Title:

Differential equations, periods, and integrality

Abstract:

Many situations in both pure mathematics and mathematical physics lead to sequences of numbers $\{A_n \mid n \geq 0\}$ that satisfy a recursion of finite length with polynomial coefficients, or equivalently, for which the generating function $\Phi(t) = \sum_{n=0}^{\infty} A_n t^n$ satisfies a linear differential equation with polynomial coefficients. Sometimes a "miracle" occurs and the numbers satisfying this recursion with appropriate initial conditions turn out always to be integers, even though nothing in the recursion suggests why this should happen. An example is given by the Apéry numbers satisfying the 3-term recursion $(n+1)^3 A_{n+1} - (34n^3 + 51n^2 + 27n + 5)u_n + n^3\, u_{n-1} = 0$ with $A_0 = 1$ and $A_1 = 5$, which played the central role in Apéry's famous proof of the irrationality of $\zeta(3)$; here it looks as though the denominator of A_n could be as large as $n!^3$, but in fact all of the A_n are integers. If one changed the coefficients of the ODE even slightly, then this would not happen. A well-known conjecture says that this integrality can only occur if the differential equation of which $\Phi(t)$ is a solution is of Picard-Fuchs type, or equivalently, if $\Phi(t)$ is a period function (= integral of some differential form on the total space of a family of varieties over the projective t-line). We will discuss various aspects of this question, including its relationship to modular forms, and then in a little more detail a particularly interesting case arising from Witten's 1-point 5-spin intersection numbers on the moduli space of curves of genus g. This is joint work with Di Yang.