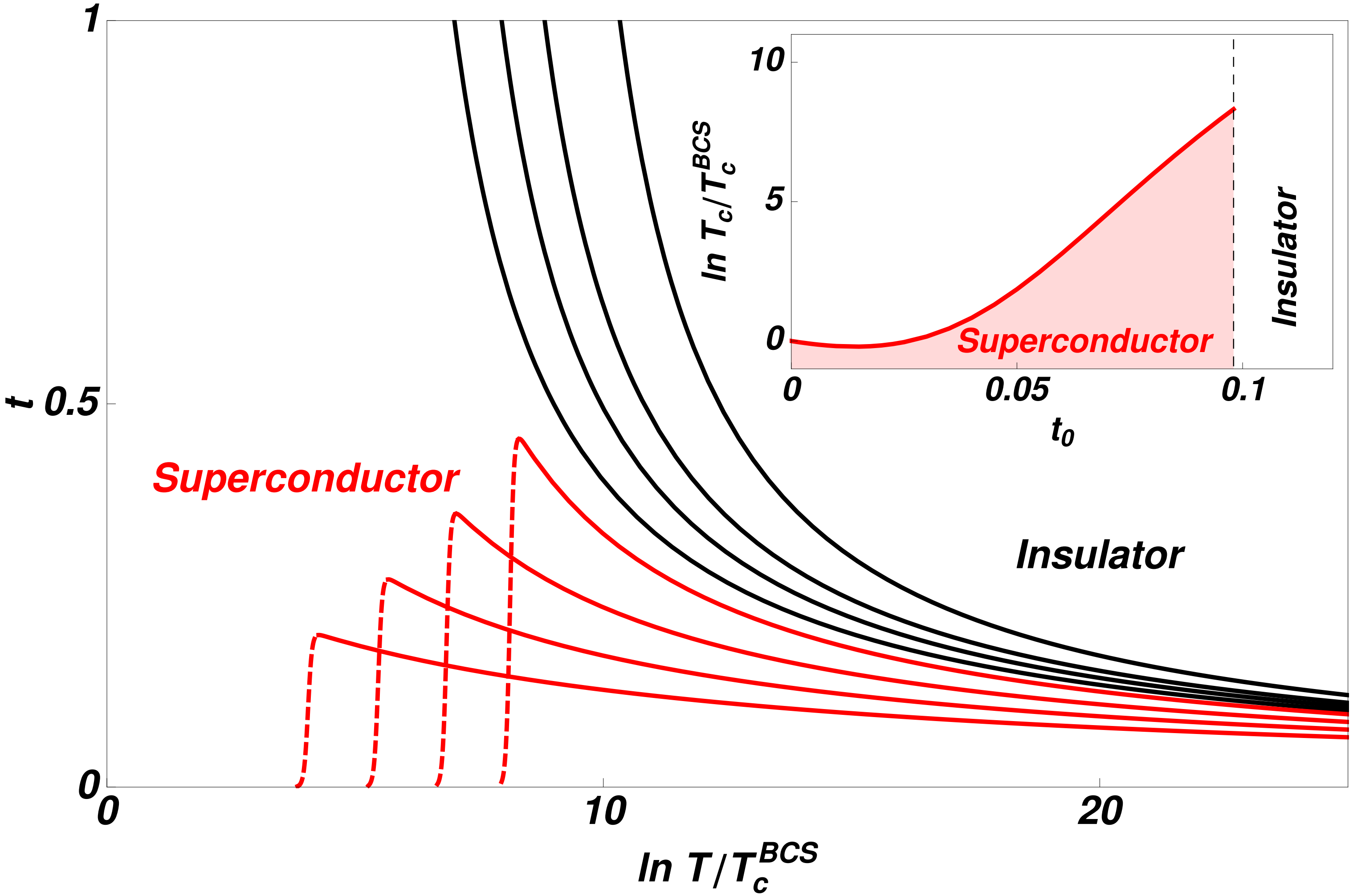
of superconductivity by multifractality

in the intermediate disorder range,

$|\gamma_0|^{-1/2} \lesssim G_0 \lesssim |\gamma_0|^{-1}$



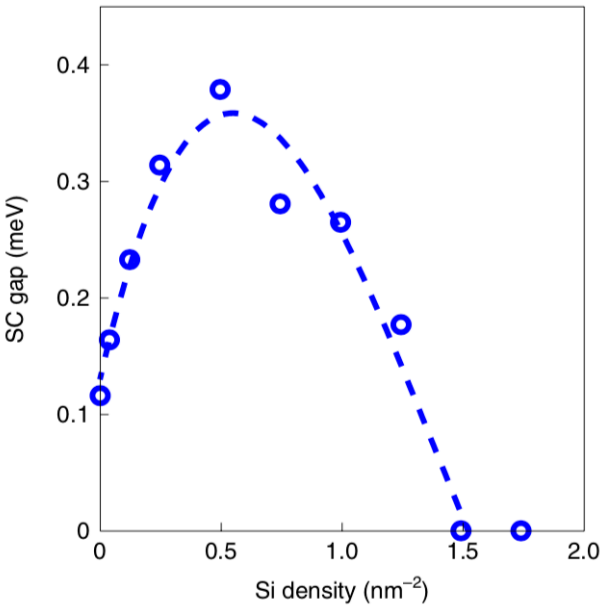
SIT in disordered 2D system

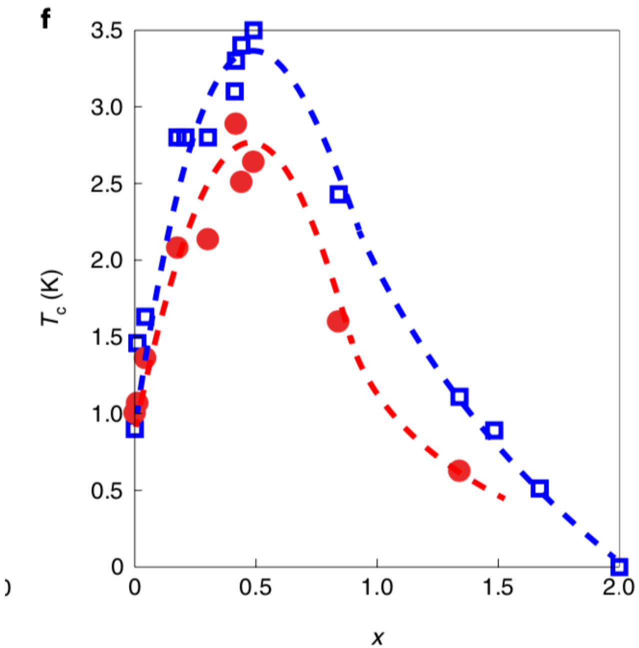


Burmistrov, Gornyi, ADM, 2012 ... 2015 ... 2021

Andriyakhina, Burmistrov, 2022

Experiments: Superconductivity in disordered $NbSe_2$





Zhao et al, Nature Physics 2019

Enhancement of T_c by disorder in monolayer $NbSe_2$ superconductor

explained as enhancement of superconductivity due to multifractality

Related experiment with analogous result:

Rubio-Verdu et al, Nano Letters 2020

Dephasing at metal-insulator and quantum Hall transitions

e-e interaction $\longrightarrow$ dephasing at finite T

$\longrightarrow$ smearing of the transition

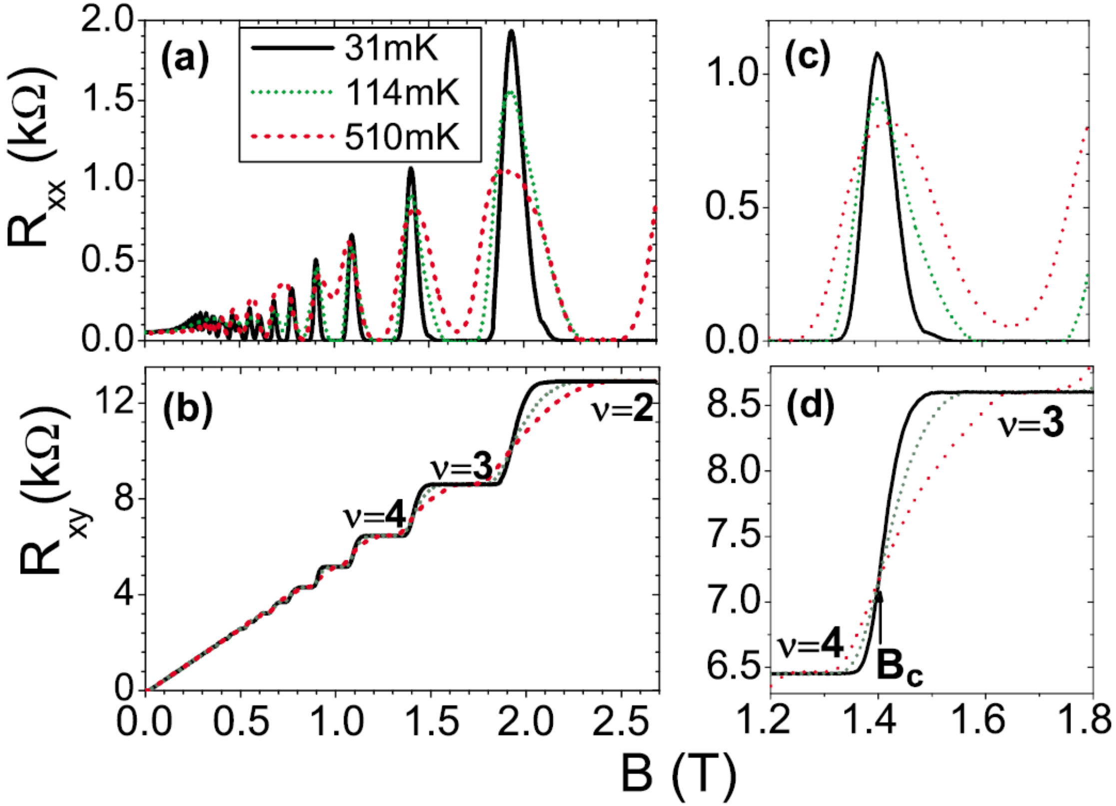
dephasing length $L_\phi \propto T^{-1/z_T}$, local. length $\xi \propto |n - n_c|^{-\nu}$

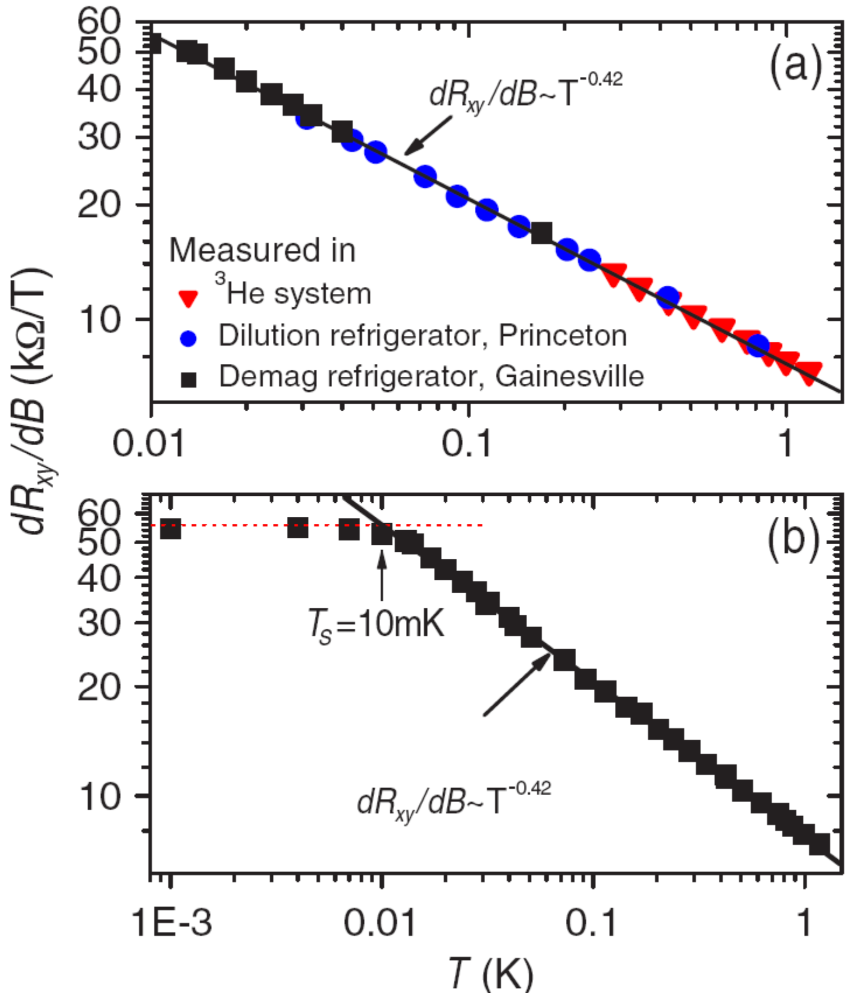
$\longrightarrow$ transition width $\delta n \propto T^{\kappa}$, $\kappa = 1/\nu z_T$

Consider short-range e-e interaction,
which is an appropriate model in various situations:

- long-range Coulomb interaction negligible
  because of large dielectric constant
- 2D: screening by metallic gate
- interacting neutral particles (e.g. cold atoms)

Temperature scaling of quantum Hall transition

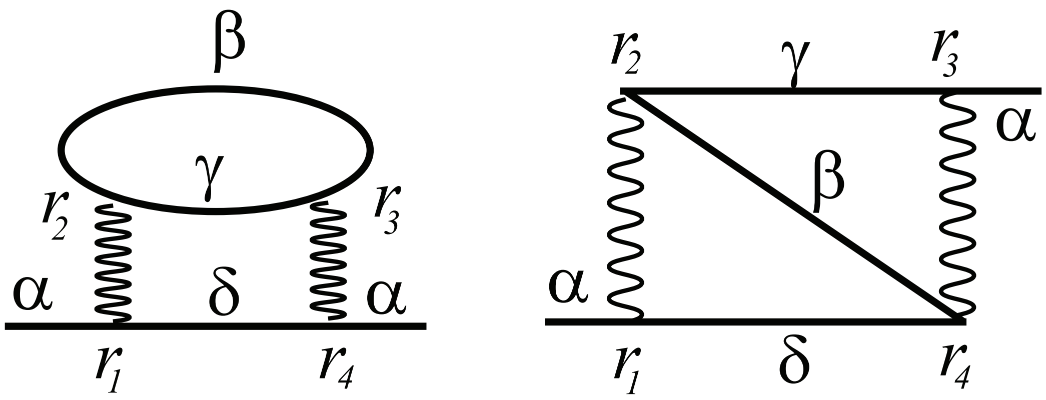




Transition width exponent $\kappa = 1/\nu z_T = 0.42 \pm 0.01$

Wei, Tsui, Paalanen, Pruisken, PRL'88 ; Li et al., PRL'05, PRL'09

Interaction-induced dephasing



Short-range interaction:

$\longrightarrow$ **dephasing rate** $\tau_\phi^{-1} \propto T^p$ **with** $p = 1 + 2\Delta_{(1,1)}/d$

$\Delta_{(1,1)} > 0 \longrightarrow$ **dephasing suppressed by multifractality**

dephasing length $L_\phi \propto T^{-1/z_T}$ $z_T = d/p$

Transition width exponent $\kappa = \frac{1}{\nu z_T} = \frac{1 + 2\Delta_{(1,1)}/d}{\nu d}$

Lee, Wang, PRL 1996 ; Wang, Fisher, Girvin, Chalker, PRB 2000

Burmistrov, Bera, Evers, Gornyi, ADM, Annals Phys. 2011

Scaling at QH transition: Theory and experiment

- **Theory (short-range interaction):**

$\longrightarrow$ **dephasing rate** $\tau_\phi^{-1} \propto T^p$ **with** $p = 1 + 2\Delta_{(1,1)}/d$

dephasing length $L_\phi \propto T^{-1/z_T}$ $z_T = d/p$

Transition width exponent $$\kappa = \frac{1}{\nu z_T} = \frac{1 + 2\Delta_{(1,1)}/d}{\nu d}$$

$\Delta_{(1,1)} \simeq 0.62 \longrightarrow p \simeq 1.62 \longrightarrow z_T \simeq 1.23$

$\nu \simeq 2.59$ **(Ohtsuki, Slevin '09)** $\longrightarrow \kappa \simeq 0.314$

- **Experiment (long-range $1/r$ Coulomb interaction):**

$\kappa = 0.42 \pm 0.01$

Difference in κ fully consistent with short-range and Coulomb ($1/r$) problems being in different universality classes

Delocalization by inelastic processes

Inelastic processes $\longrightarrow$ dephasing of quantum interference
$\longrightarrow$ cutoff for localization effects $\longrightarrow$ finite conductivity

Low-T transport is via hopping over localized states

External bath with continuous spectrum (e.g., phonons)
$\longrightarrow$ takes care about mismatch in energies of localized states

Problem of "many-body localization" (MBL):

assume that all single-particle states are localized
(e.g., 1D or quasi-1D, or 2D, or a tight-binding model of any d with sufficiently strong disorder)

What happens at finite T in the absence of external bath?
Localization, conductivity, other observables – ?

Can the system serve as its own thermal bath?

MBL vs Ergodicity

Problem of many-body localization (MBL):

Can the system serve as its own thermal bath?

Closely related questions:

Ergodicity? Thermalization?

These questions can be posed also for a many-body quantum system without any spatial structure: "quantum dot".

In this case, one can speak about Fock-space MBL.

Ergodicity and MBL in excited states of many-body systems

Spatially extended systems with short-range interaction

Gornyi, Mirlin, Polyakov, PRL 95, 206603 (2005)
Basko, Aleiner, Altshuler, Ann Phys 321, 1126 (2006)
Oganesyan, Huse, PRB 75, 155111 (2007)

Quantum dots

Altshuler, Gefen, Kamenev, Levitov, PRL 78, 2803 (1997)
Mirlin, Fyodorov, PRB 56, 13393 (1997)
Jacquod, Shepelyansky, PRL 79, 1837 (1997)

Spatially extended systems with power-law interaction

Burin, arXiv:cond-mat/0611387; PRB 91, 094202 (2015)
Yao, Laumann, Gopalakrishnan, Knap, Müller, Demler, Lukin, PRL 2014
Gutman, Protopopov, Burin, Gornyi, Santos, Mirlin, PRB 93, 245427 (2016)

and many further papers

Questions that are addressed:

- MBL transition – ? Is the critical disorder independent on the system size L, or else, how does it scale with L?
- Properties of the localized and delocalized phases? Transport; dynamics; statistics of various observables; ...
- Can an intermediate "non-ergodic delocalized" phase emerge?
- Critical behavior at the transition? Properties of the critical regime?

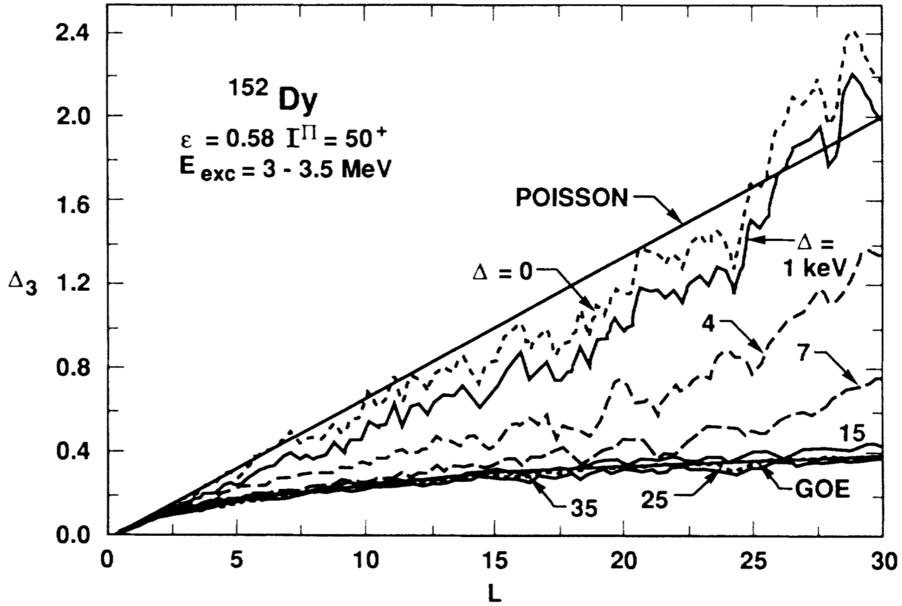
Onset of quantum chaos in nuclei

Aberg '90, '92

Two-body random interaction model, highly excited states;

Level statistics:

crossover from Poisson to Wigner-Dyson



Criterion conjectured on the basis of numerics:

By comparing these results to the average level distances shown in fig. 6 we conclude that chaos seems to set in when the average size of the two-body matrix element is

$$\Delta \approx \left(\tfrac{1}{2} - \tfrac{1}{3}\right) \bar{d}_{2p2h} \,. \tag{22}$$

Δ – interaction matrix element,

d_{2p2h} – level spacing of Fock-space basis states directly connected to the given one by interaction

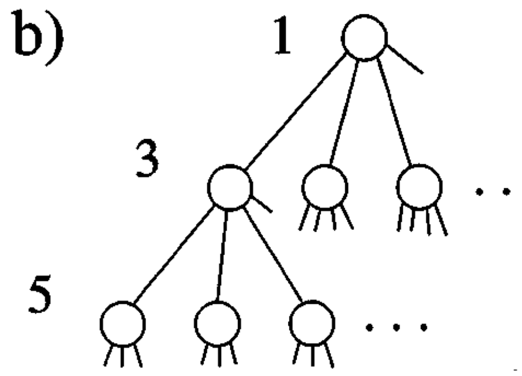
Fock-space many-body localization in quantum dots

Altshuler, Gefen, Kamenev, Levitov '97

Two-body random interaction model,

hot electron decay;

Fock space localization:
approximation by Cayley tree



MBL transition in quantum dots with increasing energy:
from Fock-space localized (no ergodicity, Poisson)
to delocalized (ergodicity, Wigner-Dyson) states

ADM, Fyodorov '97

Jacquod, Shepelyansky '97

...

Gornyi, ADM, Polyakov, Burin '17

Monteiro, Micklitz, Tezuka, Altland '20

Herre, Karcher, Tikhonov, ADM '23

Relation to localization on tree-like graphs:
Random regular graphs (RRG)

Many-body quantum dot models

Fermionic quantum dot

(in the basis of exact eigenstates of the non-interacting problem):

$$\hat{H} = \sum_i \varepsilon_i \hat{c}_i^\dagger \hat{c}_i + \sum_{ijkl} V_{ijkl} \left(\hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l + h.c. \right).$$

ε_i – random energies from $[-W, W]$

Interaction matrix elements V_{ijkl} — Gaussian random variables with zero mean and variance unity

Consider $n/2$ fermions occupying n orbitals.

Spin quantum dot:

$$\hat{H} = \sum_{i=1}^{n} \varepsilon_i \hat{S}_i^z + \sum_{i,j=1}^{n} \sum_{\alpha,\beta \in \{x,y,z\}} V_{ij}^{\alpha\beta} \left(\hat{S}_i^\alpha \hat{S}_j^\beta + h.c. \right).$$

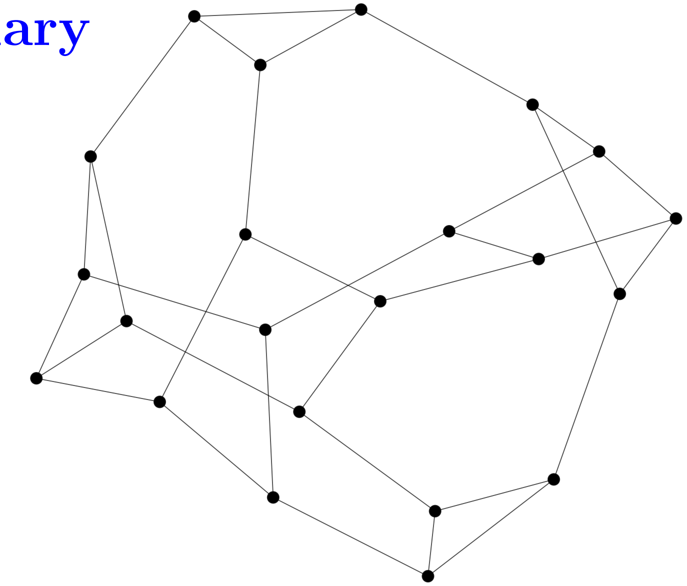
ε_i — random fields from $[-W, W]$

interaction matrix elements $V_{ij}^{\alpha\beta}$ – gaussian random variables with zero mean and variance unity

Anderson localization on random regular graphs (RRG)

Random regular graph – random graph with constant connectivity $m+1$

Locally tree-like (as Bethe lattice) but without boundary



Typical size of loops $\sim \ln N$

$$\mathcal{H} = \sum_{\langle i,j \rangle} \left(c_i^+ c_j + c_j^+ c_i \right) + \sum_{i=1} \varepsilon_i c_i^+ c_i$$

$\varepsilon_i \longrightarrow$ **disorder** W

Relation to the MBL problem:

Hilbert space size $N \sim m^L$ **where** L **is "linear size"**

Sites $\longleftrightarrow$ **many-body basis states,** **links** $\longleftrightarrow$ **interaction matrix elements**

Approaches to Anderson model on RRG

- **Direct numerics: Exact diagonalization**
- **Field theory, Large N** $\longrightarrow$ **saddle point**

  $\longrightarrow$ **self-consistency equation**

  - **Analytical solution**
  - **Numerical solution via pool method (population dynamics)**

Anderson localization on random regular graphs (RRG): Analytical solution

Tikhonov, ADM 2019

Supersymmetric field-theoretical approach

Saddle-point approximation controlled by large size N of the graph

$\longrightarrow$ self-consistency equation for the distribution of local Green functions (known from the problem on an infinite Cayley tree)

- highly accurate determination of W_c

- ergodicity of delocalized phase $W < W_c$ $\qquad N \gg N_\xi(W)$

N_ξ – correlation volume, $\qquad \ln N_\xi \propto (W_c - W)^{-1/2}$

- Wigner-Dyson level statistics
- Wave function statistics: Inverse participation ratio (IPR) $P_2 = \langle \sum_i |\psi(i)^4| \rangle$

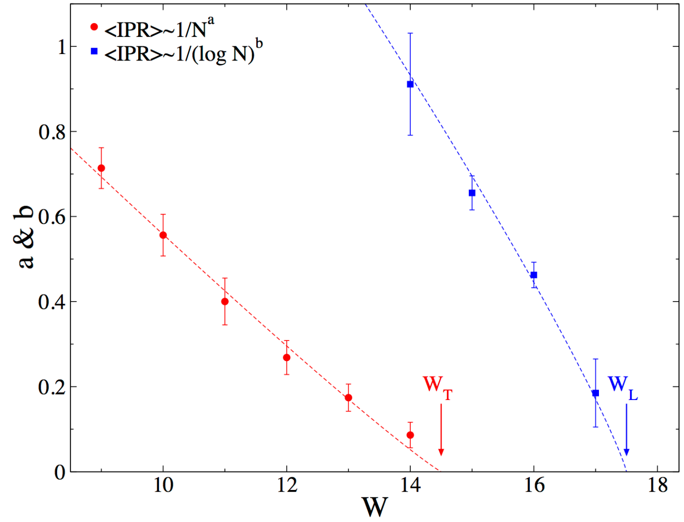
$$P_2 \simeq N_\xi(W)/N$$

- wave function correlations in delocalized and localized phases, ...

ADM, Fyodorov '91 related results for sparse random matrix model ($\sim$ RRG with fluctuating connectivity)

Anderson localization on RRG: Analysis of numerics requires great care

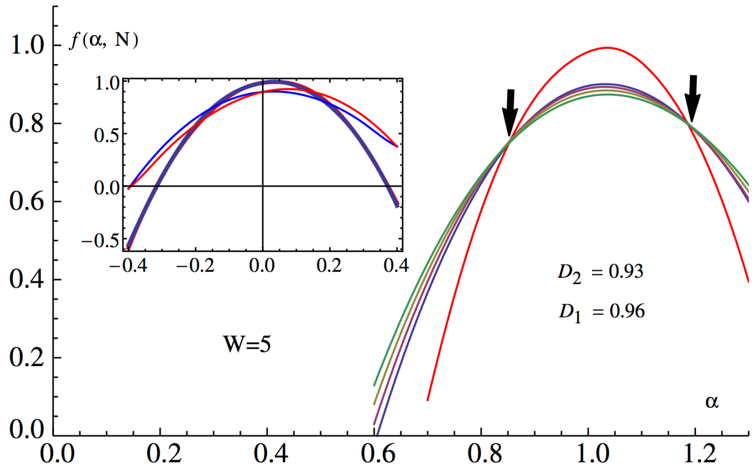
Biroli, Ribeiro-Teixeira, Tarzia, arXiv:1211.7334

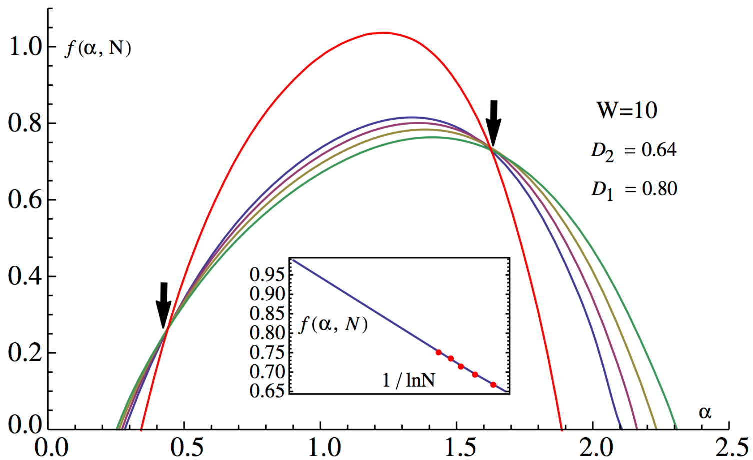


apparent fractality of IPR

$\longrightarrow$ **non-ergodicity of delocalized phase ?!**

De Luca, Altshuler, Kravtsov, Scardicchio, Phys Rev Lett ’14





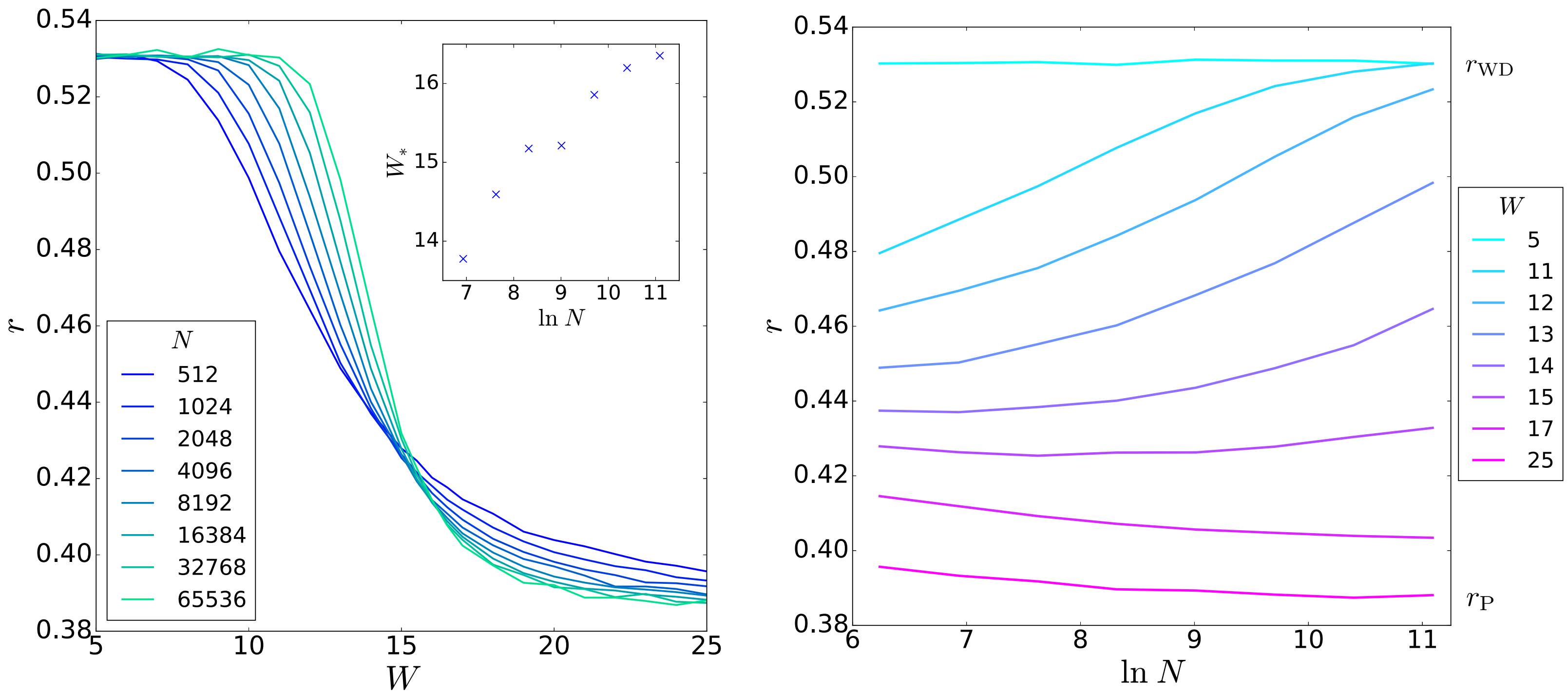
“We conclude that the nonergodicity and multifractality persist in the entire region of delocalized states $0 < W < W_c$”

Ergodicity of delocalized phase on RRG

Tikhonov, ADM, Skvortsov, 2016

Level statistics:

mean adjacent gap ratio r



- $r \to r_{\mathrm{WD}}$ in the large-N limit in the delocalized phase
- crossing point W_* drifts towards stronger disorder:

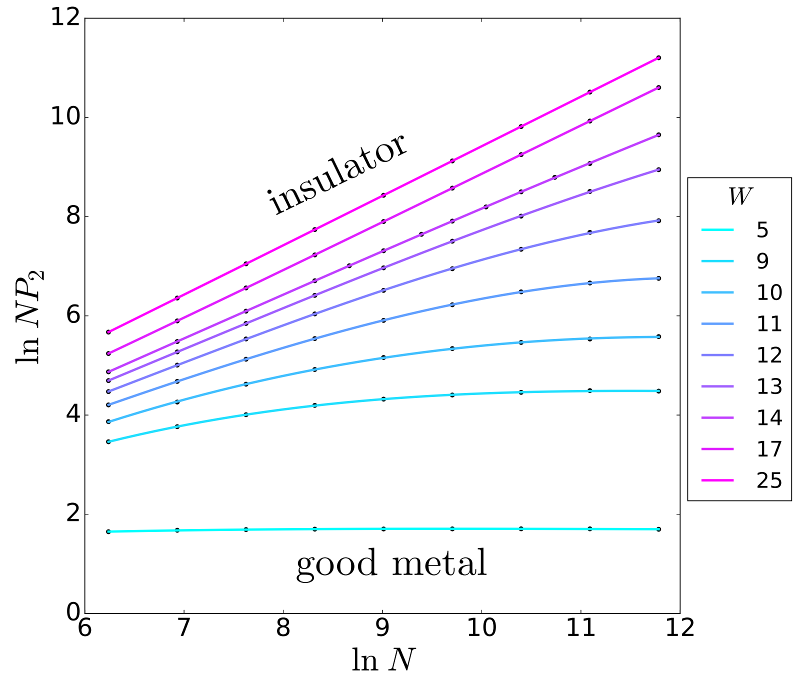
$W_* \simeq 14\ (N = 512) \longrightarrow W_* \simeq 16\ (N = 65\,536) \longrightarrow W_c\ (N = \infty)$

Equivalently: for given W non-monotonic dependence $r(N)$

Reason: critical point on tree-like structures (or at $d \to \infty$) has quasi-localized character (Poisson statistics, IPR $\propto N^0$)

Ergodicity of delocalized phase on RRG: Eigenfunction statistics

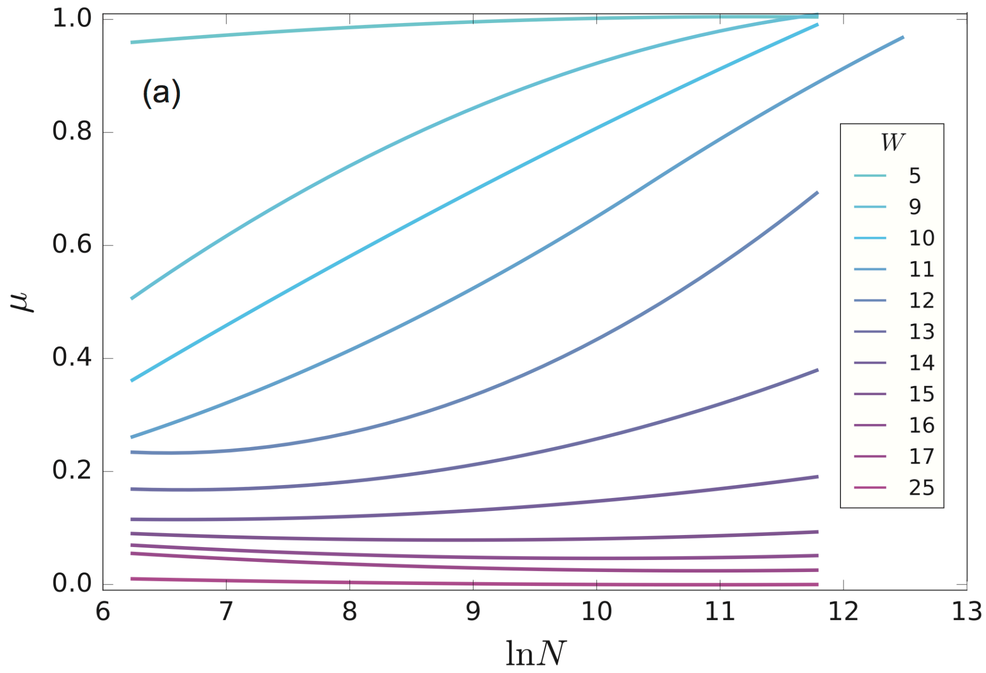
IPR $P_2(W, N)$

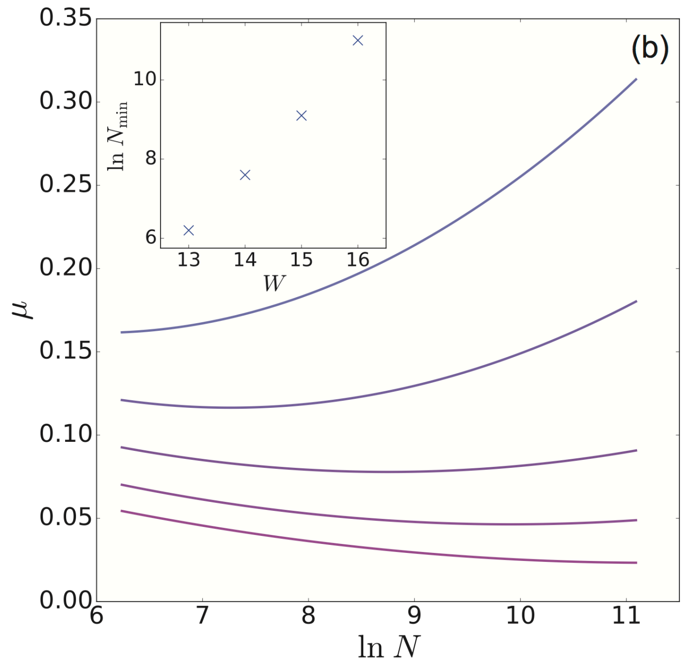


"flowing fractal exponent"

$$\mu(W, N) = -\partial \ln P_2(W, N)/\partial \ln N$$

- **Ergodicity:** $\mu \to 1$ at $N \to \infty$ for $W < W_c$
- **non-monotonic** dependence $\mu(N)$





RRG: Field-theoretical approach

$\langle \mathcal{O} \rangle = \int \prod_k [d\Phi_k] e^{-\mathcal{L}(\Phi)} U_{\mathcal{O}}(\Phi)$ $\qquad$ $\Phi_{i,s} = (S^{(1)}_{i,s}, S^{(2)}_{i,s}, \chi_{i,s}, \chi^*_{i,s})$ – supervector

Doubling $\Phi_i = (\Phi_{i,1},\ \Phi_{i,2})$ for retarded (R) and advanced (A) Green functions

$$e^{-\mathcal{L}(\Phi)} = \int \prod_i d\epsilon_i \gamma(\epsilon_i) e^{\frac{i}{2}\Phi_i^\dagger \hat{\Lambda}(E-\epsilon_i)\Phi_i + \frac{i\omega}{4}\Phi_i^\dagger \Phi_i} \prod_{\langle i,j \rangle} e^{-i\Phi_i^\dagger \Phi_j} \qquad \Lambda = \mathrm{diag}(1,-1)_{RA}$$

RRG, connectivity $p = m+1$, distributions of energies $\gamma(\epsilon)$ and hoppings $h(t)$

$$\langle Z \rangle = \int \prod_i d\Phi_i \frac{dx_i}{2\pi} e^{ipx_i} \exp\left\{ \sum_i \left[\frac{i}{2}\Phi_i^\dagger \hat{\Lambda}(E - J_i \hat{K})\Phi_i + \frac{i}{2}\left(\frac{\omega}{2} + i\eta\right) \Phi_i^\dagger \Phi_i \right.\right.$$
$$\left.\left. + \ln \tilde{\gamma}(\frac{1}{2}\Phi_i^\dagger \hat{\Lambda}\Phi_i) \right] + \frac{p}{2N} \sum_{i \neq j} \left[e^{-i(x_i+x_j)} \tilde{h}(\Phi_i^\dagger \hat{\Lambda} \Phi_j) - 1 \right] \right\}$$

Functional generalization of Hubbard-Stratonovich transformation

$\longrightarrow$ integral over functions $g(\Phi)$: $\qquad \langle \mathcal{O} \rangle = \int Dg\, U_{\mathcal{O}}(g) e^{-N\mathcal{L}(g)}$

$$\mathcal{L}(g) = \tfrac{m+1}{2} \int d\Psi d\Psi' g(\Psi) C(\Psi, \Psi') g(\Psi') - \ln \int d\Psi\, F_g^{(m+1)}(\Psi)$$

$$F_g^{(s)}(\Psi) = \exp\left\{ \tfrac{i}{2} E \Psi^\dagger \hat{\Lambda} \Psi + \tfrac{i}{2}\left(\tfrac{\omega}{2} + i\eta\right) \Psi^\dagger \Psi \right\} \tilde{\gamma}(\tfrac{1}{2}\Psi^\dagger \hat{\Lambda} \Psi) g^s(\Psi)$$

Field theory for RRG model: Saddle-point treatment

$$\langle \mathcal{O} \rangle = \int Dg\, U_{\mathcal{O}}(g) e^{-N\mathcal{L}(g)}$$

Large N $\longrightarrow$ saddle-point treatment

IPR $\quad P_2 = \frac{1}{\pi\nu} \lim_{\eta\to 0} \eta \, \langle G_R(j,j) G_A(j,j) \rangle \qquad G_{R,A}(j,j) = \langle j | (E - \mathcal{H} \pm i\eta)^{-1} | j \rangle$

$$\langle G_R(j,j) G_A(j,j) \rangle = \int Dg\, U(g) e^{-N\mathcal{L}(g)}$$

$$U(g) = \int [d\Psi] \, \tfrac{1}{16} \left(\Psi_1^\dagger \hat{K} \Psi_1 \right) \left(\Psi_2^\dagger \hat{K} \Psi_2 \right) F_g^{(m+1)}(\Psi)$$

$g_0(\Psi) = \int d\Phi \, \tilde{h}(\Phi^\dagger \hat{\Lambda} \Psi) F_{g_0}^{(m)}(\Phi)$ saddle-point equation

identical to the self-consistency equation for infinite Bethe lattice (BL) !

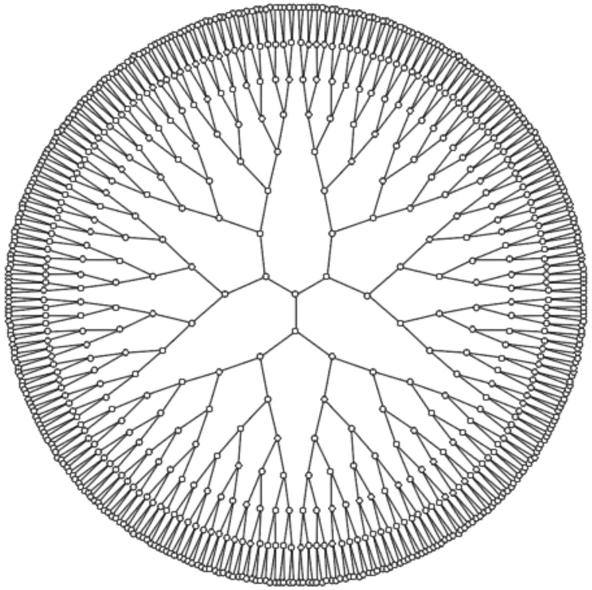
ADM, Fyodorov 1991

Symmetry $\longrightarrow$ $g_0(\Psi) = g_0(x,y); \quad x = \Psi^\dagger \Psi, \quad y = \Psi^\dagger \hat{\Lambda} \Psi$

Laplace (x) - Fourier (y) transf.: $g_0(x,y) \longleftrightarrow$ distribution of Im G and Re G

self-consistency equation in the form of Abou-Chacra, Thouless, Anderson 1973

Bethe lattice and self-consistency equation



Self-consistency eq. for the distribution of local Green functions

$$G^{(m)} \stackrel{d}{=} \frac{1}{E + i\eta - \varepsilon - V^2 \sum_{i=1}^{m} G_i^{(m)}}$$

$G^{(m)} = G_{\mathrm{R}}(0,0;E) = \langle 0|(E - \mathcal{H} + i\eta)^{-1}|0\rangle$

$\stackrel{d}{=}$ – equality in distribution for random variables

η – infinitesimal positive

Field theory for RRG model: Inverse Participation Ratio

- $W \geq W_c$ localized phase and critical point:

single saddle-point $g_0(\Phi) = g_0(x, y)$, characteristic $x \sim \eta^{-1}$

$$\longrightarrow \quad U(g_0) = \frac{C}{\eta}, \quad C \sim 1 \quad \longrightarrow \quad P_2 = \frac{C}{\pi\nu} \sim 1$$

- $W < W_c$ delocalized phase: spontaneous symmetry breaking

manifold of saddle points

$$g_0(\Psi) \longrightarrow g_{0T}(\Psi) = g_0(\hat{T}\Psi) = g_0(\Psi^\dagger \bar{\hat{T}}\hat{T}\Psi,\ \Psi^\dagger\hat{\Lambda}\Psi) \qquad \bar{\hat{T}}\hat{\Lambda}\hat{T} = \hat{\Lambda}$$

$$\langle G_R(j,j) G_A(j,j)\rangle = \int Dg e^{-N\mathcal{L}(g)} U(g) = \int d\mu(\hat{T})\, U(g_{0T})\, e^{-\frac{\pi}{2}N\eta\nu \mathrm{Str}\left[\bar{\hat{T}}\hat{T}\right]}$$

$$P_2 = \frac{1}{\pi\nu} \lim_{\eta\to 0} \eta \langle G_R(j,j) G_A(j,j)\rangle = \frac{12}{N}\frac{g_{0,xx}^{(m+1)}}{\pi^2\nu^2} = \frac{3}{N}\frac{\left\langle \nu^2 \right\rangle_{\mathrm{BL}}}{\nu^2} \qquad N \gg N_\xi$$

Near the transition: $\left\langle \nu^2 \right\rangle_{\mathrm{BL}} / \nu^2 = N_\xi \gg 1$ — correlation volume $\quad P_2 = 3\frac{N_\xi}{N}$

Ergodicity! Exact relations between RRG and infinite BL problems.

Generalized to correlation functions at arbitrary distance r
and of different eigenstates (energy separation ω)

Critical behavior

Correlation volume N_ξ $\longrightarrow$ correlation length ξ

Critical behavior: $\xi \sim (W_c - W)^{-\nu_{\rm del}}$ critical index $\nu_{\rm del} =?$

Self-consistency equation $\longrightarrow$ $m\lambda_\beta = 1$

λ_β – largest eigenvalue of certain integral operator

$\lambda_\beta(W) \simeq \frac{1}{2} - c_1\,(W - W_c) + c_2\left(\beta - \frac{1}{2}\right)^2$, has minimum at $\beta = 1/2$

Localized phase, $W > W_c$: β real

Critical point, $W = W_c$: $m\lambda_{1/2} = 1$ Abou-Chacra et al, 1973

Delocalized phase, $W < W_c$: spontaneous symmetry breaking

β becomes complex: $\beta = \frac{1}{2} \pm i\sigma\,,$ $\sigma \simeq \sqrt{\frac{c_1}{c_2}}(W_c - W)^{1/2}$

Correlation length $\ln N_\xi \simeq \dfrac{\pi}{\sigma}$ $\longrightarrow$ critical index $\boxed{\nu_{\rm del} = 1/2}$

$m = 2 \longrightarrow c_1 \simeq 1.59,\quad c_2 \simeq 0.0154 \longrightarrow \ln N_\xi \simeq 31.9\,(W_c - W)^{-1/2}$

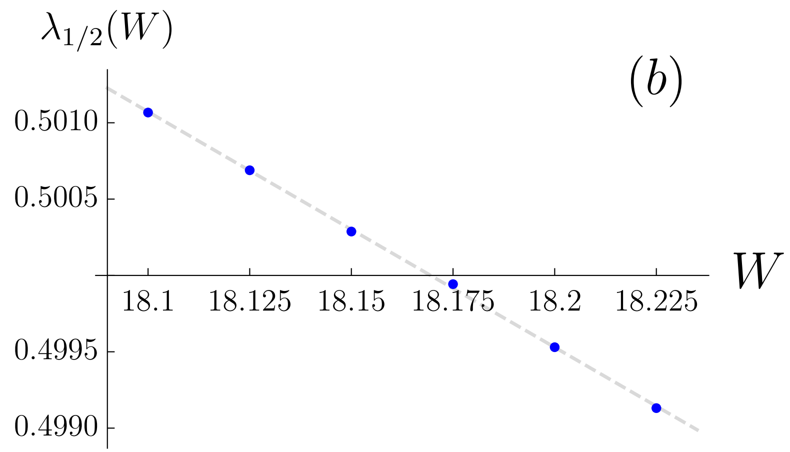
ADM, Fyodorov, 1991, Tikhonov, ADM, 2019

Critical behavior: Numerical confirmation of $\nu_{\rm del} = 1/2$

Tikhonov, ADM, 2019

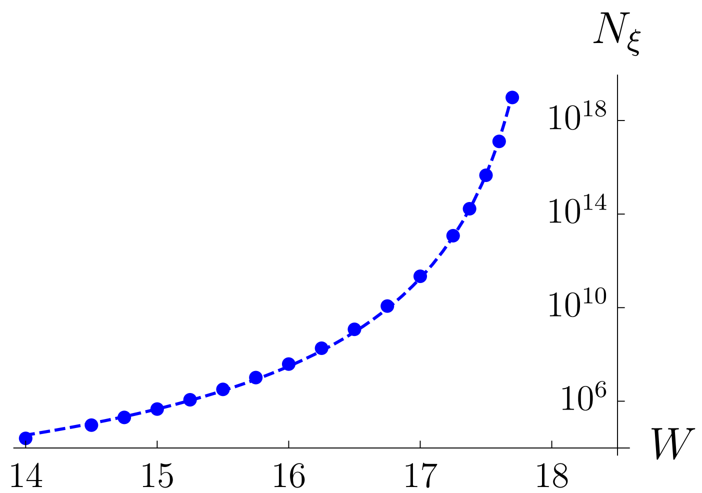
- **accurate determination of W_c from the equation $m\lambda_{1/2} = 1$**

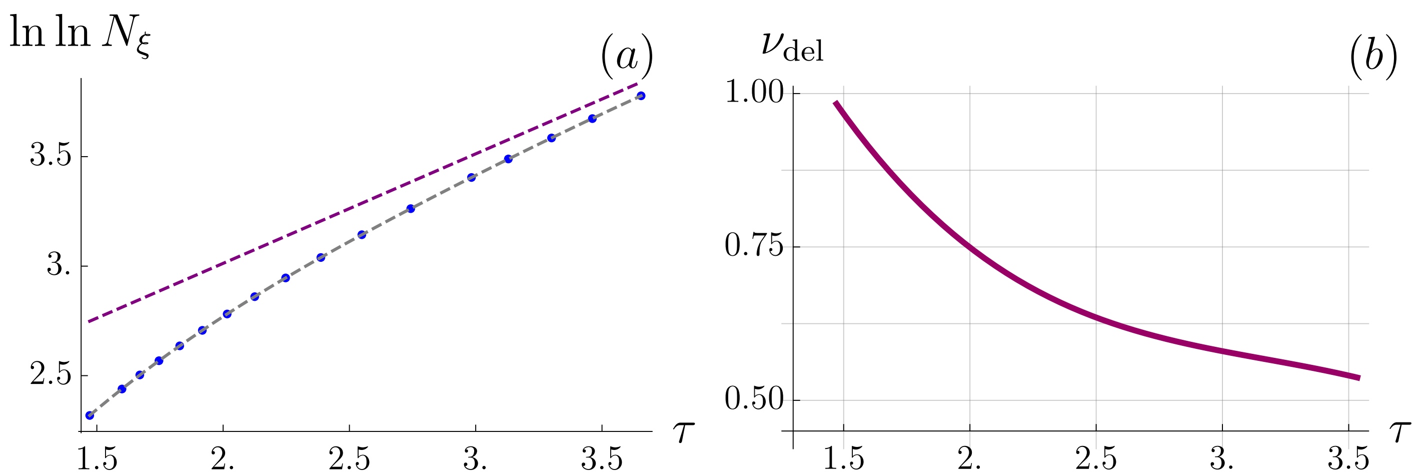
$$\boxed{W_c = 18.17 \pm 0.01} \qquad m = 2$$



- **solve self-consistency eq. by pool method $\longrightarrow$ determine N_ξ**

$$\ln N_\xi \sim (W_c - W)^{-\nu_{\rm del}} \longrightarrow \frac{\partial \ln\ln N_\xi}{\partial \ln \tau} = \nu_{\rm del} \qquad \tau = -\ln(1 - W/W_c)$$





$m = 2 \longrightarrow$ **asymptotics** $\ln N_\xi = 31.9\,(W_c - W)^{-1/2}$

$$\boxed{\nu_{\rm del} = 1/2}$$

Correlations of different wave functions on RRG

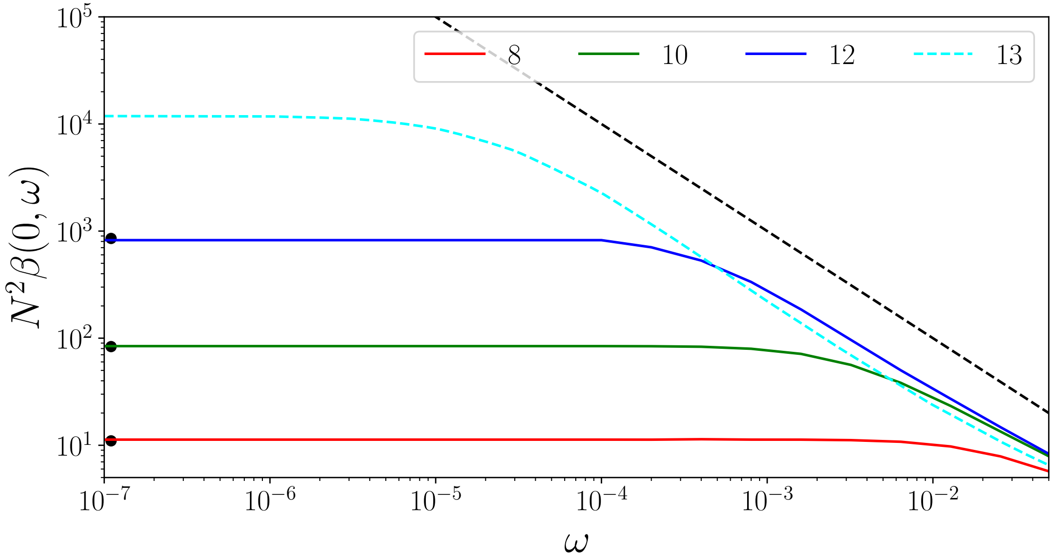
$$\beta(\omega) = \left\langle \left| \psi_k(j) \psi_l(j) \right|^2 \right\rangle, \qquad E_k = E + \omega/2, \qquad E_l = E - \omega/2$$

Fourier transf. $\longrightarrow$ return probability $p(t)$ $\longrightarrow$ quantum dynamics

- **Delocalized phase,** $N \gg N_\xi$

$$N^2\beta(\omega) \sim \begin{cases} N_\xi, & \omega < \omega_\xi \\ \dfrac{1}{\omega \ln^{3/2} \omega^{-1}}, & \omega > \omega_\xi \end{cases}$$

$\omega_\xi \sim N_\xi^{-1}$ (with log correction)



$$N^2\beta(\omega \to 0) = \frac{N}{3} P_2 = \frac{\langle \nu^2 \rangle_{\mathrm{BL}}}{\nu^2}$$

Tikhonov, ADM, 2019

Outstanding agreement between exact diagonalization, analytical results, and population dynamics.

A further manifestation of ergodicity of the delocalized phase

Wave functions correlations: Localized phase

Single particle problem in d dimensions, localized phase

Cuevas, Kravtsov, 2007

$$N^2\beta(\omega) \sim \zeta^{d-d_2} \ln^{d-1}(\delta_\zeta/\omega)\,, \qquad \omega < \delta_\zeta \equiv \zeta^{-d}$$

ζ – localization length

Logarithmic enhancement of correlations at $d > 1$ due to Mott resonances

What to expect in the localized phase on a tree-like graph (RRG)?

Tikhonov, ADM, 2019

A simplistic estimate:

Decay of an eigenstate $|\psi^2(r)| \sim m^{-r} \exp\{-r/\zeta(W)\}$

This is an average but assume that all eigenstates decay in this way

$\longrightarrow$ calculate probability of resonance at frequency ω $\longrightarrow$

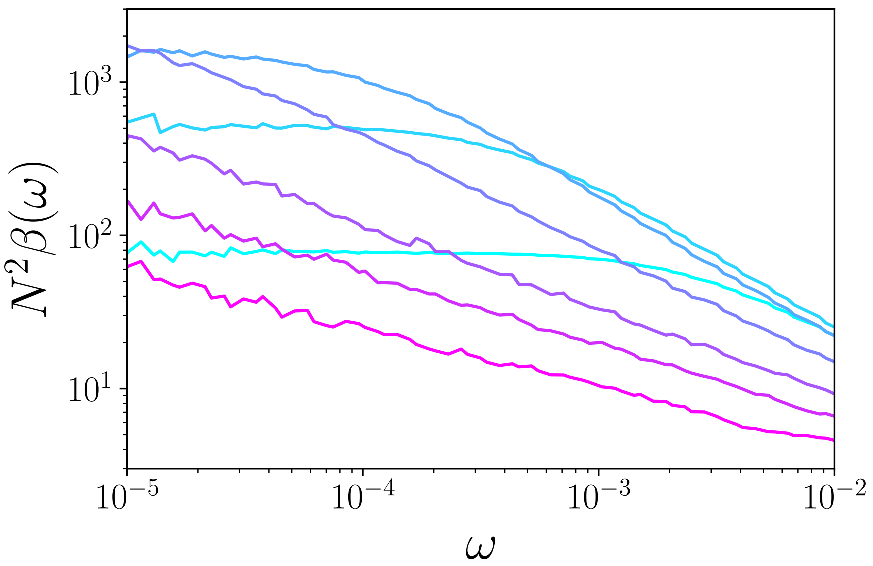
$$\boxed{N^2\beta(\omega) \sim \omega^{-\mu(W)}} \qquad \mu(W) \simeq \frac{\zeta \ln m}{\zeta \ln m + 1} \begin{cases} \to 1\,, & W \to W_c + 0 \\ \sim \frac{1}{\ln(W/W_c)}\,, & W \gg W_c \end{cases}$$

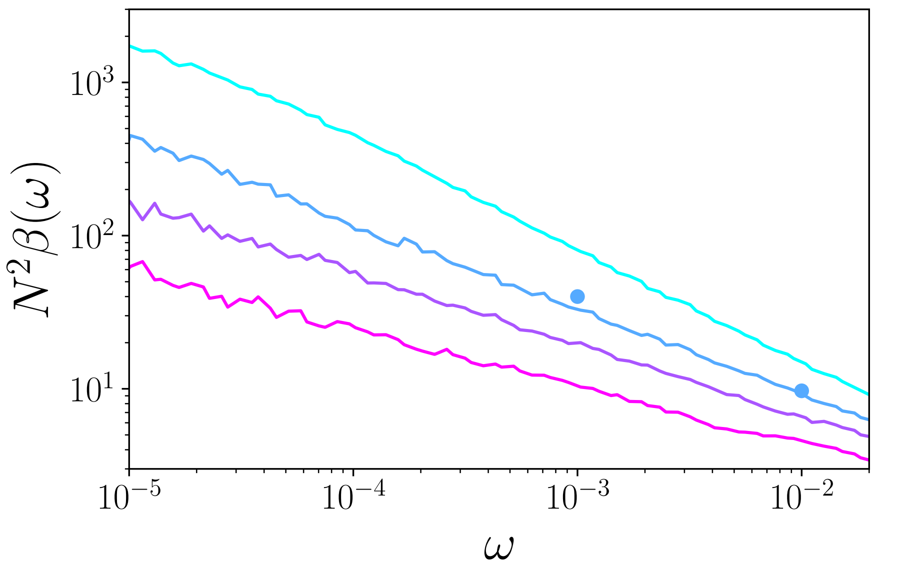
In fact, wave functions $|\psi^2(r)|$ strongly fluctuate.

But a more accurate analysis yields qualitatively the same result.

Eigenstate correlations on RRG: From ergodic to localized phase

Tikhonov, ADM, 2019



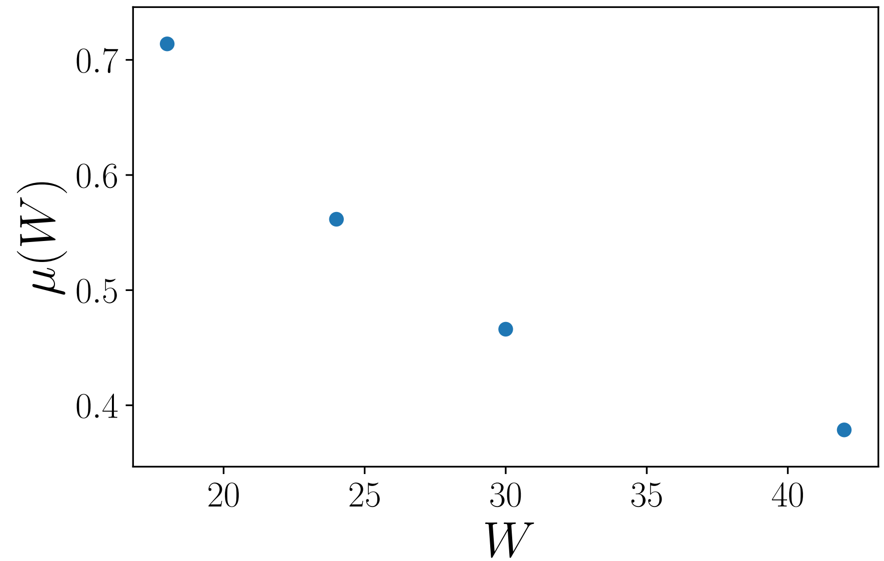


$\boldsymbol{W = 10, 12, 14, 18, 24, 30, 42}$

Localized side: $\boldsymbol{W = 18, 24, 30, 42}$

$$\boldsymbol{N^2\beta(\omega) \sim \omega^{-\mu(W)}}$$

"Fractal" scaling at $W > W_c$



Correlation of adjacent wavefunctions on RRG

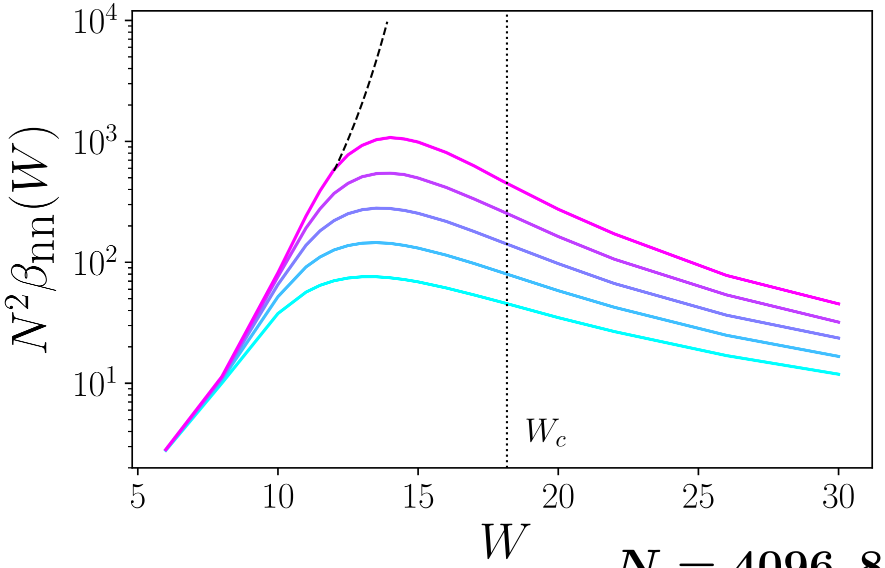
$$\beta_{\rm nn} = \left\langle \left|\psi_k(j)\,\psi_{k+1}(j)\right|^2 \right\rangle \simeq \beta(\omega \sim \Delta)$$

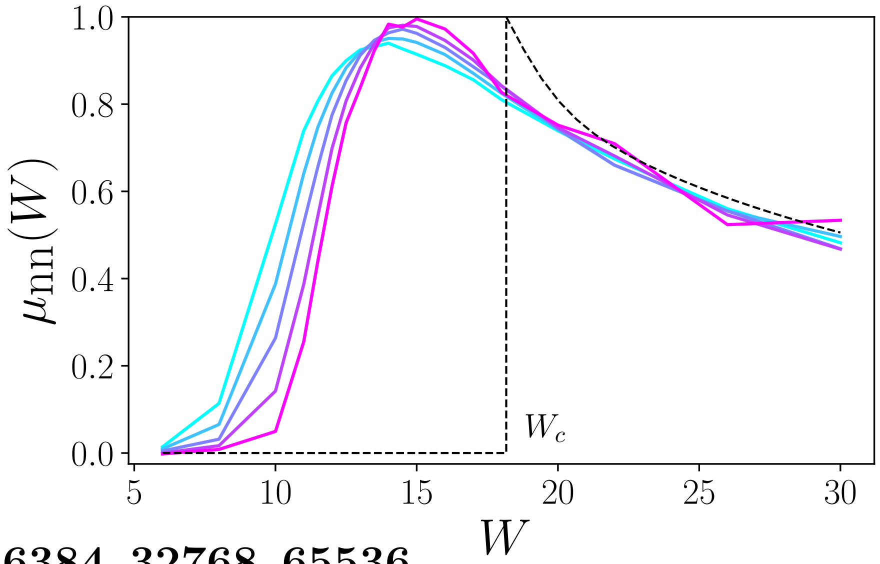
Ergodic, $N \gg N_\xi$: $\quad N^2\beta_{\rm nn} = N_\xi$

Localized: $\quad N^2\beta_{\rm nn} \sim N^{\mu(W)}$

$$\mu_{\rm nn}(W,N) = \frac{\partial \ln\left(N^2\beta_{\rm nn}\right)}{\partial \ln N}$$

$$\mu_{\rm nn}(W,N) \xrightarrow{N\to\infty} \begin{cases} 0, & W < W_c \\ \mu(W), & W > W_c \end{cases}$$





$N = 4096, 8192, 16384, 32768, 65536$

Non-monotonic behavior of $\beta_{\rm nn}(W)$ and $\mu_{\rm nn}(W)$ around the transition

Maximum at $W_{\rm peak}(N)$

apparent N-dependent crit. point

$W_{\rm peak}(N\to\infty) = W_c \simeq 18.17$

$\log_2 N$	12	13	14	15	16
$W_{\rm peak}(N)$	13.70	13.78	13.89	14.06	14.28
$\nu(N)$	4.31	3.52	2.22	1.42	0.96

Quantum dots and random graphs

Spin quantum dot

$$\hat{H} = \sum_{i=1}^{n} \varepsilon_i \hat{S}_i^z + \sum_{i,j=1}^{n} \sum_{\alpha,\beta\in\{x,y,z\}} V_{ij}^{\alpha\beta} \left(\hat{S}_i^\alpha \hat{S}_j^\beta + h.c. \right)$$

$n \gg 1$ spins

ε_i — random fields from $[-W, W]$

interaction $V_{ij}^{\alpha\beta}$ – random with zero mean and variance unity

Fock-space coordination number: $m \simeq n^2/2$

Similarly, for fermionic quantum dot

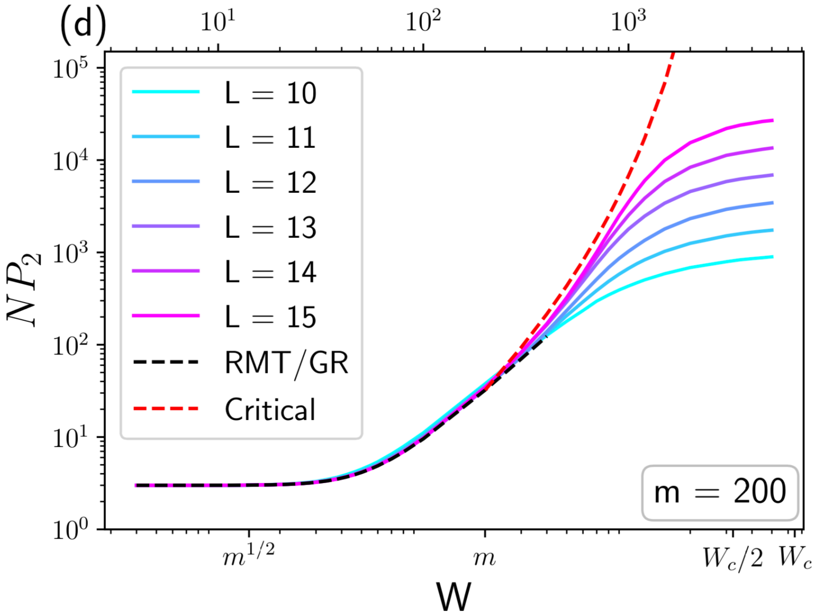
$$\hat{H} = \sum_{i} \varepsilon_i \hat{c}_i^\dagger \hat{c}_i + \sum_{ijkl} V_{ijkl} \left(\hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l + h.c. \right)$$

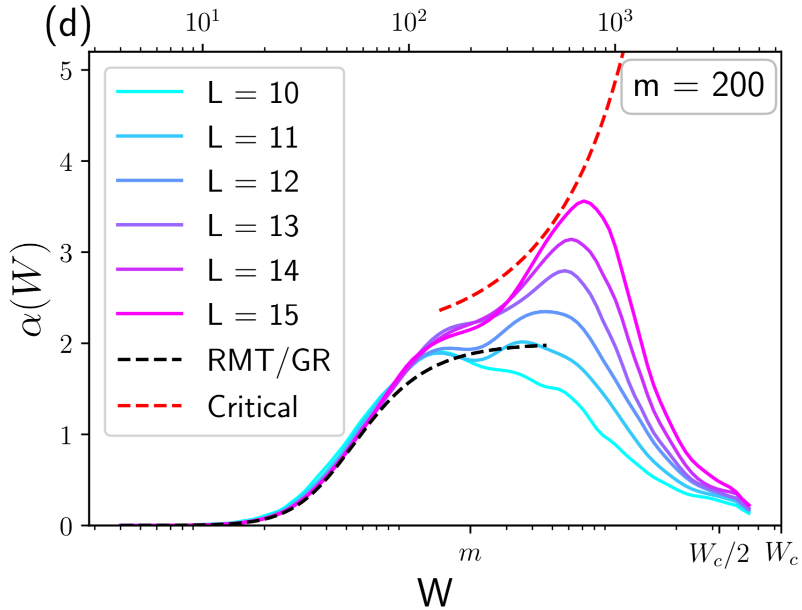
Fock-space coordination number: $m \simeq n^4/64$

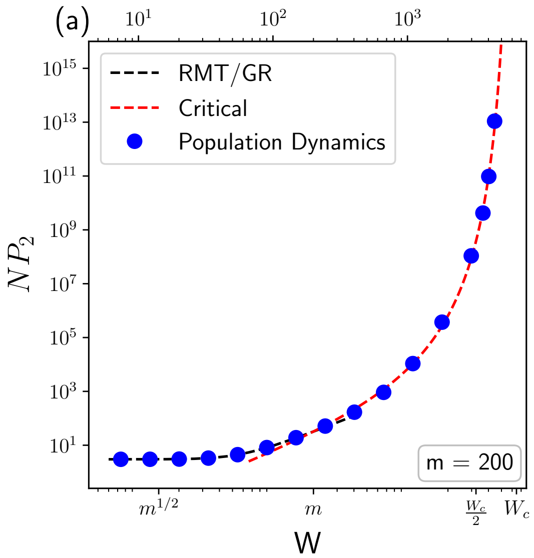
$\longrightarrow$ consider first RRG model with large m

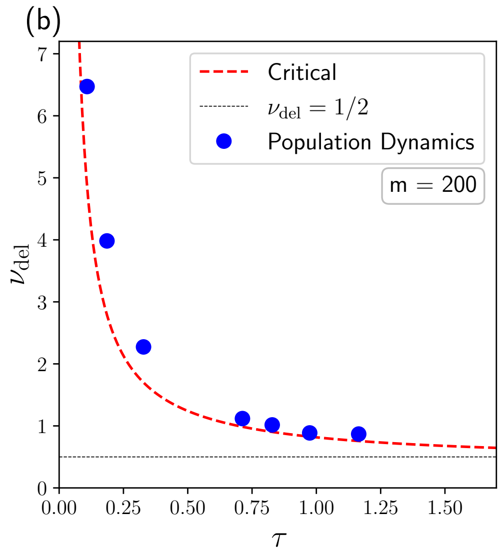
RRG with large connectivity $m \gg 1$

Herre, Karcher, Tikhonov, ADM '23









Critical disorder: $W_c = 4m \ln \dfrac{W_c}{2}$

$\alpha(W) = d\ln(NP_2)/d\ln W$

excellent agreement between analytical predictions, exact diagonalization, and population dynamics

Critical regime $W_c/2 < W < W_c$ and $N \gg N_\xi$ inaccessible for exact diagonalization

Quantum dot critical disorder: Analytical expectations

How does the critical disorder W_c in quantum dots scale with n ?

RRG approximation $\longrightarrow$ $W_c \sim m \ln m$

m – coordination number

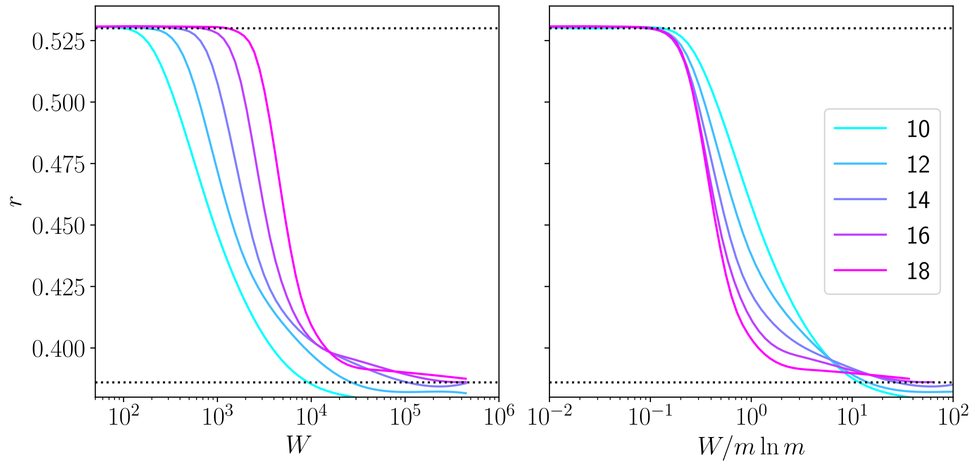
$m \approx n^2/2$ for spin quantum dot; $m \approx n^4/64$ for fermionic quantum dot

But the RRG approximation neglects small-scale loops that might reduce W_c.
How important are they?

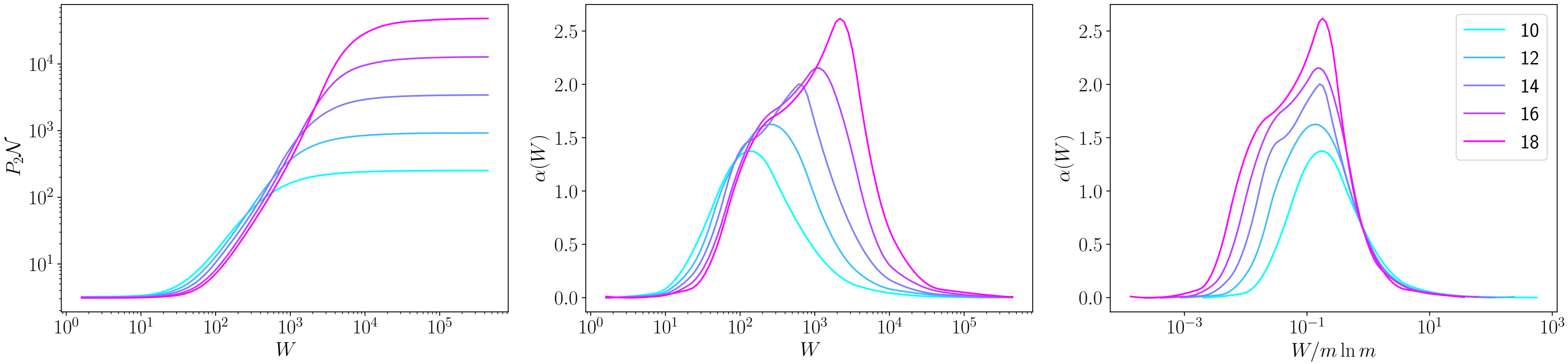
Gornyi, ADM, Polyakov, Burin '17: $W_c \sim m \ln^{\mu} m$ with $\mu \leq 1$.

Fermionic quantum dot: Exact diagonalization

Herre, Karcher, Tikhonov, ADM ’23



level statistics (gap ratio)



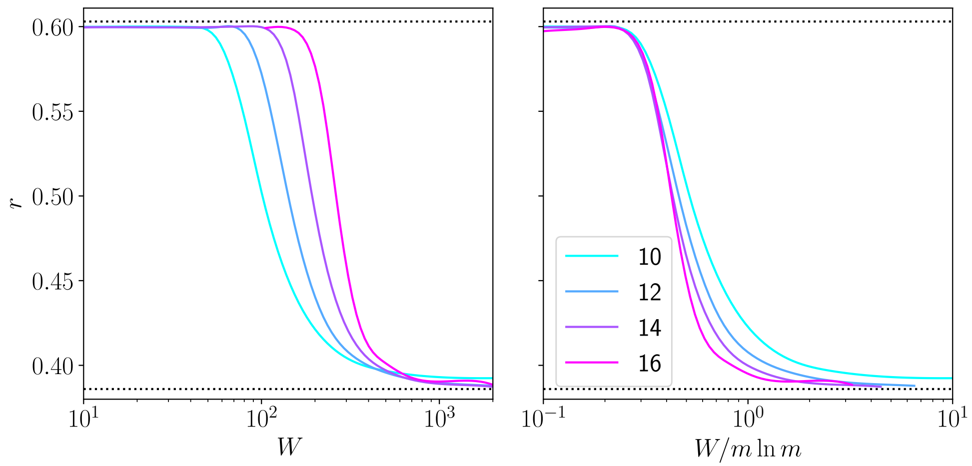
IPR and its log derivative $\alpha(W) = d\ln(NP_2)/d\ln W$

Data consistent with MBL transition at $W_c \sim m \ln m$ as on RRG.

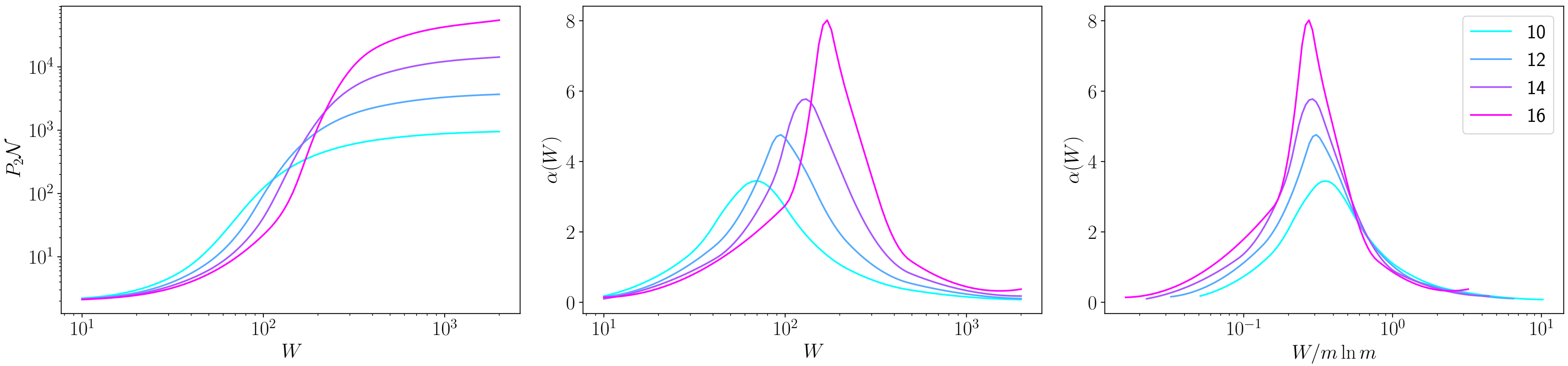
$m \approx n^4/64$ – coordination number

Spin quantum dot: Exact diagonalization

Herre, Karcher, Tikhonov, ADM '23



level statistics (gap ratio)



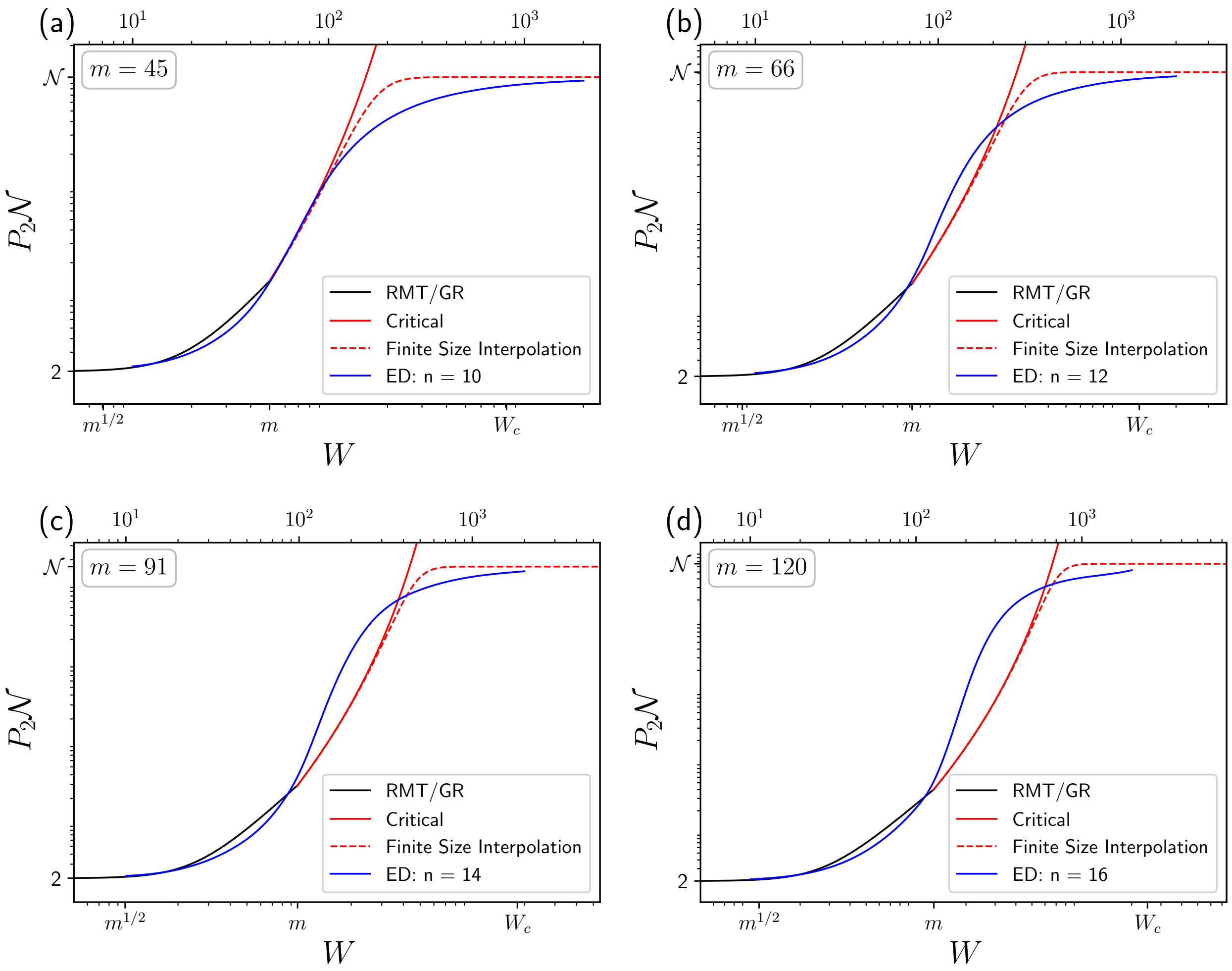
IPR and its log derivative $\alpha(W) = d\ln(NP_2)/d\ln W$

Data consistent with MBL transition at $W_c \sim m \ln m$ as on RRG.

$m \approx n^2/2$ – coordination number

Spin quantum dot vs RRG

Herre, Karcher, Tikhonov, ADM '23



For $W > m$, the data for spin quantum dot deviate from those for RRG towards faster localization.

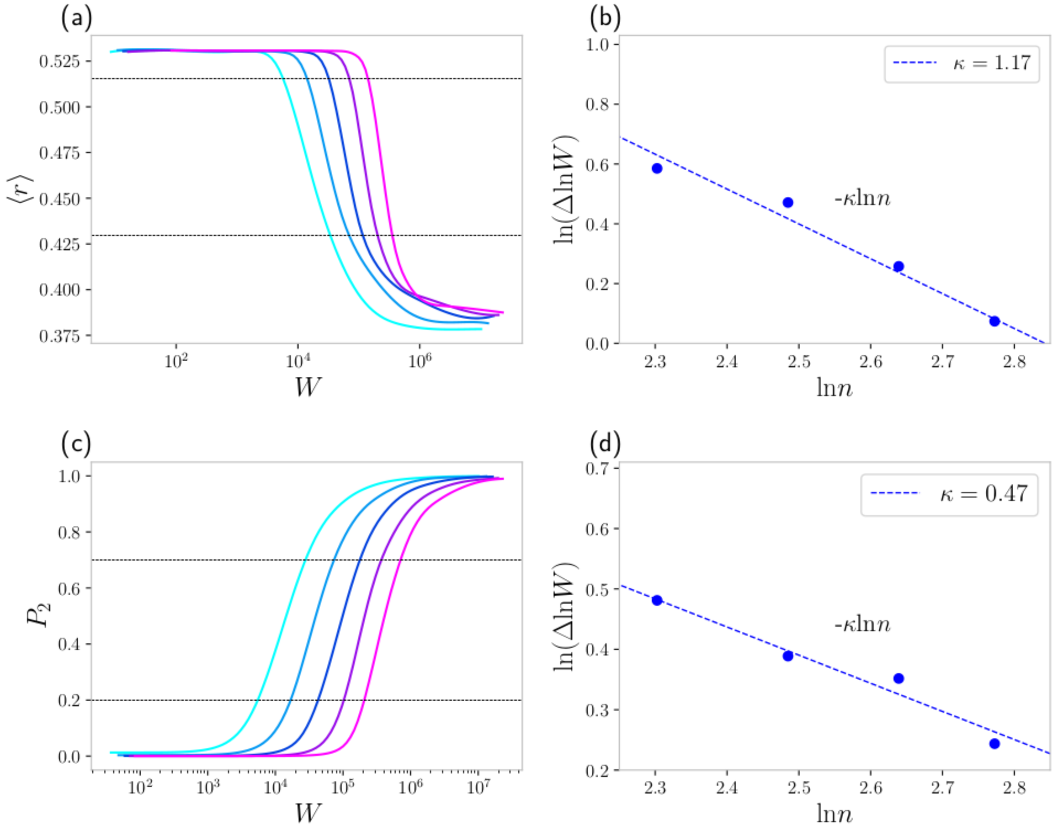
Critical regime for quantum dot models may be different from that on RRG. Remains to be understood.

Fock-space MBL transition in quantum dots

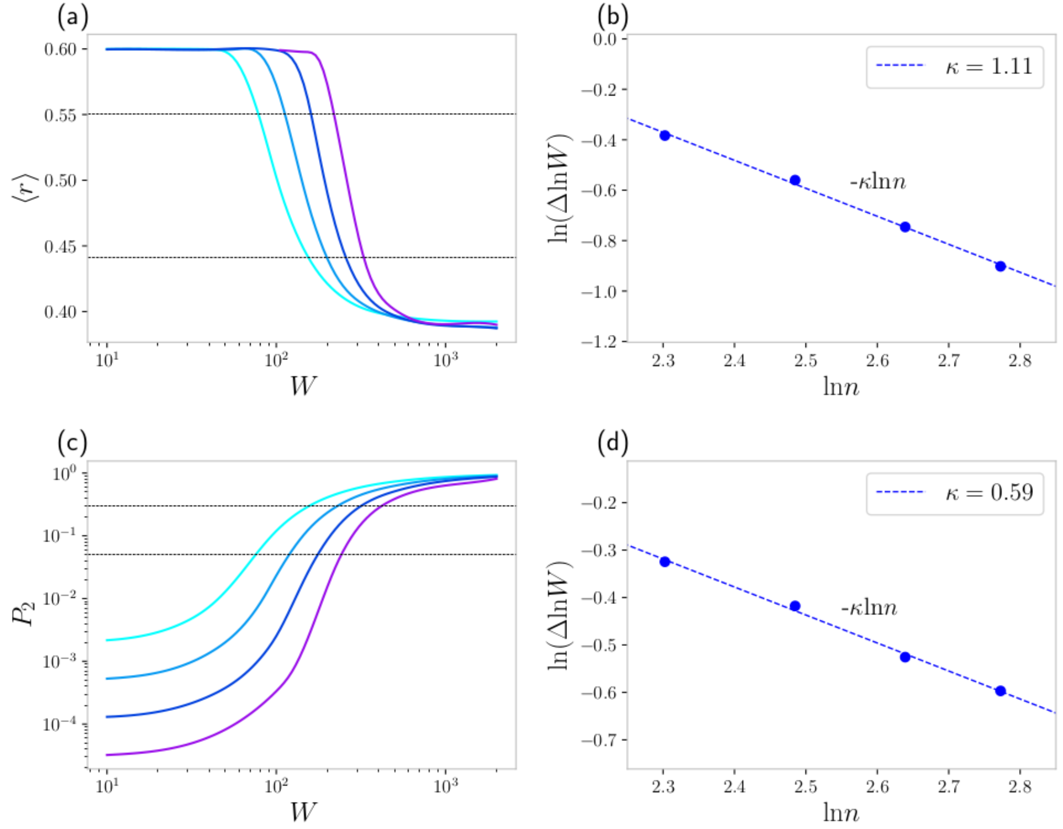
Herre, Karcher, Tikhonov, ADM '23

For finite n, there is a crossover from ergodicity to Fock-space MBL.

It becomes a sharp transition at $n \to \infty$ if $\Delta W / W \approx \Delta(\ln W) \to 0$



fermionic quantum dot



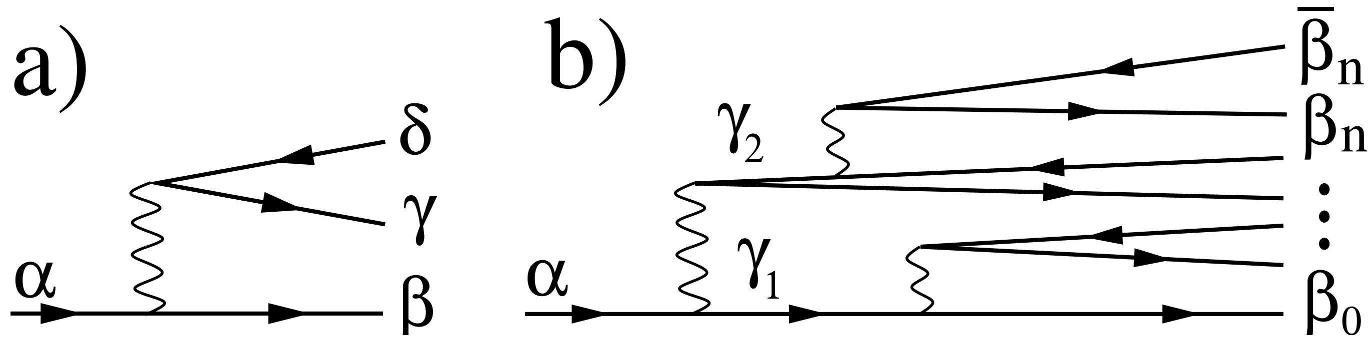
spin quantum dot

$\Delta(\ln W) \sim n^{-\kappa} \longrightarrow$ sharp Fock-space-MBL transition

From Fock- to real-space localization. Short-range interaction

Gornyi, ADM, Polyakov '05

Single-particle excitation decay processes:



Lowest order process: $e \to eeh$

$$\longrightarrow \quad \text{Golden rule} \quad \tau_\phi^{-1} \sim V^2/\Delta_\xi^{(3)}$$

$V \sim \alpha\Delta_\xi$ – interaction matrix element, α - interaction strength,

Δ_ξ - single-particle level spacing in localization volume,

$\Delta_\xi^{(3)} \sim \Delta_\xi^2/T$ - three-particle level spacing in localization volume

But: for $V < \Delta_\xi^{(3)}$, i.e. $T < T_3$, where $T_3 \sim \Delta_\xi/\alpha$

$\longrightarrow$ no decay (no hybridization) to the lowest order

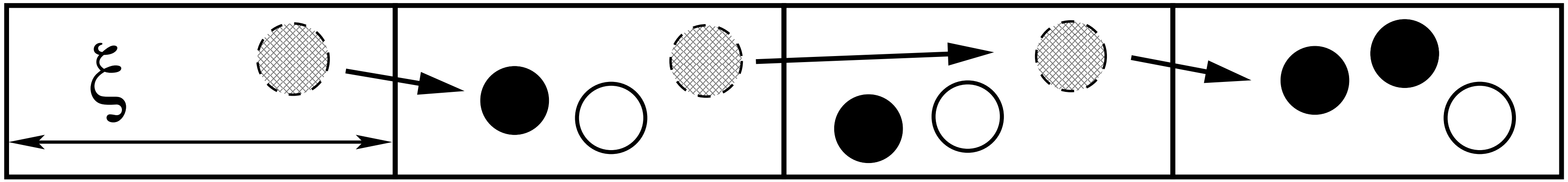
Localization transition

Gornyi, ADM, Polyakov '05

Higher orders? $\longrightarrow$ **have to analyze** $V^{(n)}/\Delta^{(2n+1)}$

$$V^{(n)} = \sum_{\text{diagrams}} \sum_{\gamma_1,\cdots\gamma_{n-1}} V_1 \prod_{i=1}^{n-1} \frac{V_{i+1}}{E_i - \epsilon_{\gamma_i}}$$

$\longrightarrow$ **optimal processes ("ballistic", "forward approximation"):**
a "string" with a few excitations per localization volume



$$\frac{V^{(n)}}{\Delta^{(2n+1)}} \sim \left(\frac{T}{T_3}\right)^n \qquad \text{(logarithms omitted)}$$

$\longrightarrow$ **Many-Body Localization transition at** $T = T_c \sim T_3$

Mapping onto Bethe lattice

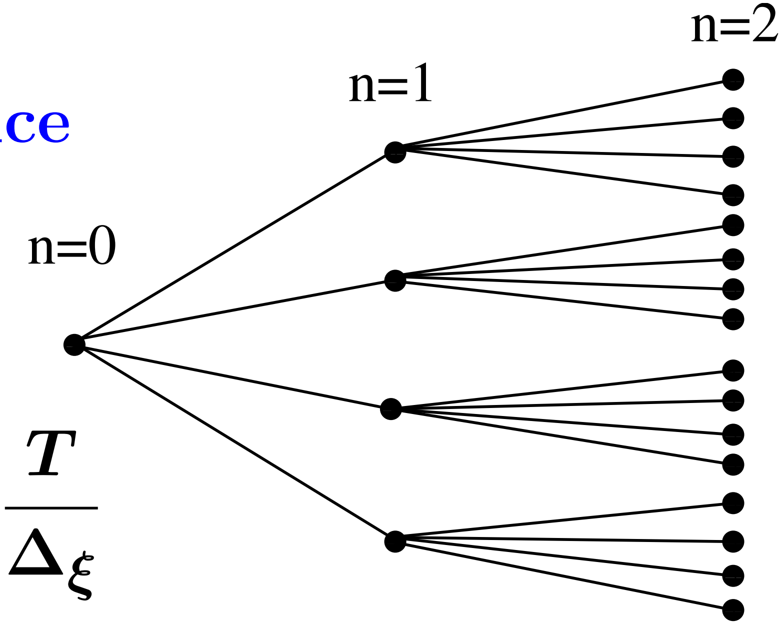
Gornyi, ADM, Polyakov '05

Interacting problem in Fock space

$\longrightarrow$ Anderson model on the Bethe lattice

$\longrightarrow$ Metal-Insulator Transition at

$$\Delta/V = 4\ln K$$



K: Coordination number $K \sim \dfrac{\Delta_\xi}{\Delta_\xi^{(3)}} \sim \dfrac{T}{\Delta_\xi}$

Δ: Level spacing of $n = 1$ states: $\Delta = \Delta_\xi^{(3)}$

V: hopping matrix element:
interaction matrix element $V \sim \alpha\Delta_\xi$

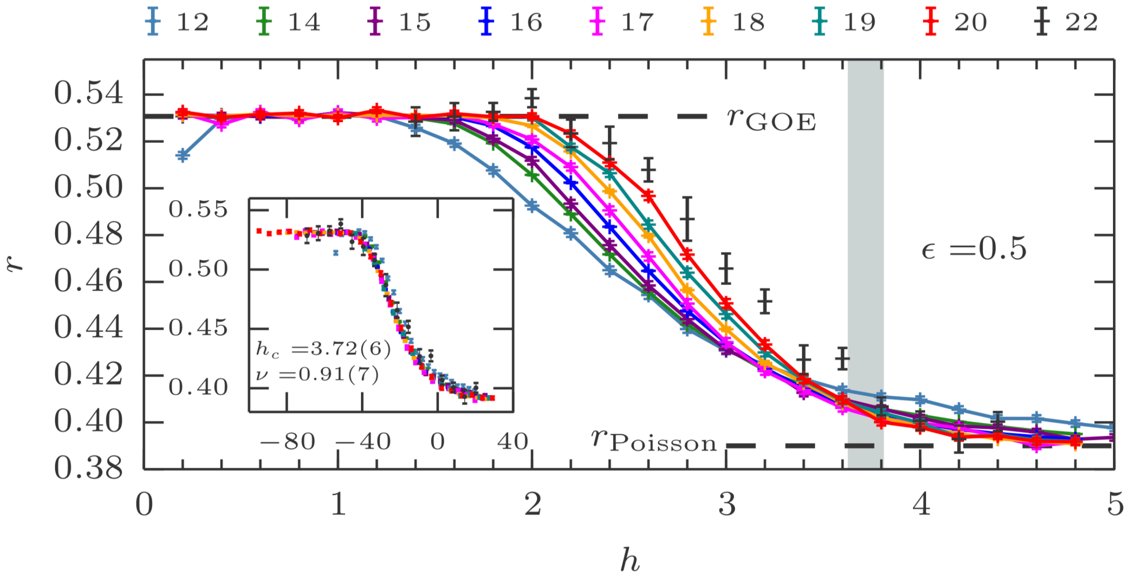
$\longrightarrow$ MBL transition temperature $$T_c = \frac{\Delta_\xi}{\alpha \ln \alpha^{-1}}$$

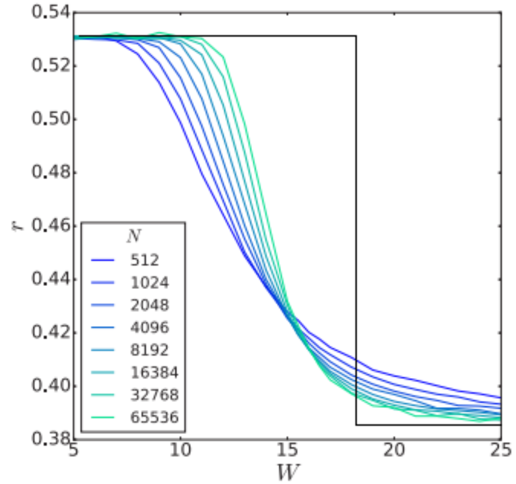
Basko, Aleiner, Altshuler '06: analogous result from SCBA

Numerics for MBL transition in 1D. Analogies to RRG

MBL with short-range interaction: XXZ spin chain in random field

Luitz, Laflorencie, Alet, PRB (2015)   Striking similarities to RRG

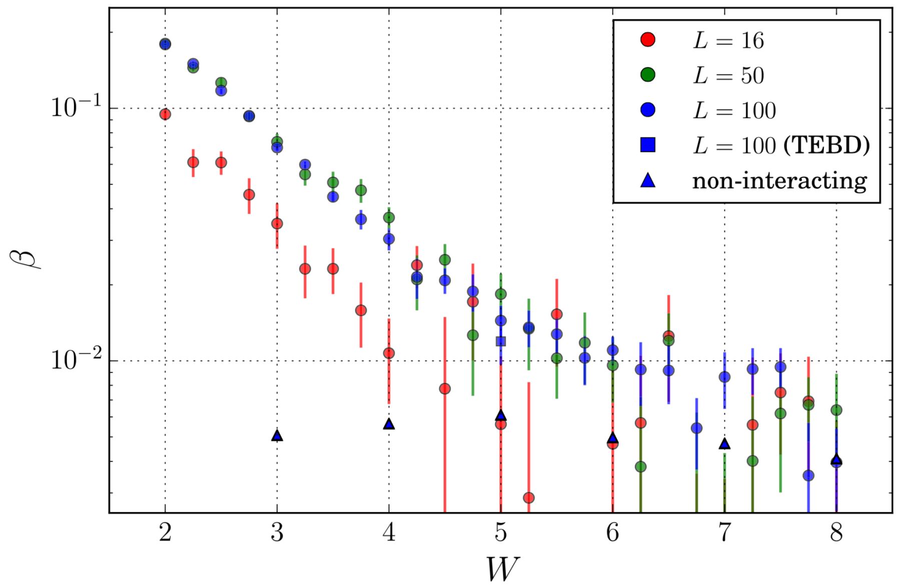




RRG

Tikhonov, ADM, Skvortsov 2016

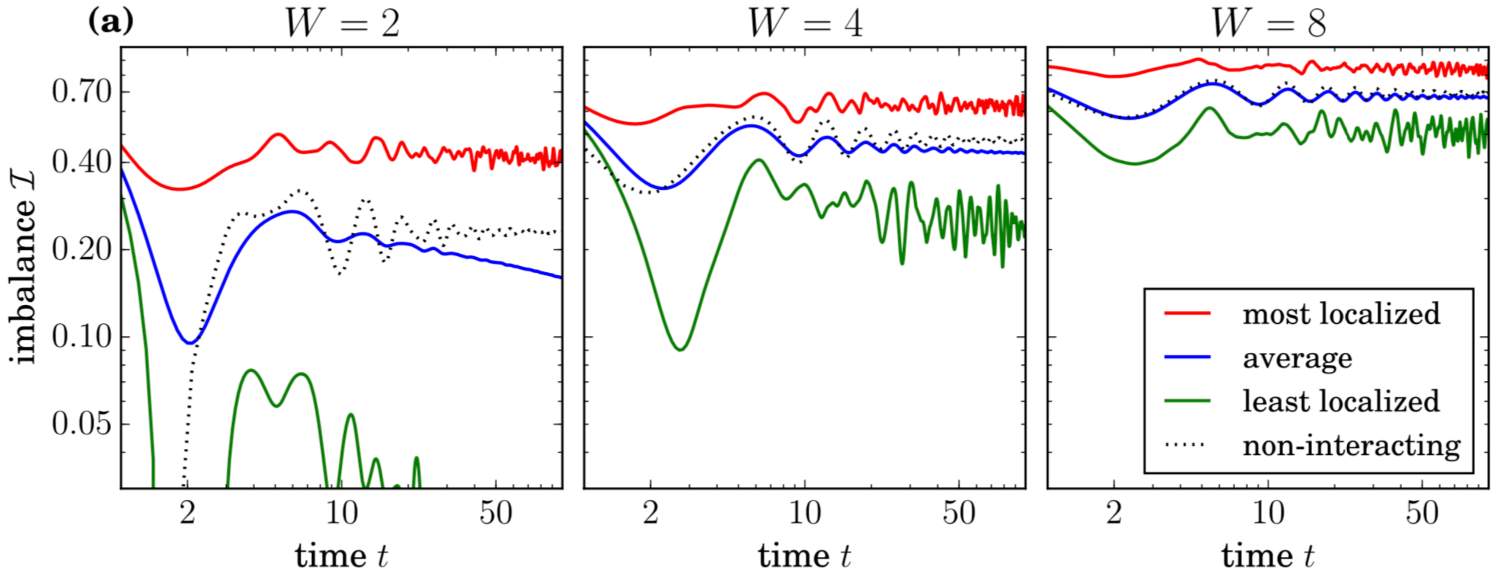
- strong drift of crossing point:
  strong finite-size effects,
  actual transition
  at considerably stronger disorder,
  as also implied by MPS-TDVP study
  Doggen et al, PRB 98, 174202 (2018)
  $W_c \simeq 5.5 - 6$ rather than $3.7 - 3.8$



- critical point similar to localized phase

MBL transition in long Heisenberg chains via MPS-TDVP

Doggen et al 2018



$L = 100$

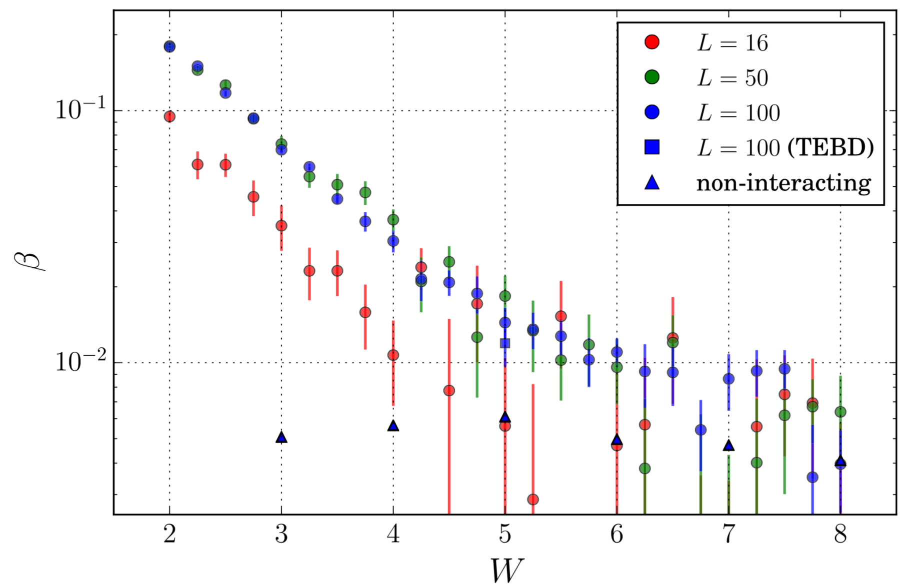
Initial Néel state.

Monitor imbalance $I(t)$ between total densities on even and odd sites.

Imbalance decay $I(t) \sim t^{-\beta}$

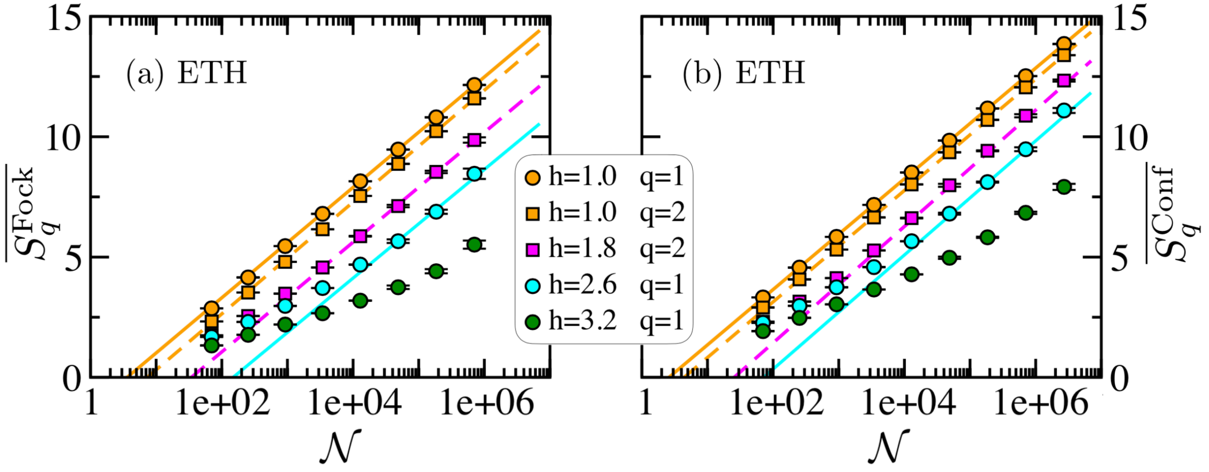
$\beta(W) = 0$ for $W > W_c$

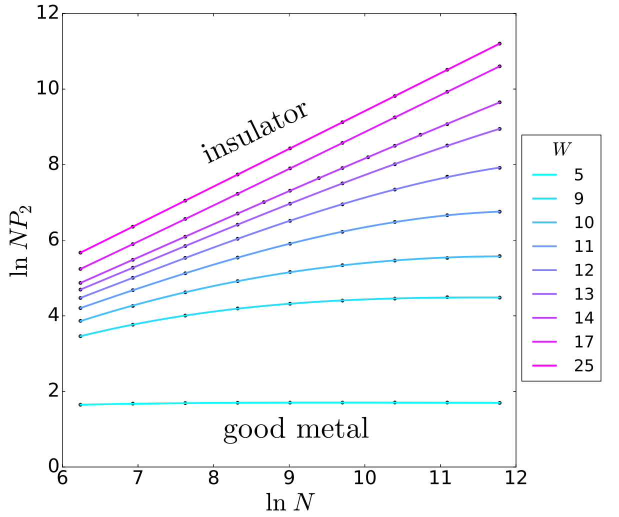
Drift from $W_c \approx 4$ for $L = 16$ to $W_c \simeq 5.5 - 6$ for $L = 50$ and 100



MBL transition in 1D. Analogies to RRG.

- ergodicity of the delocalized phase achieved for Hilbert space size $N \gg N_\xi$





RRG

Tikhonov, ADM, Skvortsov 2016

Macé, Alet, Laflorencie, PRL 2019

- asymmetry of critical behavior:

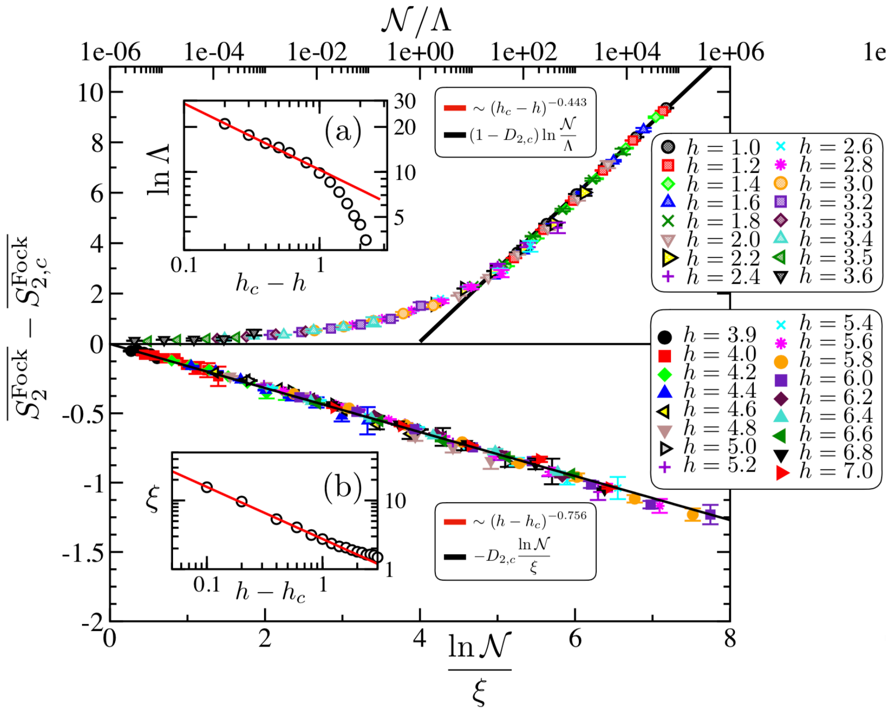
$\nu_{\mathrm{del}} \simeq 0.45$ and $\nu_{\mathrm{loc}} \simeq 0.76$

to be compared to

$\nu_{\mathrm{del}} = 1/2$ and $\nu_{\mathrm{loc}} = 1$ (RRG)

Numerically found exponents for MBL are close to those for RRG and strongly violate Harris criterion

$\longrightarrow$ MBL systems too small to exhibit asymptotic critical behavior



Many-body localization transition: Role of rare regions

- **An avalanche instability may destroy MBL: a thermal seed (rare region of weak disorder) grows and "swallows" the whole system.**

  As a consequence, it was found that the critical disorder $W_c(L)$ grows with L in $d > 1$.

  In particular, $W_c(L) \sim \exp(c \ln^{1/3} L)$ in 2D.

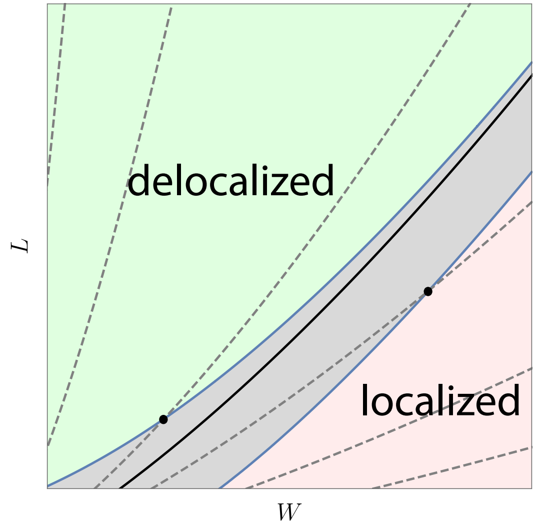
- **Slow, subdiffusive transport in 1D on the ergodic side of the transition is attributed to Griffiths effects (rare regions of strong disorder).**

- **Phenomenological strong-randomness RG for the MBL transition was proposed, which includes the above rare-region physics. It leads to BKT-type transition.**

MBL with long-range interaction and RRG

Random spin chain with $1/r^{\alpha}$ interaction, $d < \alpha < 2d$

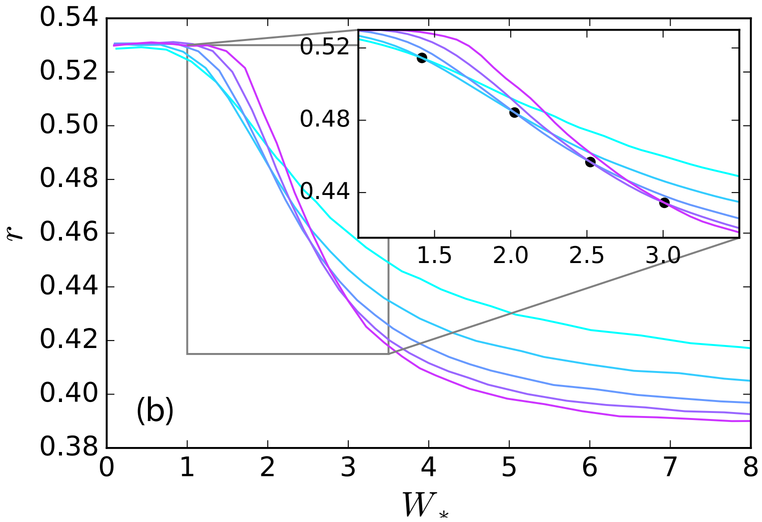
Mapping to RRG $\longrightarrow$ $\boxed{W_c \sim L^{2d-\alpha} \ln L}$

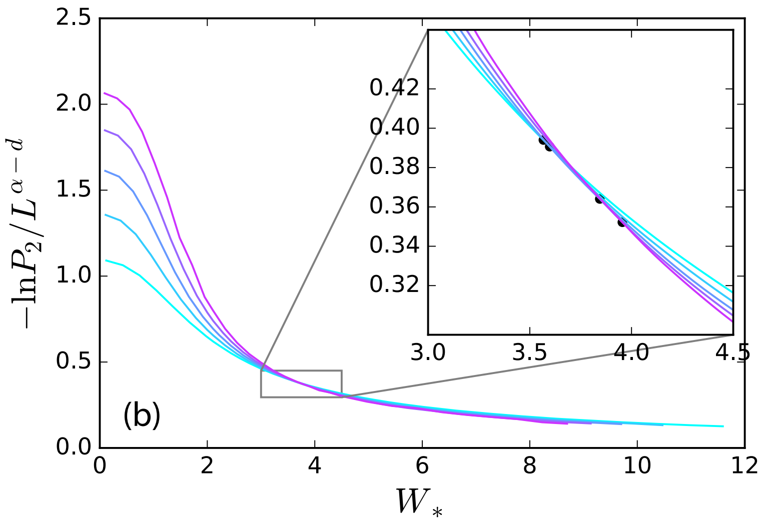


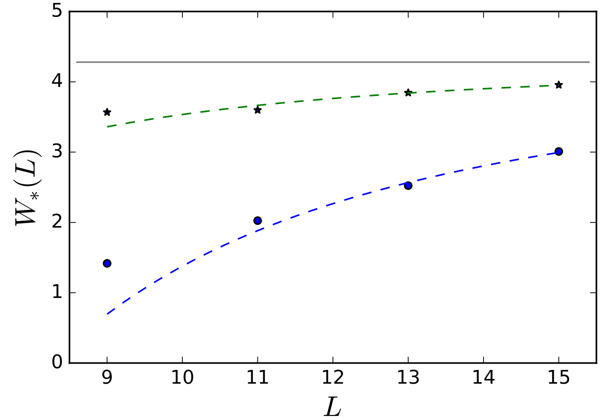
Agreement with exact diagonalization

$d = 1\,, \qquad \alpha = 3/2$

- **Scaling of transition point**
- **Delocalized side: Ergodicity**
- **Critical point** $\longrightarrow$ **drift towards larger** $W_* = W/L^{1/2} \ln L$





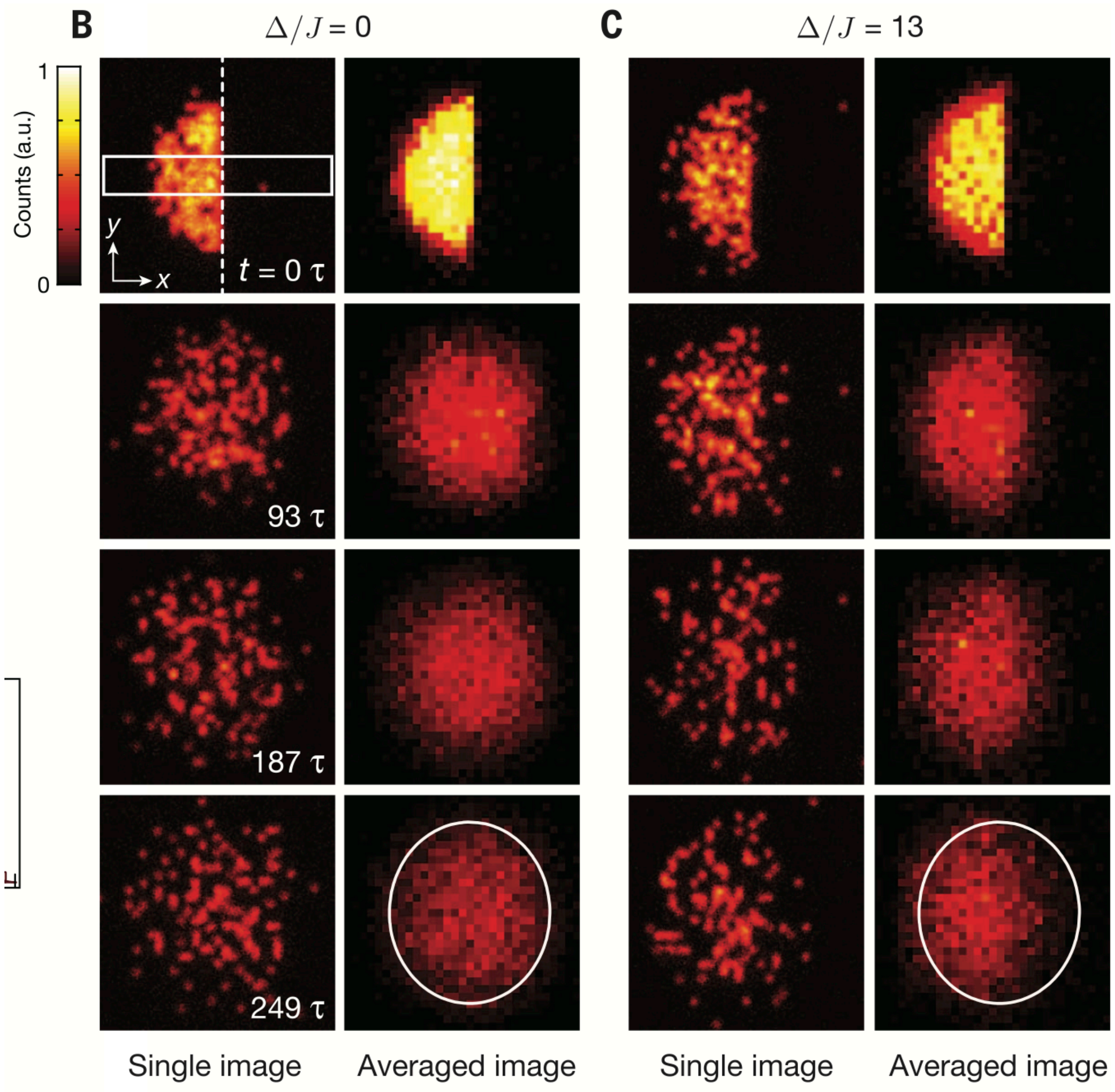


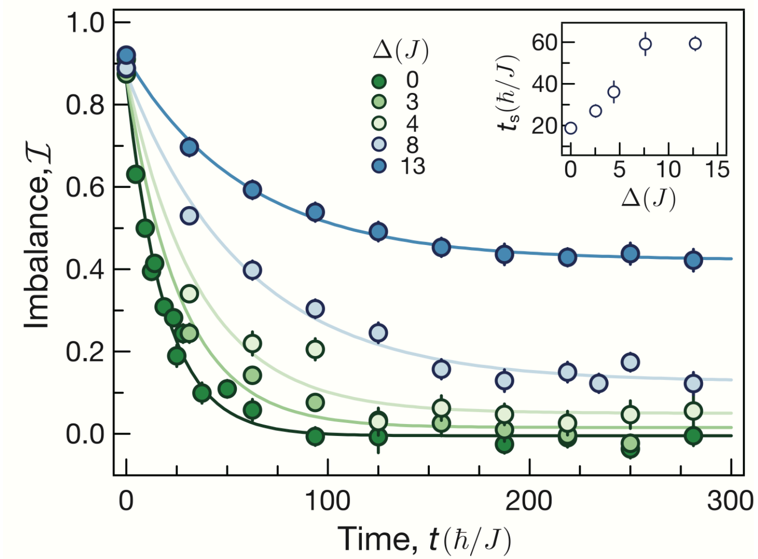
extrapolation to $W_{*c} \simeq 4.3$

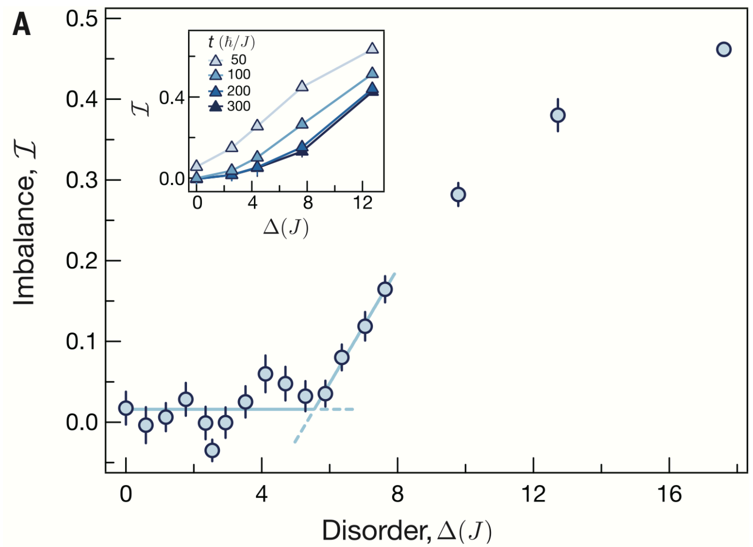
Many-body localization transition: Experiments

Cold atoms in 1D and 2D optical lattices

Schreiber et al, Science 2015; Choi et al, Science 2016 (group of I. Bloch)



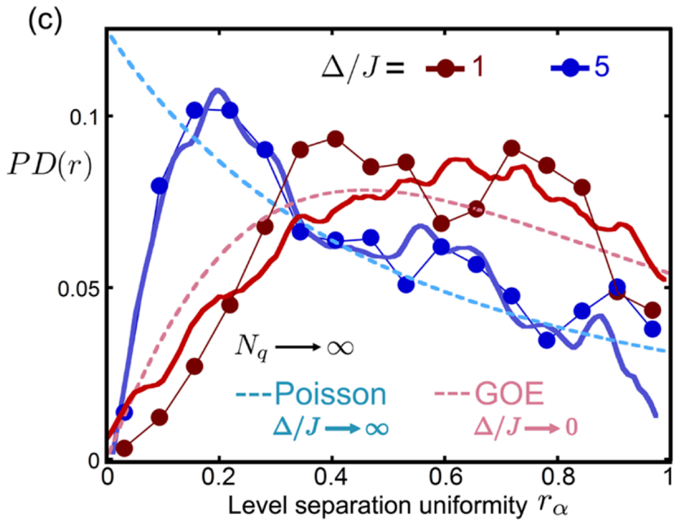


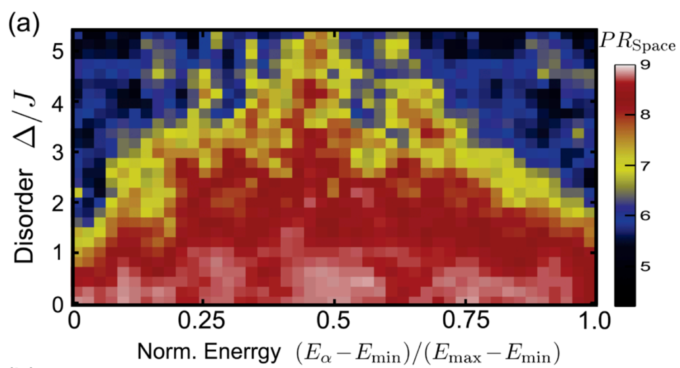


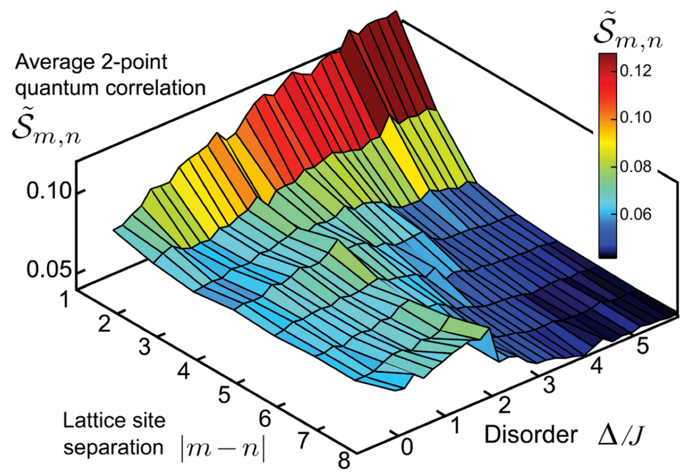
Experiment: MBL transition in a system of coupled superconducting qubits

Roushan, ..., Martinis, Science 2017

Spectroscopy of a chain of 9 superconducting qubits







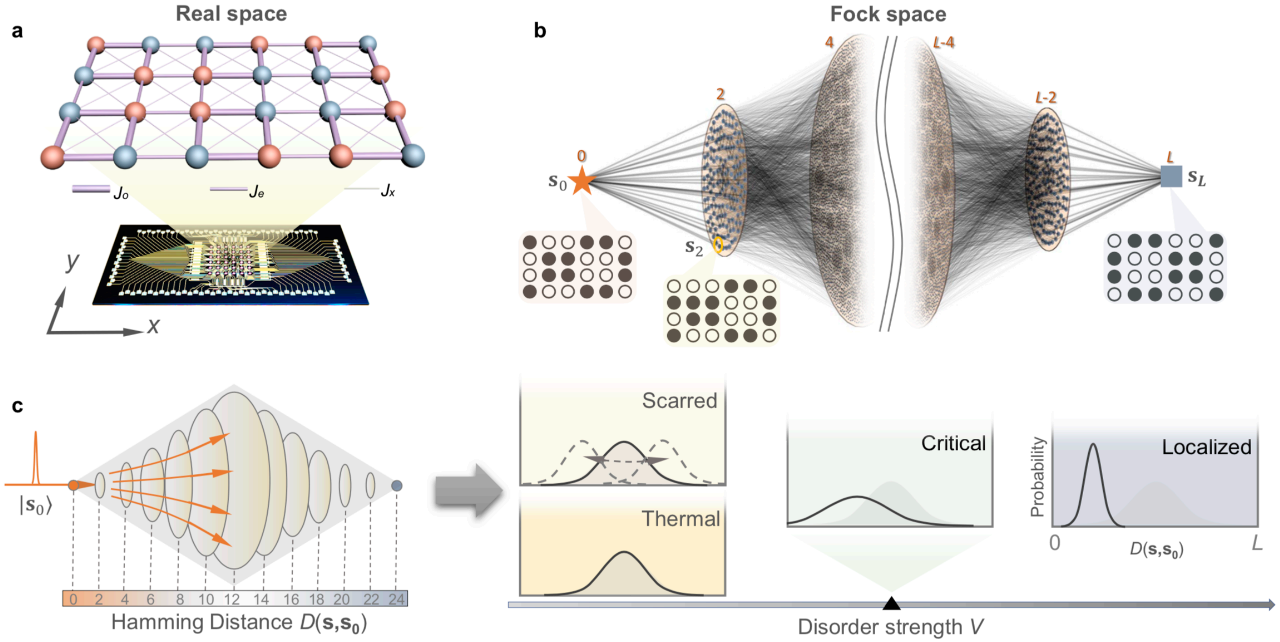
level statistics

spatial IPR of eigenstates

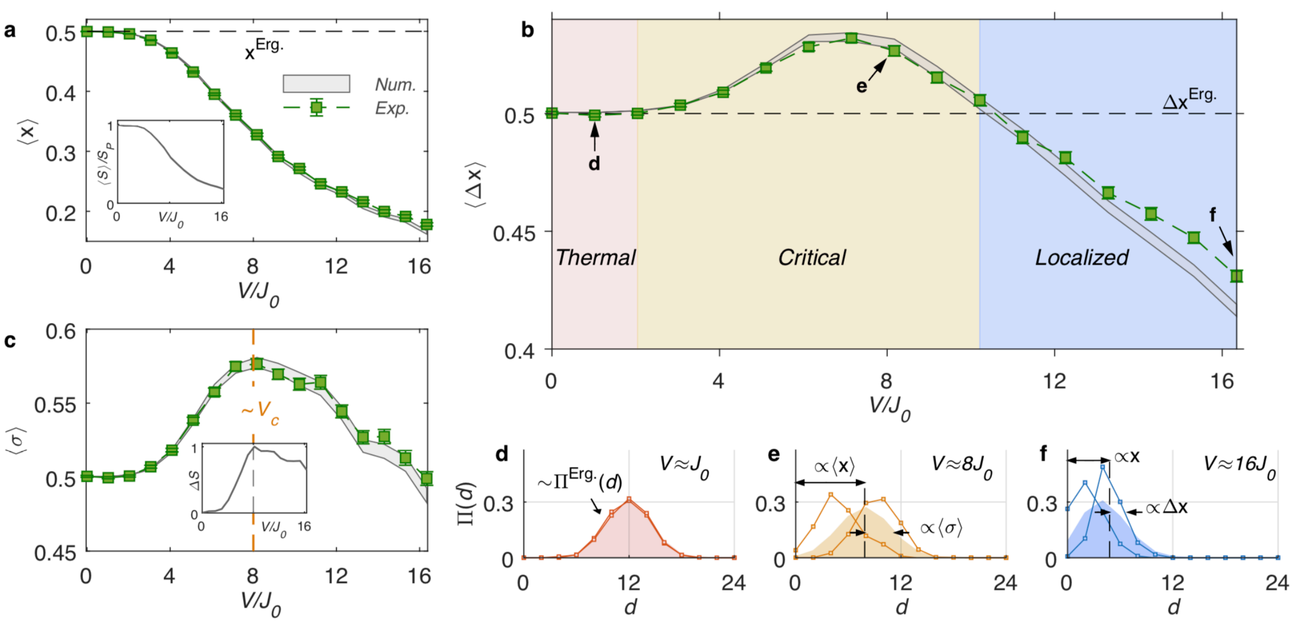
spatial correlations

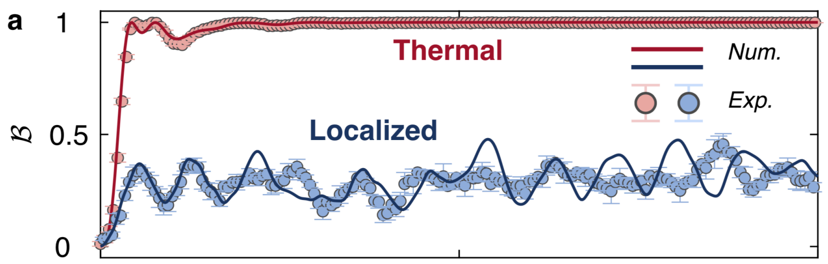
Exp.: MBL transition in 2D superconducting quantum processor

Yao et al, arXiv:2211.05803



Observation of Fock-space dynamics in 6 × 4 qubit array

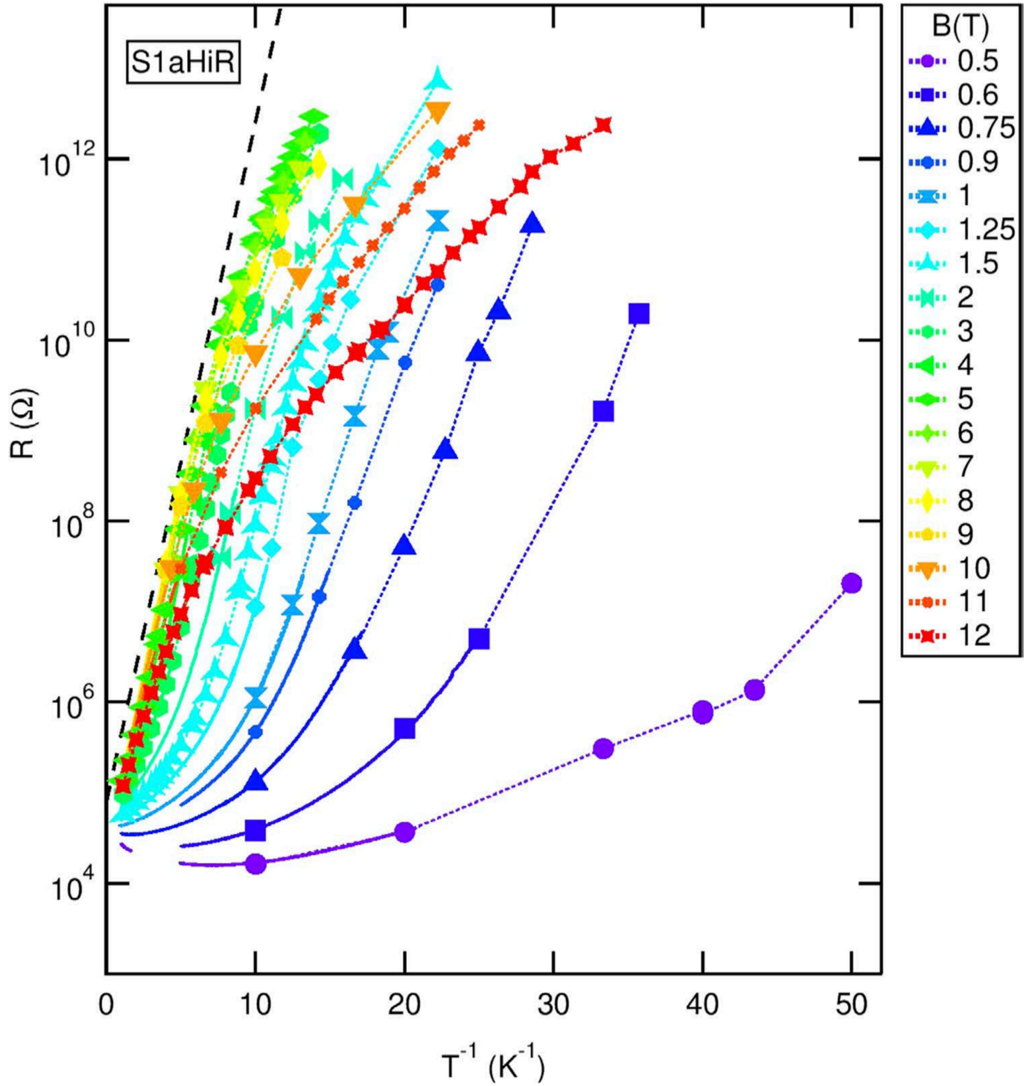




Experiment: Indication of MBL transition near SIT

Insulating side of superconductor-insulator transition in 2D films of InO

Ovadia et al '15 (group of D. Shahar)



vicinity of SIT

$\longrightarrow$ **large localization length**

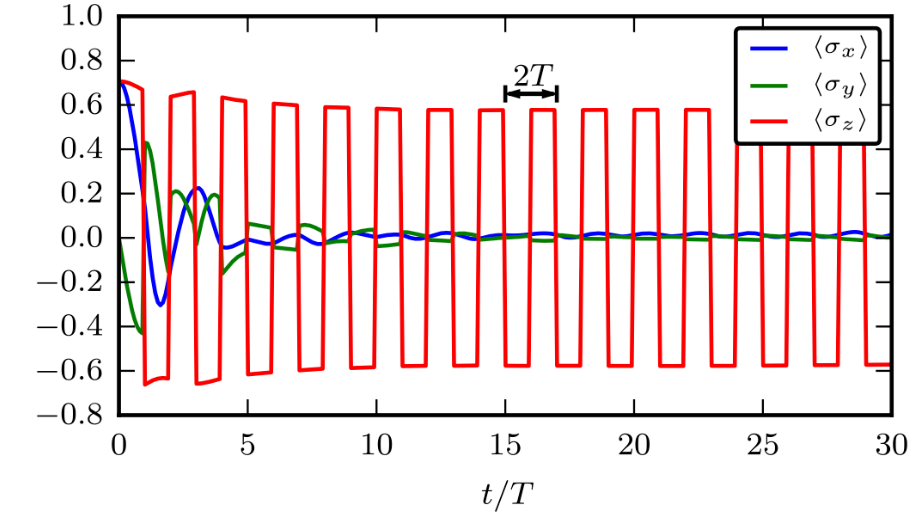
$\longrightarrow$ **large dielectric constant**

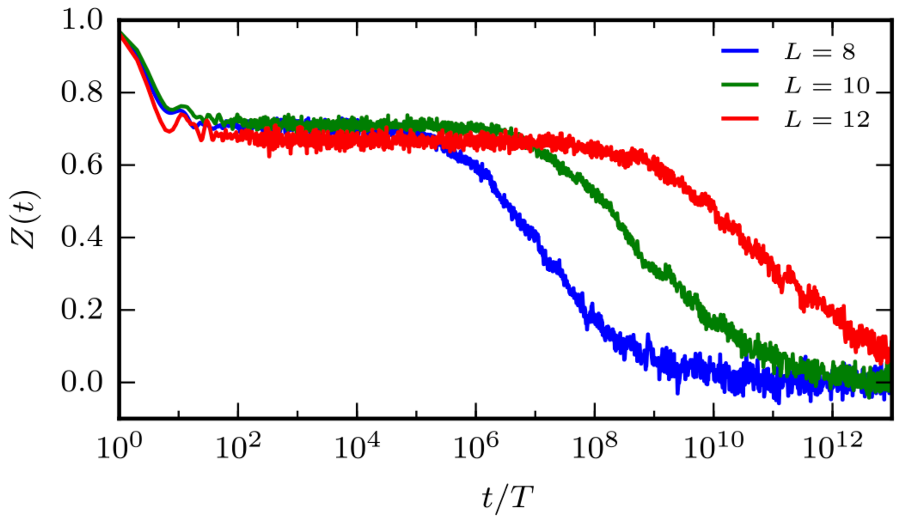
$\longrightarrow$ **strong screening of Coulomb interaction**

$\longrightarrow$ **room for finite-T MBL transition regime?**

Time crystals

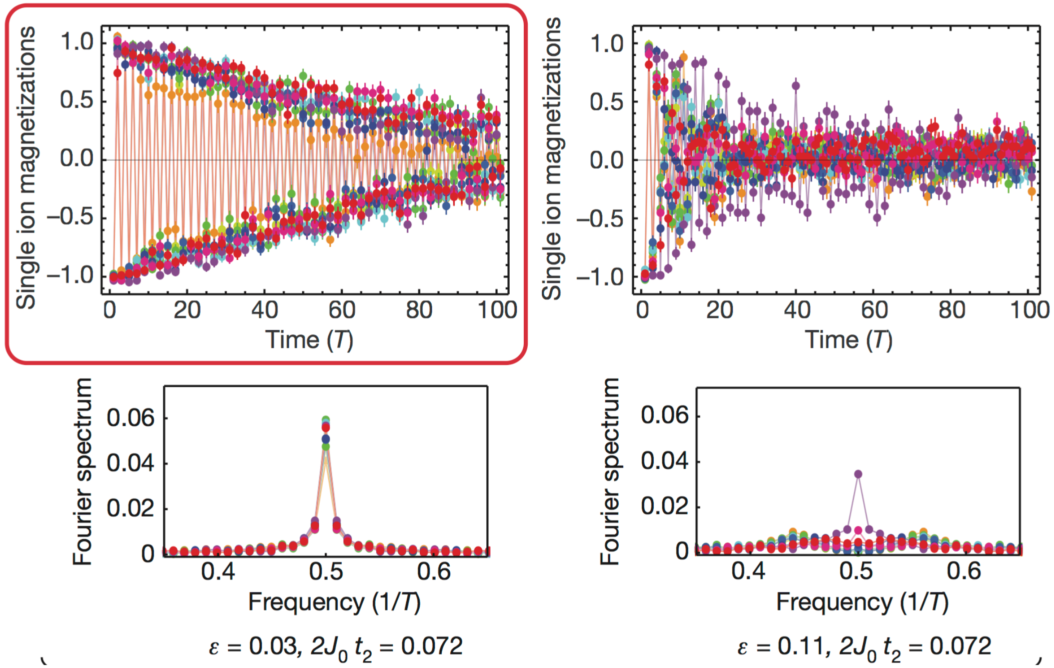
spontaneous breaking of discrete time translation symmetry: subharmonic oscillations in a Floquet driven MBL system

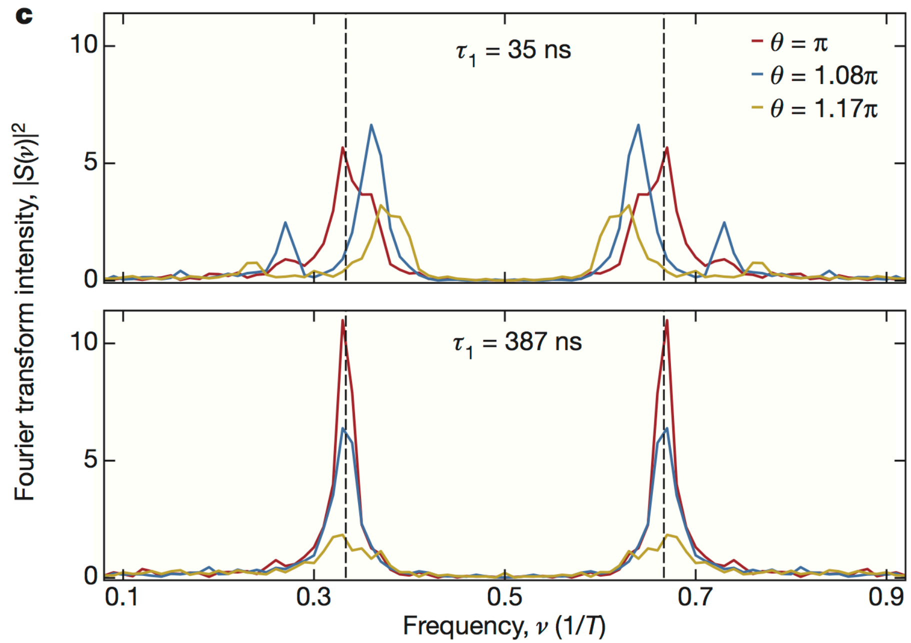




Else, Bauer, Nayak, PRL'16

Time crystals: Experiments





trapped atomic ions

Zhang, Monroe, Nature'17

NV impurity spins in diamond

Choi, ..., Lukin, Nature'17

What we have discussed:

- **I. Disorder and localization**
  - **disorder: diagrammatics, quantum interference, localization**
  - **field theory: non-linear σ-model; quasi-1D geometry: exact solution**
- **II. Criticality and multifractality**
  - **RG, metal-insulator transition, criticality**
  - **Multifractality of wave functions**
- **III. Symmetries and topologies**
  - **symmetry classification of disordered electronic systems**
  - **topological insulators and superconductors; disordered Dirac fermions**
- **IV. Interaction**
  - **electron-electron-interaction: dephasing and renormalization**
  - **Interplay of disorder and interaction; superconductor-insulator transition**
- **V. Localization on tree-like graphs (Random Regular Graphs)**
- **VI. Many-body localization**

Thank you!