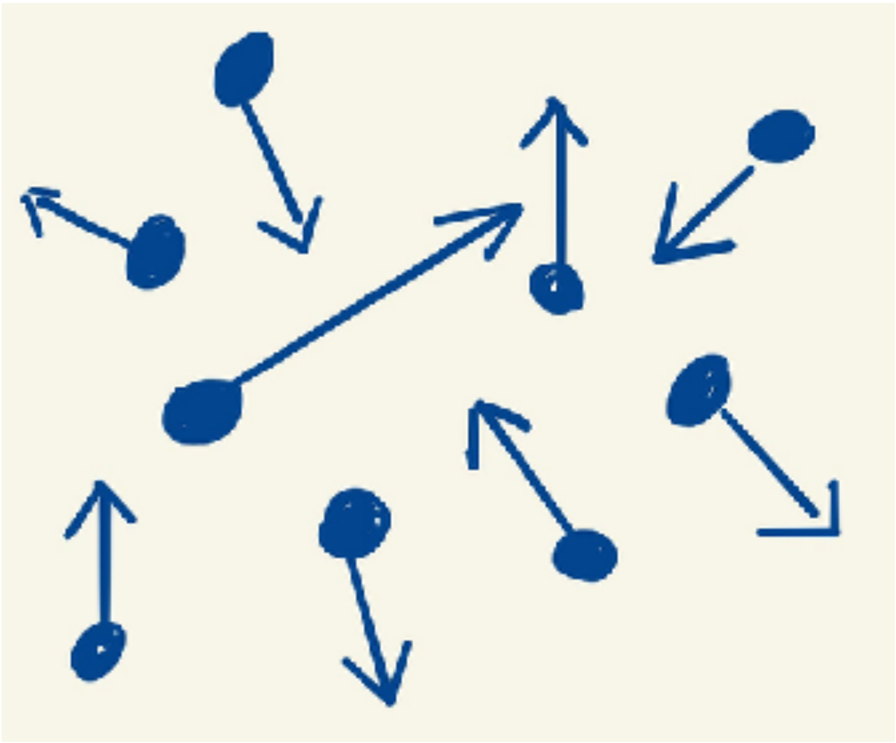
Leonardo Senatore (ETH)

Two different applications of amplitude techniques to Cosmology

Leonardo Senatore (ETH)

On the Effective Field Theory of Large Scale Structure

What is a fluid?



wikipedia: credit
National Oceanic and Atmospheric Administration/
Department of Commerce

$$\partial_t \rho_\ell + \partial_i \left(\rho_\ell v_\ell^i\right) = 0$$

$$\partial_t v_\ell^i + v_\ell^j \partial_j v_\ell^i + \frac{1}{\rho_\ell} \partial_i p_\ell = \text{viscous terms}$$

– From short to long

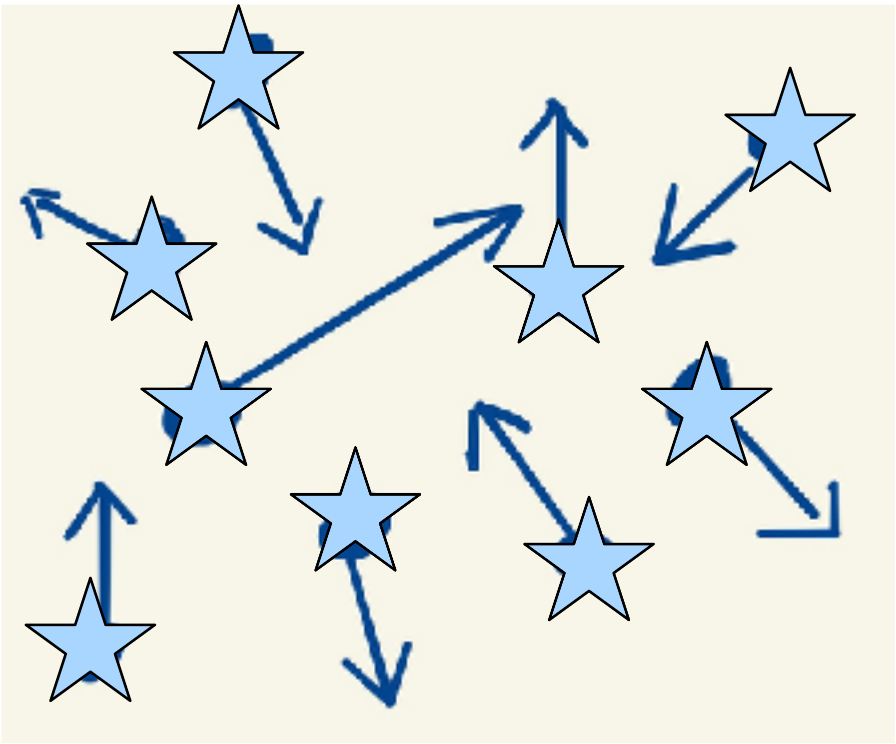
– The resulting equations are simpler

– Description arbitrarily accurate

  – construction can be made without knowing the nature of the particles.

– short distance physics appears as a non trivial stress tensor for the long-distance fluid

Do the same for matter in our Universe



credit NASA

with Baumann, Nicolis and Zaldarriaga **JCAP 2012**
with Carrasco and Hertzberg **JHEP 2012**

$$\nabla^2 \Phi_\ell = H^2 \left(\delta\rho_\ell / \rho \right)$$
$$\partial_t \rho_\ell + H \rho_\ell + \partial_i \left(\rho_\ell v_\ell^i \right) = 0$$
$$\partial_t v_\ell^i + v_\ell^j \partial_j v_\ell^i + \partial_i \Phi_\ell = \partial_j \tau^{ij}$$

– From short to long

– The resulting equations are simpler

– Description arbitrarily accurate

  – construction can be made without knowing the nature of the particles.

– short distance physics appears as a non trivial stress tensor for the long-distance fluid

$$\tau_{ij} \sim \delta_{ij}\, \rho_{\text{short}} \left(v^2_{\text{short}} + \Phi_{\text{short}} \right)$$

Dealing with the Effective Stress Tensor

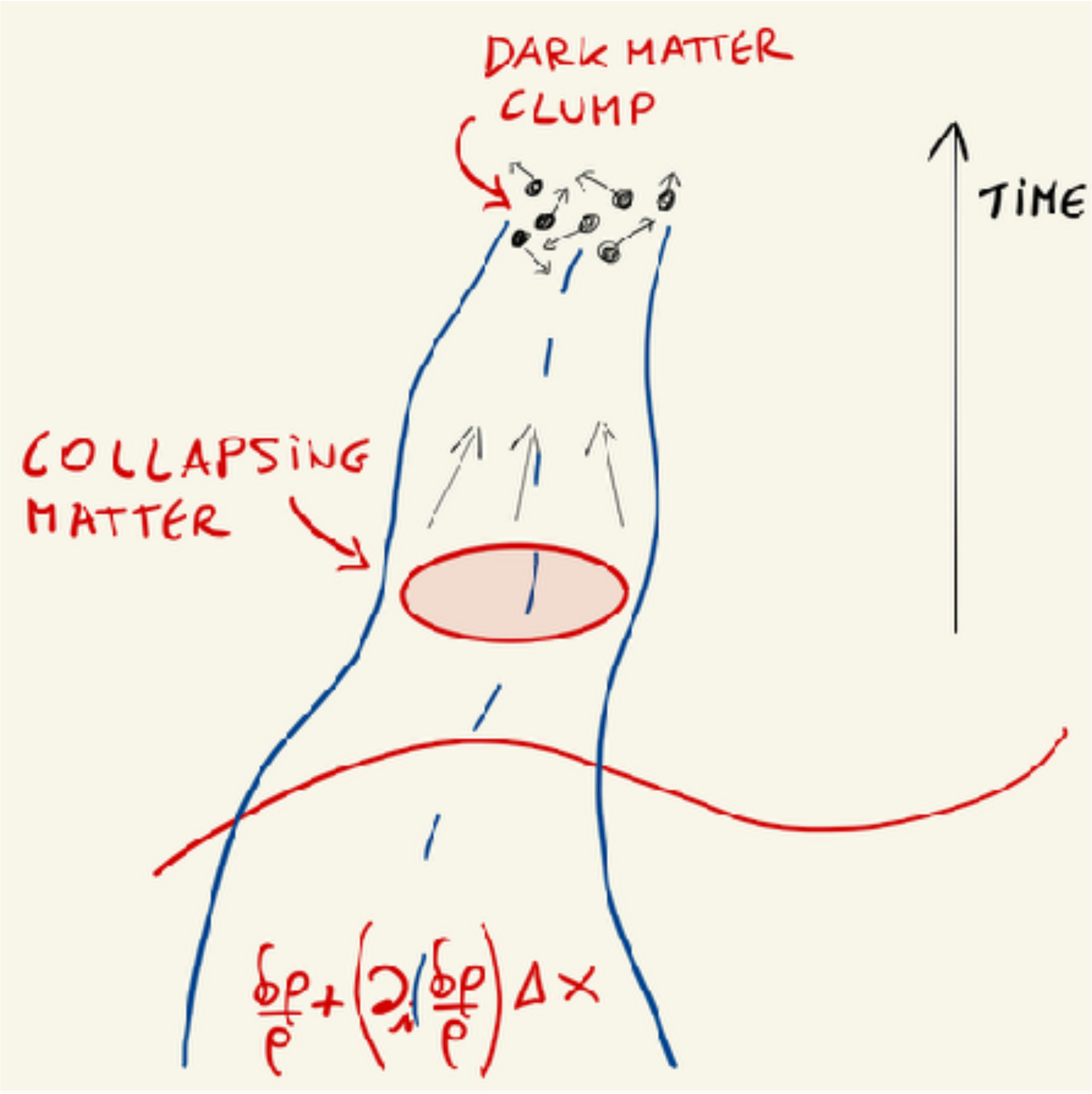
- Using our lessons from quantum field theory, we express the stress tensor with all the terms allowed by the symmetries.

- Equations with only long-modes

$$\partial_t v_\ell^i + v_\ell^j \partial_j v_\ell^i + \partial_i \Phi_\ell = \partial_j \tau^{ij}$$

$$\tau_{ij} \sim \delta\rho_\ell/\rho + \ldots$$

every term allowed by symmetries



Perturbation Theory within the EFT

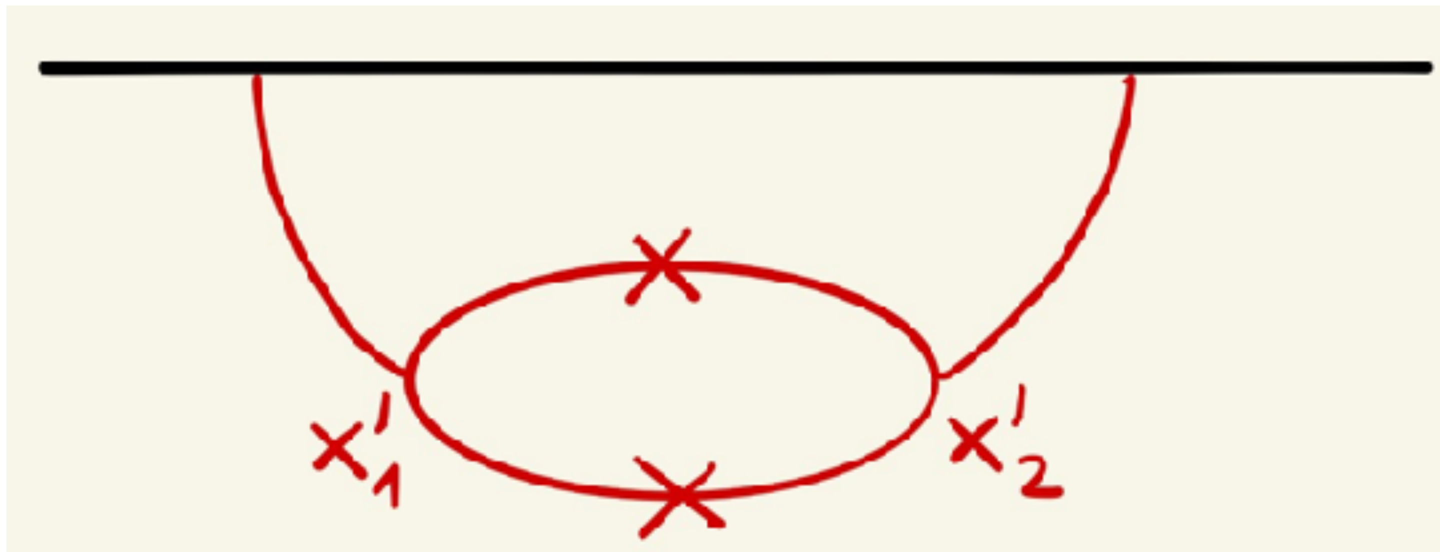
- In the EFT we can solve iteratively $\delta_\ell, v_\ell, \Phi_\ell \ll 1$, where $\delta_\ell = \dfrac{\delta\rho_\ell}{\rho}$

$$\nabla^2\Phi_\ell = H^2\left(\delta\rho_\ell/\rho\right)$$
$$\partial_t\rho_\ell + H\rho_\ell + \partial_i\left(\rho_\ell v_\ell^i\right) = 0$$
$$\partial_t v_\ell^i + v_\ell^j\partial_j v_\ell^i + \partial_i\Phi_\ell = \partial_j\tau^{ij}$$

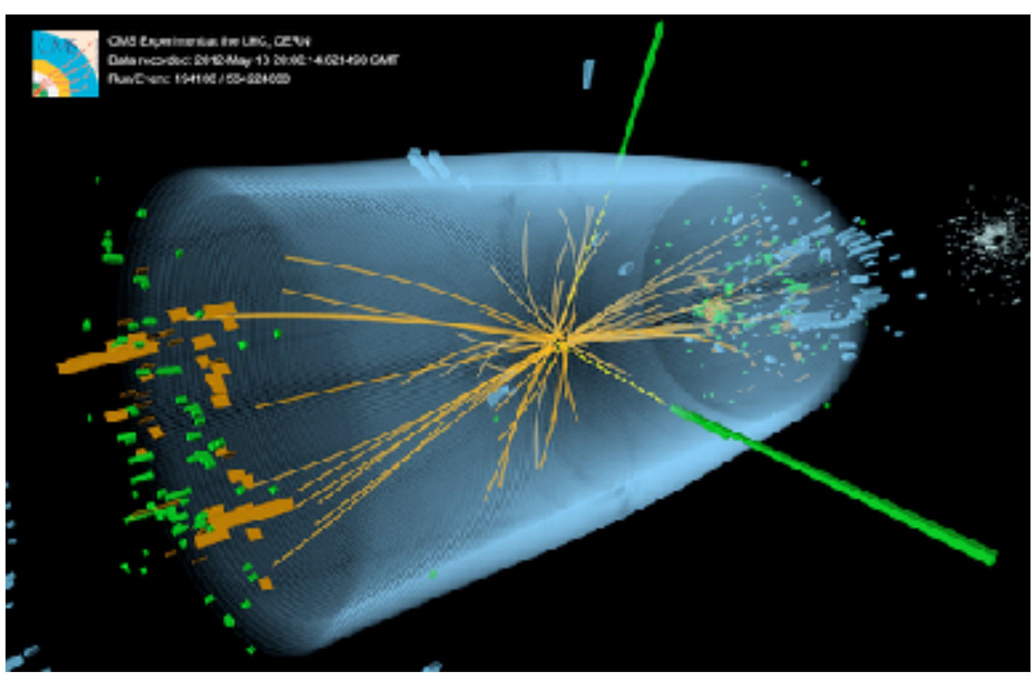
$$\tau_{ij} \sim \delta\rho_\ell/\rho + \ldots$$

Perturbation Theory within the EFT

- Solve iteratively some non-linear eq. $\delta_\ell = \delta_\ell^{(1)} + \delta_\ell^{(2)} + \ldots \ll 1$
- We end up with the same Feynman diagrams as Prof. Feynman:



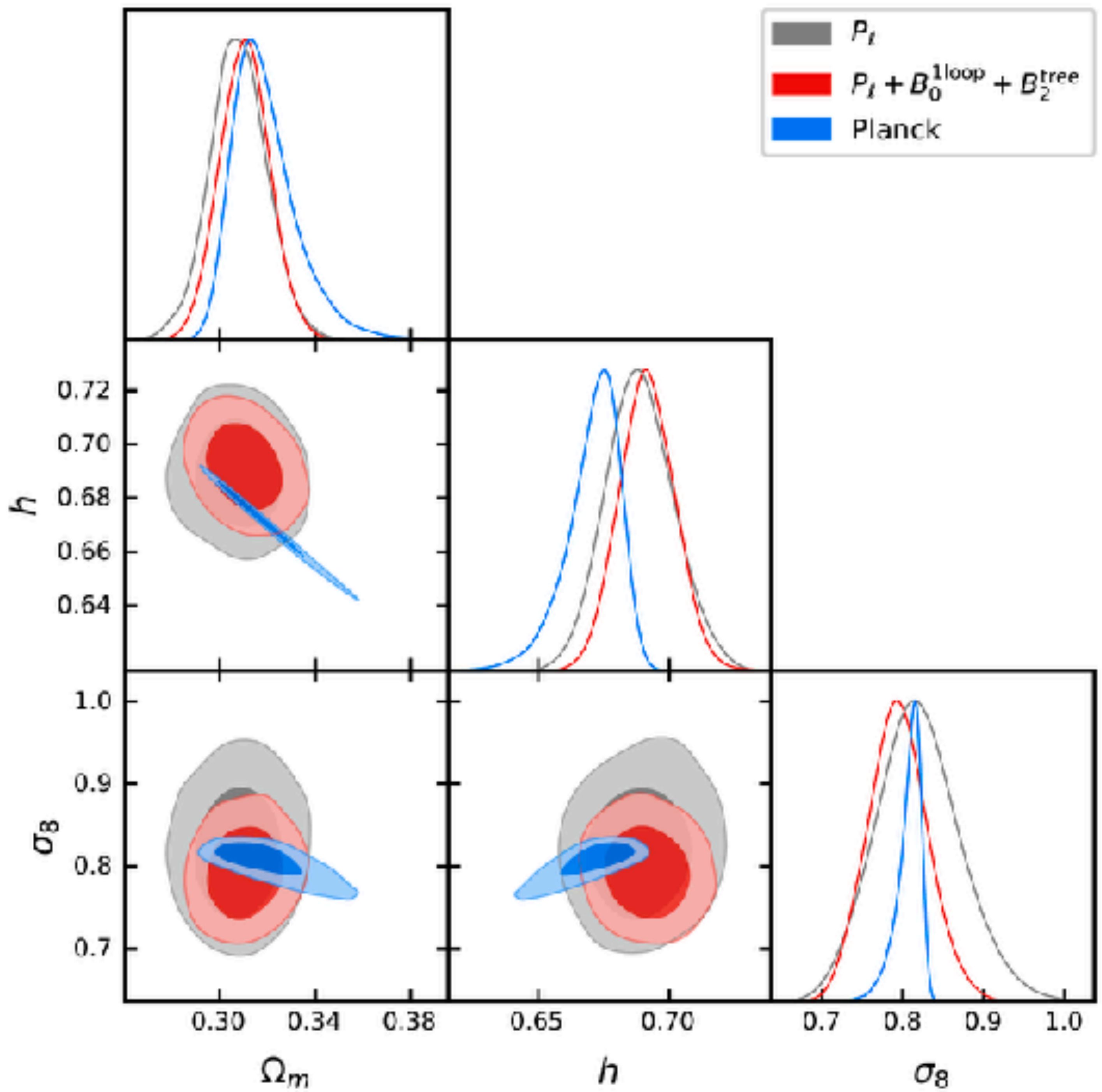
- but this time `lines' are not `quantum elementary particles', but `waves of galaxies'
- and so we seek help of the expert of particle physics



see eg. Simonovic, Baldauf, Zaldarriaga, Carrasco, Kollmeier **2018**
with Anastasiou, Braganca, Zheng **2212**

Application to data has teeth

- Though non-linearities are non-negligible, we can compute them, at long distances.
    - … it actually took a decade-long of work to do that…
- Bounds similar to CMB:



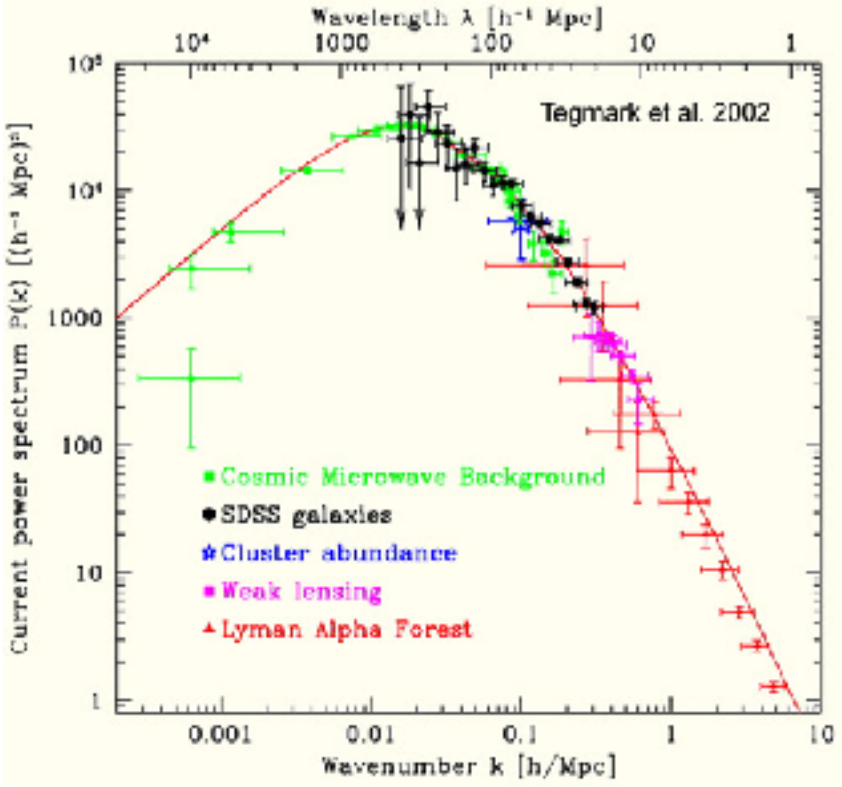
Evaluational/Computational Challenge

with Anastasiou, Braganca, Zheng **2212**

The best approach so far

Simonovic, Baldauf, Zaldarriaga, Carrasco, Kollmeier **2018**

- Nice trick for fast evaluation of the loops integrals
- The power spectrum is a numerically computed function
- Decompose linear power spectrum



$$P_{11}(k) = \sum_n c_n\, k^{\mu + i\alpha\, n}$$

- Loop can be evaluated analytically

$$P_{1-\mathrm{loop}}(k) = \int_{\vec{q}} K(\vec{q}, \vec{k})\ P_{11}(k-q)\, P_{11}(q) =$$

$$= \sum_{n_1, n_2} c_{n_1} c_{n_2} \left(\int_{\vec{q}} K(\vec{q}, \vec{k})\ k^{\mu + i\alpha n_1}\ k^{\mu + i\alpha n_2} \right) = \sum_{n_1, n_2} c_{n_1} c_{n_2} M_{n_1, n_2}(k)$$

–using quantum field theory techniques

– $M_{n_1 n_2}$ is cosmology independent $\Rightarrow$ so computed once

Computational Challenge

Philcox, Ivanov, Cabass, Simonovic, Zaldarriaga **2022**

- Two difficulties:

$$P_{1-\text{loop}}(k) = \int_{\vec{q}} K(\vec{q},\vec{k})\ P_{11}(k-q)\ P_{11}(q) =$$

$$= \sum_{n_1,n_2} c_{n_1} c_{n_2} \left(\int_{\vec{q}} K(\vec{q},\vec{k})\ k^{\mu+i\alpha n_1}\ k^{\mu+i\alpha n_2} \right) = \sum_{n_1,n_2} c_{n_1} c_{n_2} M_{n_1,n_2}(k)$$

  - integrals are complicated due to fractional, complex exponents
  - many functions needed, the matrix $M_{n_1 n_2 n_3}$ for bispectrum is about 50Gb, so, ~impossible to load on CPT for data analysis

- In order to ameliorate (solve) these issues, we use a different basis of functions.

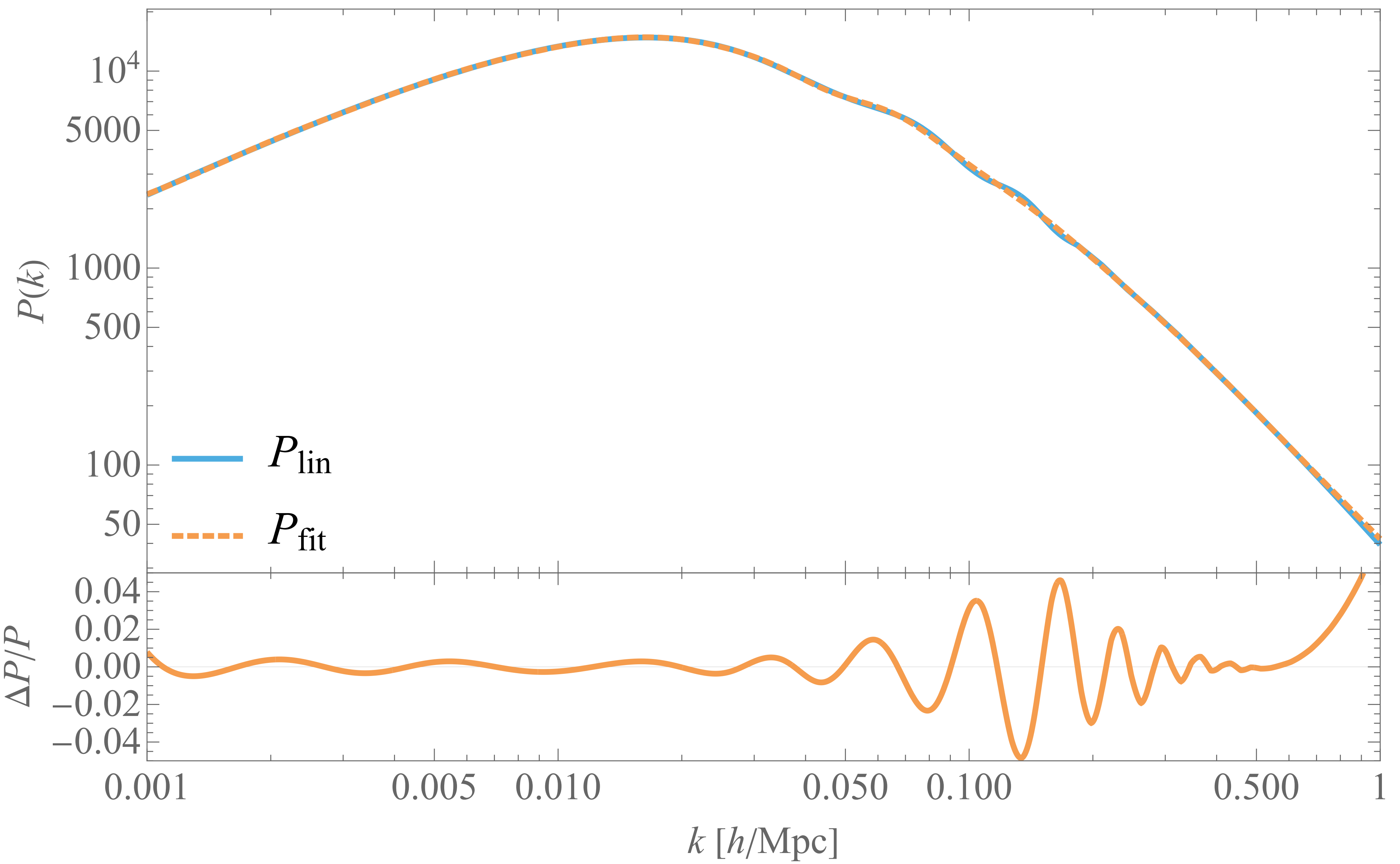
Complex-Masses Propagators

with Anastasiou, Braganca, Zheng
2212

- Use as basis:

$$f(k^2, k^2_{\rm peak}, k^2_{\rm UV}, i, j) \equiv \frac{\left(k^2/k_0^2\right)^i}{\left(1+\frac{(k^2-k^2_{\rm peak})^2}{k^4_{\rm UV}}\right)^j},$$

- With just 16 functions:



Complex-Masses Propagators

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- This basis is equivalent to massive propagators to integer powers

$$\frac{1}{\left(1+\frac{(k^2-k_{\text{peak}}^2)^2}{k_{\text{UV}}^4}\right)^j} = \frac{k_{\text{UV}}^{4j}}{\left(k^2-k_{\text{peak}}^2-i\,k_{\text{UV}}^2\right)^j\left(k^2-k_{\text{peak}}^2+i\,k_{\text{UV}}^2\right)^j},$$

$$\frac{k_{\text{UV}}^2}{\left(k^2-k_{\text{peak}}^2-i\,k_{\text{UV}}^2\right)\left(k^2-k_{\text{peak}}^2+i\,k_{\text{UV}}^2\right)} = -\frac{i/2}{k^2-k_{\text{peak}}^2-i\,k_{\text{UV}}^2}+\frac{i/2}{k^2-k_{\text{peak}}^2+i\,k_{\text{UV}}^2}$$

- So, each basis function:

$$f(k^2,k_{\text{peak}}^2,k_{\text{UV}}^2,i,j) = \sum_{n=1}^{j} k_{\text{UV}}^{2(n-i)}\,k^{2i}\left(\frac{\kappa_n}{(k^2+M)^n}+\frac{\kappa_n^*}{(k^2+M^*)^n}\right)$$

Complex-Masses Propagators

with Anastasiou, Braganca, Zheng **2212**

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$$\frac{1}{\left(1+\frac{(k^2-k_{\text{peak}}^2)^2}{k_{\text{UV}}^4}\right)^j} = \frac{k_{\text{UV}}^{4j}}{\left(k^2-k_{\text{peak}}^2-i\,k_{\text{UV}}^2\right)^j\left(k^2-k_{\text{peak}}^2+i\,k_{\text{UV}}^2\right)^j},$$

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Complex-Mass propagator

- So, each basis function:

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Complex-Masses Propagators

with Anastasiou, Braganca, Zheng
2212

- We end up with integral like this:

$$L(n_1, d_1, n_2, d_2, n_3, d_3) = \int_q \frac{(\boldsymbol{k}_1 - \boldsymbol{q})^{2n_1} \boldsymbol{q}^{2n_2} (\boldsymbol{k}_2 + \boldsymbol{q})^{2n_3}}{((\boldsymbol{k}_1 - \boldsymbol{q})^2 + M_1)^{d_1} (\boldsymbol{q}^2 + M_2)^{d_2} ((\boldsymbol{k}_2 + \boldsymbol{q})^2 + M_3)^{d_3}}$$

- with integer exponents.
- First we manipulate the numerator to reduce to:

$$T(d_1, d_2, d_3) = \int_q \frac{1}{((\boldsymbol{k}_1 - \boldsymbol{q})^2 + M_1)^{d_1} (\boldsymbol{q}^2 + M_2)^{d_2} ((\boldsymbol{k}_2 + \boldsymbol{q})^2 + M_3)^{d_3}},$$

- Then, by integration by parts, we find (i.e. Babis teach us how to) recursion relations

$$\int_q \frac{\partial}{\partial q_\mu} \cdot (q_\mu t(d_1, d_2, d_3)) = 0$$

$$\Rightarrow \quad (3 - d_{1223})\hat{0} + d_1 k_{1s} \widehat{1^+} + d_3 (k_{2s}) \widehat{3^+} + 2 M_2 d_2 \widehat{2^+} - d_1 \widehat{1^+} \widehat{2^-} - d_3 \widehat{2^-} \widehat{3^+} = 0$$

- relating same integrals with raised or lowered the exponents (easy terminate due to integer exponents).

Complex-Masses Propagators

with Anastasiou, Braganca, Zheng
2212

- We end up to three master integrals:
- Tadpole:

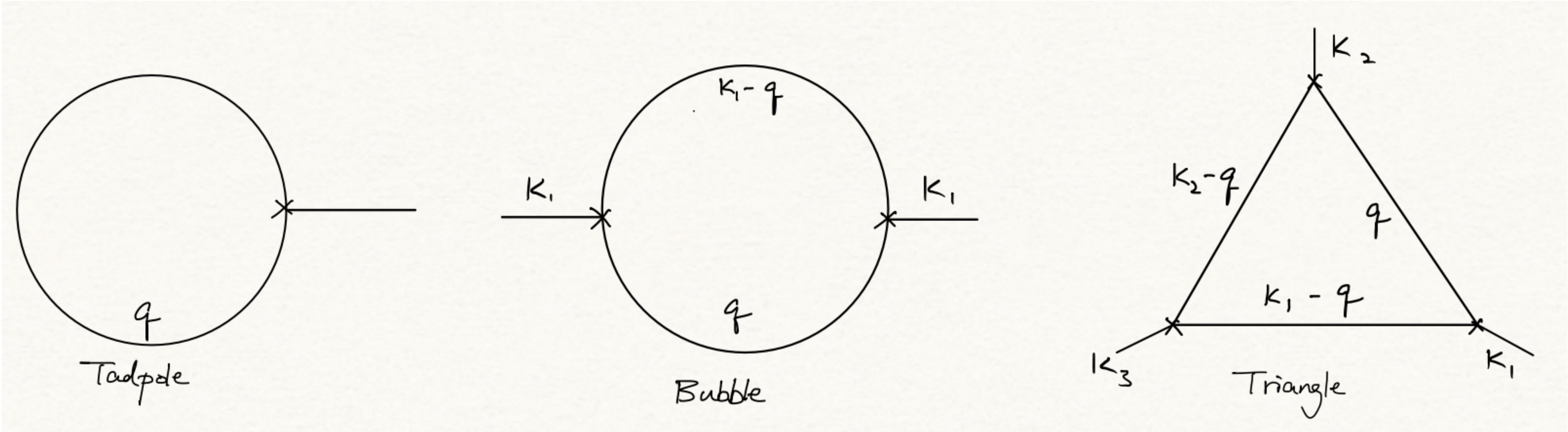
$$\text{Tad}(M_j, n, d) = \int \frac{d^3\boldsymbol{q}}{\pi^{3/2}} \frac{(\boldsymbol{p}_i^2)^n}{(\boldsymbol{p}_i^2 + M_j)^d}$$

- Bubble:

$$B_{\text{master}}(k^2, M_1, M_2) = \int \frac{d^3\boldsymbol{q}}{\pi^{3/2}} \frac{1}{(q^2 + M_1)(|\boldsymbol{k} - \boldsymbol{q}|^2 + M_2)}$$

- Triangle:

$$T_{\text{master}}(k_1^2, k_2^2, k_3^2, M_1, M_2, M_3) = \int \frac{d^3\boldsymbol{q}}{\pi^{3/2}} \frac{1}{(q^2 + M_1)(|\boldsymbol{k}_1 - \boldsymbol{q}|^2 + M_2)(|\boldsymbol{k}_2 + \boldsymbol{q}|^2 + M_3)},$$



Complex-Masses Propagators

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- The master integrals are evaluated with Feynman parameters, but with great care of branch cut crossing, which happens because of complex masses.

- Bubble Master:

$$B_{\text{master}}(k^2, M_1, M_2) = \frac{\sqrt{\pi}}{k} i[\log\left(A(1, m_1, m_2)\right) - \log\left(A(0, m_1, m_2)\right)$$
$$- 2\pi i H(\text{Im}\, A(1, m_1, m_2)) H(-\text{Im}\, A(0, m_1, m_2))] \,,$$

$$A(0, m_1, m_2) = 2\sqrt{m_2} + i(m_1 - m_2 + 1)\,,$$
$$A(1, m_1, m_2) = 2\sqrt{m_1} + i(m_1 - m_2 - 1)\,,$$
$$m_1 \;=\; M_1/k^2 \;\text{ and }\; m_2 \;=\; M_2/k^2$$

- Triangle Master:

$$F_{\text{int}}(R_2, z_+, z_-, x_0) = s(z_+, -z_-)\frac{\sqrt{\pi}}{\sqrt{|R_2|}}\, \frac{\arctan\left(\frac{\sqrt{z_+ - x}\sqrt{x_0 - z_-}}{\sqrt{x_0 - z_+}\sqrt{x - z_-}}\right)}{\sqrt{x_0 - z_+}\sqrt{x_0 - z_-}}\Bigg|_{x=0}^{x=1} .$$

- Very simple expressions with simple rule for branch cut crossing.

- In 3d, these are the same integrals for all n-point functions!

Leonardo Senatore (ETH)

Positivity bounds on effective field theories with spontaneously broken Lorentz

with Creminelli, Janssen **JHEP 2022**

EFT's & Positivity bounds

- EFT's are the common framework to describe phenomena below a certain energy.
- Given a set of DOF, write down all operators allowed by the symmetries
- Is every operator possible? With arbitrary prefactor?
- The seminal work of Allan Adams et al, **2006** showed that, by assuming unitarity, locality and Lorentz invariance of the UV completion, there are bounds on some coefficients.
  - This is very interesting theoretically and experimentally.
  - Much much work has followed since then, and is happening today.

e.g. Caron Hout and Van Duong **2020**

EFT's & Positivity bounds

- Is it possible to extend such a program to theories with Lorentz invariance, and in particular boosts, are spontaneously broken?
  - Typical regime for Cosmology and Condensed matter
- Why that would be interesting?
  - Cosmology:
    - Not so many data
    - Peculiar looking theories:
      - Galileons, Ghost Condensate
        - » While strange behaviors in Lorentz invariant limit, not clear the broken phase can be ruled out.
  - Condensed Matter
- One could perhaps argue that these kinds of Lagrangians are much more numerous to probe experimentally.

EFT's & Positivity bounds

– Using that the Lorentz-breaking EFT is originating from a Lorentz preserving one is not easy.

– Normal bounds are based on $2 \to 2$ scattering. But in Lorentz breaking background operators with many legs become relevant.

$$(\partial\phi)^n \to (\dot{\phi}_0)^{n-2}(\partial\delta\phi)^2$$

– not much is know about scattering $n \to m$

– Sometimes it is very hard to connect the Lorentz preserving and Lorentz breaking theories: e.g. fluids. There is no straightforward limit.

– Therefore, try to study directly the broken phase.

Review of Lorentz Invariant case

–Useful/needed properties. The S-matrix:

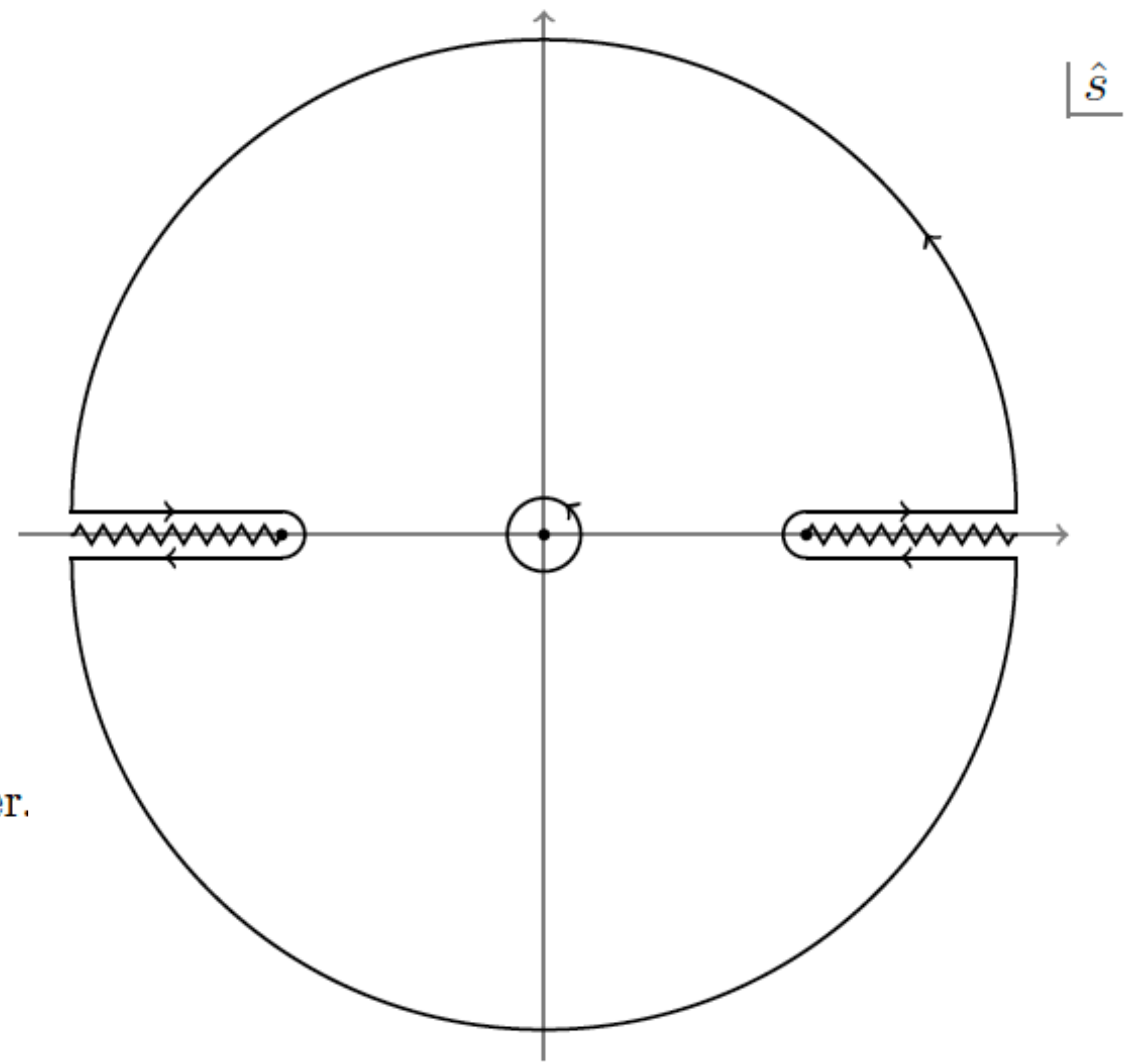
1. It is a physically well-defined function for all real s.
2. It is field redefinition independent.
3. It has an analytic continuation to the upper and lower half complex s-planes, with singularities residing only on the real axis, including unitarity cuts for energies $|s| > 4m^2$ where m is the mass gap in the theory, which is assumed to be non-zero. This property is a consequence of locality and Lorentz invariance.
4. The discontinuity across the cut on the positive real axis is $i \times$ a positive number. This is a consequence of unitarity.
5. It satisfies a crossing symmetry: $\mathcal{M}(s)^* = \mathcal{M}(4m^2 - s^*)$. This is a consequence of locality and Lorentz invariance.
6. It decays as $|\mathcal{M}(s)|/s^2 \to 0$ as $|s| \to \infty$. This property follows from the minimal requirements to derive the Froissart bound [16].

Review of Lorentz Invariant case

– The S-matrix in an EFT, in the forward limit, will take the following form

$$\hat{\mathcal{M}}(\hat{s}) = c_0 + c_2 \frac{\hat{s}^2}{\Lambda^4} + c_4 \frac{\hat{s}^4}{\Lambda^8} + \ldots \ ,$$

– Then $\oint \mathrm{d}\hat{s}\, \frac{\hat{\mathcal{M}}(\hat{s})}{\hat{s}^3} = 2\pi i \frac{c_2}{\Lambda^4} \ .$



– Deform contour by analyticity

  – Circle at infinity negligible

  – Integral along negative cut

    • =along positive cut

  – integral along positive cut=

  $i \times c_+$, with c_+ a non-negative number.

  – $\Rightarrow \ c_2 \geq 0$.

Doing the same for Lorentz breaking EFT's

– Many difficulties

  – Most important: with boosts, the *in* and *out* states, no matter how energetic, can be mapped to the same state. So, they are defined no matter what the center of mass energy s is. So S-matrix is defined at all s.

    • Without boosts, this cannot be done. It is clearly impossible to scatter a 1 TeV phonon, because it simply does not exists (as there is a privileged reference frame)

  – Other difficulties relate to analyticity, crossing, etc. But the one above seems just a show stopper.

  – Explorations with assumptions made in e.g.

Grall and Melville **2021**
Baumann, Green and Porto **2015**

• Let us try to find the *same ingredients* that we uses for the S-matrix, but controlled.

UV/IR control

– Something that we control both in the UV and IR

– Idea: correlation functions of conserved currents (or the stress tensor), as they are defined at all energies.

– In the UV, we *assume* the theory goes to a conformal fixed point, a CFT. Currents are primary operators and their 2-point function is fixed:

$$\langle J^\mu(-k) J^\nu(k) \rangle = c_J (k^\mu k^\nu - \eta^{\mu\nu} k^2) k^{d-4} \, ,$$

  – Also, they are *field-redefinition independent*

• Which correlation function to study?

  – Since we expect causality to play a role, choose ret. or adv. Green's functions:

$$G_R^{\mu\nu}(x-y) = i\theta(x^0 - y^0) \, \langle 0 | [J^\mu(x), J^\nu(y)] | 0 \rangle \ ,$$
$$G_A^{\mu\nu}(x-y) = -i\theta(y^0 - x^0) \, \langle 0 | [J^\mu(x), J^\nu(y)] | 0 \rangle \ .$$

Analiticity

– .

$$\tilde{G}^{\mu\nu}_{R,A}(\omega,\boldsymbol{p}) = \int_{\mathbb{R}^d} \mathrm{d}^d x\, e^{-ip\cdot x} G^{\mu\nu}_{R,A}(x)\,.$$

– $G^{\mu\nu}_R(x) = 0$ for $x^0 < 0$ and for $x^2 > 0$

– $\Rightarrow$ Integration region restricted to (FLC): $x^0 > 0, x^2 < 0$

– Consider complex four-momentum p : convergence for

$$\mathrm{Re}(-ip\cdot x) < 0 \text{ or } p^{\mathsf{Im}}\cdot x < 0 \text{ as } |x| \to \infty$$

– or: $p^{\mathsf{Im}} \in \mathrm{FLC}$

– So, for $p^{\mathsf{Im}} \in \mathrm{FLC}$, $\tilde{G}^{\mu\nu}_R(\omega,\boldsymbol{p})$ is analytic.

– Analogously, $\tilde{G}^{\mu\nu}_A(\omega,\boldsymbol{p})$ is analytic in backward light cone.

Analiticity

– We explore this region by choosing: $\boldsymbol{p} = \boldsymbol{k}_0 + \omega \boldsymbol{\xi}$

– where $\boldsymbol{k}_0, \boldsymbol{\xi} \in \mathbb{R}^{d-1}$, $|\boldsymbol{\xi}| \equiv \xi < 1$, and

$$\omega^{\mathsf{Im}} > 0 \text{ for } \tilde{G}_R \text{ and } \omega^{\mathsf{Im}} < 0 \text{ for } \tilde{G}_A$$

– Let us now define:

$$\tilde{G}^{\mu\nu}(\omega) = \begin{cases} \tilde{G}^{\mu\nu}_R(\omega, \boldsymbol{p}) & \text{if } \omega^{\mathsf{Im}} \geq 0 \,, \\ \tilde{G}^{\mu\nu}_A(\omega, \boldsymbol{p}) & \text{if } \omega^{\mathsf{Im}} < 0 \,, \end{cases}$$

– This function is analytic on $\mathbb{C} \setminus \{(-\infty, -m) \cup (m, \infty)\}$

Analiticity

–. $$\tilde{G}^{\mu\nu}(\omega) = \begin{cases} \tilde{G}^{\mu\nu}_R(\omega, \boldsymbol{p}) & \text{if } \omega^{\text{Im}} \geq 0\,, \\ \tilde{G}^{\mu\nu}_A(\omega, \boldsymbol{p}) & \text{if } \omega^{\text{Im}} < 0\,, \end{cases} \qquad \mathbb{C} \setminus \{(-\infty, -m) \cup (m, \infty)\}$$

– Consider $\omega \in \mathbb{R}$:

$$\lim_{\varepsilon \to 0} \left(\tilde{G}^{\mu\nu}(\omega + i\varepsilon) - \tilde{G}^{\mu\nu}(\omega - i\varepsilon) \right) = i \int_{\mathbb{R}^d} \mathrm{d}^d x\, e^{-ip\cdot x} \langle 0 | [J^\mu(x), J^\nu(0)] | 0 \rangle \quad (8$$

$$= i \int_{\mathbb{R}^d} \mathrm{d}^d x\, e^{-ip\cdot x} \langle 0 | J^\mu(x) \left(\sum_n |P_n\rangle \langle P_n| \right) J^\nu(0) | 0 \rangle - (\mu \leftrightarrow \nu, x \leftrightarrow 0)$$

$$= i \int_{\mathbb{R}^d} \mathrm{d}^d x\, e^{-ip\cdot x} \langle 0 | e^{-i\hat{P}\cdot x} J^\mu(0) e^{i\hat{P}\cdot x} \left(\sum_n |P_n\rangle \langle P_n| \right) J^\nu(0) | 0 \rangle - (\mu \leftrightarrow \nu, x \leftrightarrow 0)$$

$$= i(2\pi)^d \sum_n \{ \delta^{(d)}(p - P_n) \langle 0 | J^\mu(0) | P_n \rangle \langle P_n | J^\nu(0) | 0 \rangle - \delta^{(d)}(p + P_n) \langle 0 | J^\nu(0) | P_n \rangle \langle P_n | J^\mu(0) | 0 \rangle$$

– Assuming a mass gap: $P_n^0 > m > 0$, the difference vanish in $|\omega| < m$, so function is analytic except for the two cuts.

– Analiticity ok

Positivity along cut

– Since we aim for a contour argument similar to S-matrix one, we need positivity along the cuts.

$$\lim_{\varepsilon\to 0}\left(\tilde{G}^{\mu\nu}(\omega+i\varepsilon)-\tilde{G}^{\mu\nu}(\omega-i\varepsilon)\right)=$$

$$i(2\pi)^d\sum_n\left\{\delta^{(d)}(p-P_n)\langle 0|J^\mu(0)|P_n\rangle\langle P_n|J^\nu(0)|0\rangle-\delta^{(d)}(p+P_n)\langle 0|J^\nu(0)|P_n\rangle\langle P_n|J^\mu(0)|0\rangle\right\}$$

– Contract with a real $V^\mu V^\nu$, divide by ω^ℓ and integrate along the positive cut. Only one $\delta-$ function contributes:

$$\frac{1}{(2\pi)^d}\int_{(m,\infty)\ \mathrm{cut}}\frac{\mathrm{d}\omega}{\omega^\ell}\tilde{G}^{\mu\nu}(\omega)V_\mu V_\nu=i\int_m^\infty\frac{\mathrm{d}\omega}{\omega^\ell}\sum_n\delta^{(d)}(p-P_n)\ \left|\langle P_n|J^\mu(0)V_\mu|0\rangle\right|^2,$$

– this is $i\times(\text{positive})$

- Similarly for negative cut:

$$\frac{1}{(2\pi)^d}\int_{(-\infty,-m)\ \mathrm{cut}}\frac{\mathrm{d}\omega}{\omega^\ell}\tilde{G}^{\mu\nu}(\omega)V_\mu V_\nu=-i\int_{-\infty}^{-m}\frac{\mathrm{d}\omega}{\omega^\ell}\sum_n\delta^{(d)}(p+P_n)\ \left|\langle P_n|J^\mu(0)V_\mu|0\rangle\right|^2,$$

- for odd ℓ, this is $i\times(\text{positive})$. Positivity ok.

Crossing Symmetry

– Useful, though not necessary, property:

$$\begin{aligned}\tilde{G}_A^{\nu\mu}(-p) &= -i\int_{\mathbb{R}^d} \mathrm{d}^d x\, e^{ip\cdot x}\theta(-x^0)\,\langle 0|[J^\nu(x), J^\mu(0)]|0\rangle \\ &= -i\int_{\mathbb{R}^d} \mathrm{d}^d x\, e^{-ip\cdot x}\theta(x^0)\,\langle 0|[J^\nu(-x), J^\mu(0)]|0\rangle \\ &= -i\int_{\mathbb{R}^d} \mathrm{d}^d x\, e^{-ip\cdot x}\theta(x^0)\,\langle 0|[J^\nu(0), J^\mu(x)]|0\rangle = \tilde{G}_R^{\mu\nu}(p)\,,\end{aligned}$$

– In particular: $\tilde{G}^{\mu\nu}(\omega) = \tilde{G}^{\nu\mu}(-\omega)$ when $\boldsymbol{k}_0 = \mathbf{0}$

– Reality of Green's function: $\tilde{G}_R^{\mu\nu}(p) = \tilde{G}_R^{\mu\nu}(-p^*)^*$

– Combining: $$\tilde{G}_R^{\mu\nu}(p) = \tilde{G}_A^{\nu\mu}(p^*)^*$$

Gauging the symmetry

– UV-IR connection

– Need to be sure we are computing, in the IR, with EFT, the same quantity that in the UV has the CFT scaling.

– Integrated-out heavy modes generate contact terms at low energies. These are not encoded in the Noether current constructed from the EFT. Therefore, neglecting them would give IR-UV mismatch.

- To keep track of contact terms: gauge the symmetry: interpret the correlation functions of currents as functional derivatives with respect to the non-dynamical gauge bosons.

- Let us be explicit. Notice

$$G_R^{\mu\nu}(x-y) = i\theta(x^0-y^0)\langle 0|[J^\mu(x),J^\nu(y)]|0\rangle = i\langle 0|\mathrm{T}\{J^\mu(x)J^\nu(y)\}|0\rangle - i\langle 0|J^\nu(y)J^\mu(x)|0\rangle \ .$$

– The last term does not produce contact terms, as only low-energy states contribute:

$$i\int_{\mathbb{R}^d} \mathrm{d}^d x\, e^{-ip\cdot x}\langle 0|J^\nu(0)J^\mu(x)|0\rangle = i(2\pi)^d\sum_n \delta^{(d)}(p+P_n)\langle 0|J^\nu(0)|P_n\rangle\langle P_n|J^\mu(0)|0\rangle$$

– but time-ordering has a convolution and so they contribute

Gauging the symmetry

$$G_R^{\mu\nu}(x-y) = i\theta(x^0-y^0)\langle 0|[J^\mu(x), J^\nu(y)]|0\rangle = i\langle 0|\mathrm{T}\{J^\mu(x)J^\nu(y)\}|0\rangle - i\langle 0|J^\nu(y)J^\mu(x)|0\rangle$$

– Time-ordered part:

$$\langle 0|\mathrm{T}\{J^\mu(x)J^\nu(y)\}|0\rangle = \frac{1}{Z}\int \mathcal{D}\phi\; e^{i\int_{\mathbb{R}^d}\mathrm{d}^d x\,\mathcal{L}(\phi)} J^\mu(x)J^\nu(y)$$

$$Z = \int \mathcal{D}\phi\; e^{i\int_{\mathbb{R}^d}\mathrm{d}^d x\,\mathcal{L}(\phi)}$$

– Non-ordered part:

– Go to Shroedinger picture:

$$\langle 0|J^\nu(y)J^\mu(x)|0\rangle = \langle 0|U(+\infty, y^0)J^\nu_{(s)}(\boldsymbol{y})U(y^0,x^0)J^\mu_{(s)}(\boldsymbol{x})U(x^0,-\infty)|0\rangle$$

– Inserting unity

$$\mathbb{1} = \int \mathcal{D}\phi(\tilde{\boldsymbol{x}})\;|\phi(\tilde{\boldsymbol{x}})\rangle\langle\phi(\tilde{\boldsymbol{x}})|$$

– and time evolution: $\langle\phi(y^0,\tilde{\boldsymbol{y}})|U(y^0,x^0)|\phi(x^0,\tilde{\boldsymbol{x}})\rangle = \int_{\phi(\tilde{\boldsymbol{x}})}^{\phi(\tilde{\boldsymbol{y}})}\mathcal{D}\phi\; e^{i\int_{x^0}^{y^0}\mathrm{d}^d x\,\mathcal{L}(\phi)}$

- We get:

$$\langle 0|J^\nu(y)J^\mu(x)|0\rangle = \frac{1}{Z}\int\mathcal{D}\phi(\tilde{\boldsymbol{x}})\int\mathcal{D}\phi(\tilde{\boldsymbol{y}})\;J^\nu(\phi(y^0,\boldsymbol{y}))J^\mu(\phi(x^0,\boldsymbol{x}))\int_{\phi(\tilde{\boldsymbol{y}})}\mathcal{D}\phi_3\;e^{i\int_{y^0}^{+\infty}\mathrm{d}^d x\,\mathcal{L}(\phi_3)}\times$$
$$\int_{\phi(\tilde{\boldsymbol{x}})}^{\phi(\tilde{\boldsymbol{y}})}\mathcal{D}\phi_2\;e^{i\int_{x^0}^{y^0}\mathrm{d}^d x\,\mathcal{L}(\phi_2)}\int^{\phi(\tilde{\boldsymbol{x}})}\mathcal{D}\phi_1\;e^{i\int_{-\infty}^{x^0}\mathrm{d}^d x\,\mathcal{L}(\phi_1)}\;. \qquad (26)$$

Gauging the symmetry

– So we can write, gauging the symmetry:

$$
\begin{aligned}
&G_R^{\mu\nu}(x,y) = \\
&= \frac{i}{Z}\Bigg(\int \mathcal{D}\phi_0 \; e^{i\int_{\mathbb{R}^d} \mathrm{d}^d x\, \mathcal{L}\left(\phi_0, A_\mu^{(0)}\right)} \; J^\mu\left(\phi_0(x)\right) J^\nu\left(\phi_0(y)\right)\Big|_{A_\mu^{(0)}=0} + \\
&\int \mathcal{D}\phi(\tilde{\boldsymbol{x}}) \int \mathcal{D}\phi(\tilde{\boldsymbol{y}}) \; J^\nu(\phi(y^0,\boldsymbol{y}))J^\mu(\phi(x^0,\boldsymbol{x})) \int_{\phi(\tilde{\boldsymbol{y}})} \mathcal{D}\phi_3 \; e^{i\int_{y^0}^{+\infty} \mathrm{d}^d x\, \mathcal{L}\left(\phi_3, A_\mu^{(3)}\right)} \times \\
&\int_{\phi(\tilde{\boldsymbol{x}})}^{\phi(\tilde{\boldsymbol{y}})} \mathcal{D}\phi_2 \; e^{i\int_{x^0}^{y^0} \mathrm{d}^d x\, \mathcal{L}\left(\phi_2, A_\mu^{(2)}\right)} \quad \int^{\phi(\tilde{\boldsymbol{y}})} \mathcal{D}\phi_1 \; e^{i\int_{-\infty}^{x^0} \mathrm{d}^d x\, \mathcal{L}\left(\phi_1, A_\mu^{(1)}\right)}\Big|_{A_\mu^{(1,2,3)}=0}\Bigg) \; .
\end{aligned}
$$

– or equivalently as functional derivative:

$$
\begin{aligned}
G_R^{\mu\nu}(x,y) = \frac{i}{Z}\Bigg(&- \frac{\delta^2}{\delta A_\mu^{(0)}(x)\delta A_\nu^{(0)}(y)} \int \mathcal{D}\phi \; e^{i\int_{\mathbb{R}^d} \mathrm{d}^d x\, \mathcal{L}\left(\phi_0, A_\mu^{(0)}\right)}\Big|_{A_\mu^{(0)}=0} - \\
&\frac{\delta^2}{\delta A_\mu^{(1)}(x)\; \delta A_\nu^{(3)}(y)} \int \mathcal{D}\phi(\tilde{\boldsymbol{x}}) \int \mathcal{D}\phi(\tilde{\boldsymbol{y}}) \; \int_{\phi(\tilde{\boldsymbol{y}})} \mathcal{D}\phi_3 \; e^{i\int_{y^0}^{+\infty} \mathrm{d}^d x\, \mathcal{L}\left(\phi_3, A_\mu^{(3)}\right)} \; \times \\
&\int_{\phi(\tilde{\boldsymbol{x}})}^{\phi(\tilde{\boldsymbol{y}})} \mathcal{D}\phi_2 \; e^{i\int_{x^0}^{y^0} \mathrm{d}^d x\, \mathcal{L}\left(\phi_2, A_\mu^{(2)}\right)} \int^{\phi(\tilde{\boldsymbol{x}})} \mathcal{D}\phi_1 \; e^{i\int_{-\infty}^{x^0} \mathrm{d}^d x\, \mathcal{L}\left(\phi_1, A_\mu^{(1)}\right)}\Big|_{A_\mu^{(1,2,3)}=0}\Bigg) \; .
\end{aligned}
$$

Gauging the symmetry

– . $$G_R^{\mu\nu}(x,y) = \frac{i}{Z}\left(-\frac{\delta^2}{\delta A_\mu^{(0)}(x)\delta A_\nu^{(0)}(y)} \int \mathcal{D}\phi\, e^{i\int_{\mathbb{R}^d} \mathrm{d}^d x\, \mathcal{L}\left(\phi_0, A_\mu^{(0)}\right)}\Bigg|_{A_\mu^{(0)}=0} - \right.$$
$$\frac{\delta^2}{\delta A_\mu^{(1)}(x)\ \delta A_\nu^{(3)}(y)} \int \mathcal{D}\phi(\tilde{\boldsymbol{x}}) \int \mathcal{D}\phi(\tilde{\boldsymbol{y}}) \int_{\phi(\tilde{\boldsymbol{y}})} \mathcal{D}\phi_3\, e^{i\int_{y^0}^{+\infty} \mathrm{d}^d x\, \mathcal{L}\left(\phi_3, A_\mu^{(3)}\right)} \times$$
$$\left. \int_{\phi(\tilde{\boldsymbol{x}})}^{\phi(\tilde{\boldsymbol{y}})} \mathcal{D}\phi_2\, e^{i\int_{x^0}^{y^0} \mathrm{d}^d x\, \mathcal{L}\left(\phi_2, A_\mu^{(2)}\right)} \int^{\phi(\tilde{\boldsymbol{x}})} \mathcal{D}\phi_1\, e^{i\int_{-\infty}^{x^0} \mathrm{d}^d x\, \mathcal{L}\left(\phi_1, A_\mu^{(1)}\right)}\Bigg|_{A_\mu^{(1,2,3)}=0} \right).$$

– This is the expression in the UV. In the IR, $e^{iS_{\mathrm{EFT}}(\phi_\ell, A_\mu)} = \int \mathcal{D}\phi_h\, e^{iS_{\mathrm{EFT}}(\phi_h, \phi_\ell, A_\mu)}$

– and we generate contact terms. They are captured by the gauge bosons dependence and therefore by the functional derivatives:

  – only from the T-ordered part, because contain the *same* gauge boson.

- UV and analyticity control.

Contour argument

– Consider, for example:

$$\tilde{G}^{00}(\omega) = \mu^{d-2}\left[c_1 \frac{1}{1 - c_s^2 \xi^2} + \frac{\omega^2}{\Lambda^2}\left(\frac{c_2}{(1 - c_s^2 \xi^2)^2} + d_1\right) + \mathcal{O}\left(\frac{\omega^4}{\Lambda^4}\right)\right]$$

Contour argument

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$$\tilde{G}^{00}(\omega) = \mu^{d-2}\left[c_1 \frac{1}{1 - c_s^2\xi^2} + \frac{\omega^2}{\Lambda^2}\left(\frac{c_2}{(1 - c_s^2\xi^2)^2} + d_1\right) + \mathcal{O}\left(\frac{\omega^4}{\Lambda^4}\right)\right]$$

non-relativistic speed

cutoff

contact terms

Contour argument

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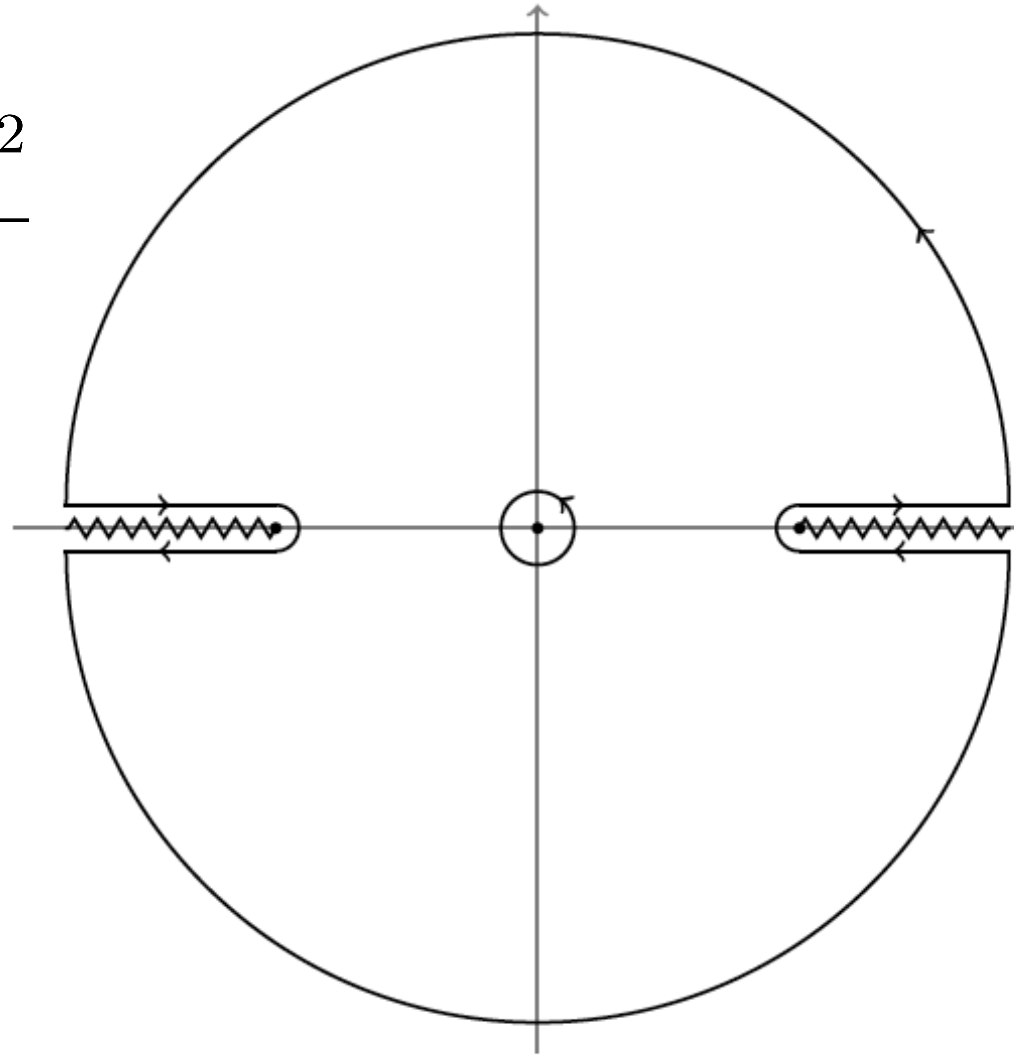
non-relativistic speed

cutoff

contact terms

–. $$\oint d\omega \frac{\tilde{G}^{00}(\omega)}{\omega^3} = 2\pi i \left(\frac{c_2}{(1-c_s^2\xi^2)^2} + d_1\right)\frac{\mu^{d-2}}{\Lambda^2}$$



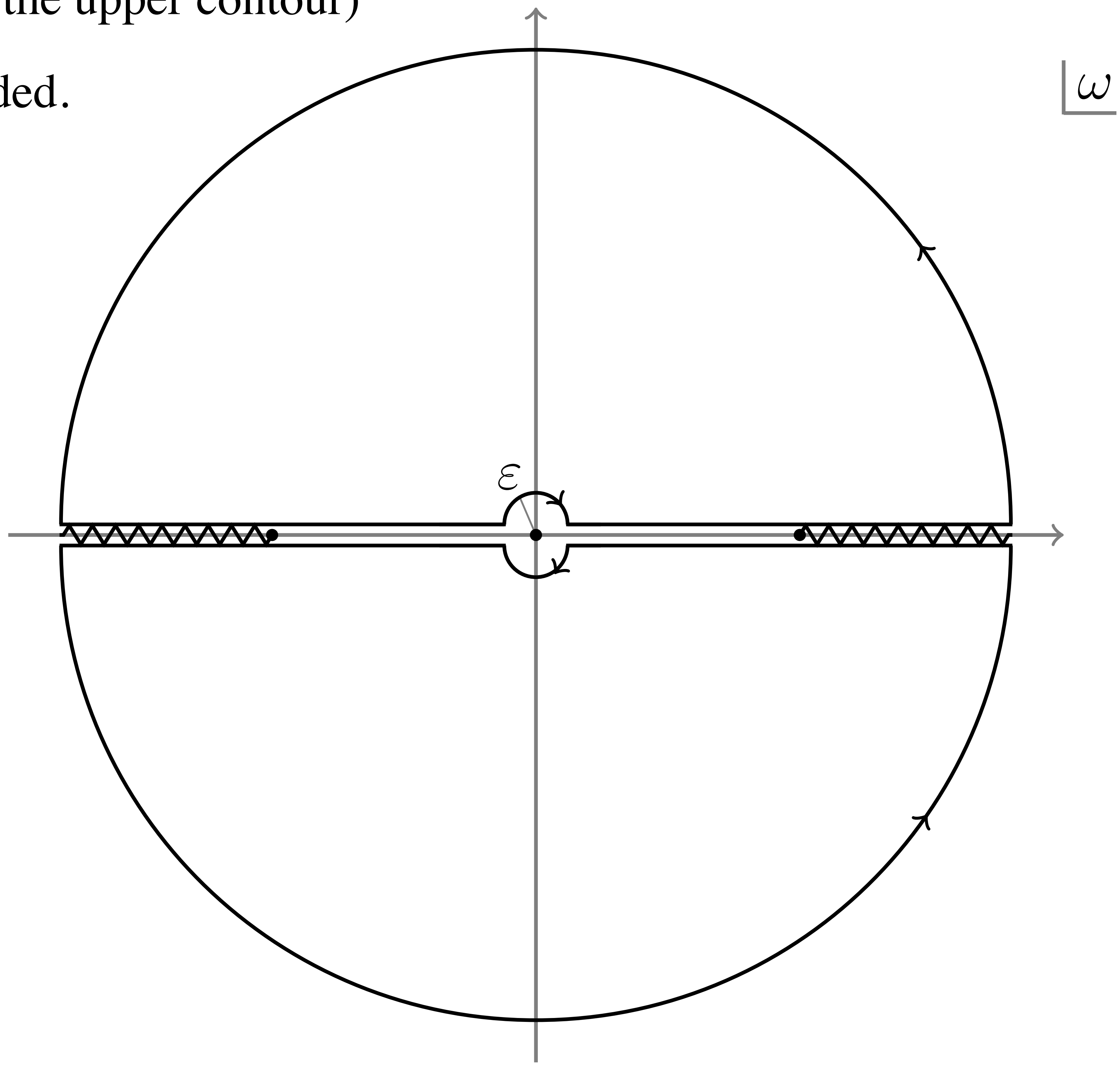
–For $d = 3$, $\tilde{G}^{00}(\omega) \sim \omega$ for $\omega \to \infty$

–circle negligible

• . $$\Rightarrow \quad \frac{c_2}{(1-c_s^2\xi^2)^2} + d_1 \geq 0$$

Without mass gap

- At loop level, the cut extends all the way to origin. One can use this contour (or, using crossing symmetry, just the upper contour)
  - So, no mass gap needed.



An example

Conformal Superfluids

- Apply setup to example of the EFT by Hellerman et al, **2015** Monin et al, **2017**
- Motivated by CFT studies, they match an operator at large charge with a state (at large charge): correlation functions of large charge operators can be computed with an EFT around this state. This state spontaneously breaks the symmetry, and also breaks, due to finite chemical potential, also time translations.
- An EFT can be constructed, using the non-linear realization of symmetries. The full symmetry is (could be an inflationary model!)

$$SO(d,2)\times U(1) \quad \text{broken to} \quad \text{rotations and spacetime translations}$$

- Simplest construction: Cuomo, **2021**
  - Write diff. invariant action with Wyel invariant metric: $\hat{g}_{\mu\nu} \equiv g_{\mu\nu}|g^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi|$
  - and $\chi = \mu t + \pi(t,x)$
  - (we will Gauge it)

- Leading operator: $S^{(1)} = \frac{c_1}{6}\int d^3x\sqrt{-\hat{g}} = \frac{c_1}{6}\int d^3x\sqrt{-g}|\partial\chi|^3$

JJ calculation

– The EFT action reads, at NLO:

$$\mathcal{L} = \frac{c_1}{6}|\nabla\chi|^3 - 2c_2\frac{(\partial|\nabla\chi|)^2}{|\nabla\chi|} + c_3\left(2\frac{(\nabla^\mu\chi\partial_\mu|\nabla\chi|)^2}{|\nabla\chi|^3} + \partial_\mu\left(\frac{\nabla^\mu\chi\nabla^\nu\chi}{|\nabla\chi|^2}\right)\partial_\nu|\nabla\chi|\right)$$

$$-\frac{b}{4}\frac{F_{\mu\nu}F^{\mu\nu}}{|\nabla\chi|} + \frac{d}{2}\frac{F^\mu{}_i F^{\nu i}}{|\nabla\chi|^3}\nabla_\mu\chi\nabla_\nu\chi\,,$$

$$\nabla_\mu\chi \equiv \partial_\mu\chi - A_\mu\,,$$

$$|v| \equiv \sqrt{-v_\mu v^\mu}\,.$$

– Gauge symmetry: $\pi(x) \to \pi(x) + \Lambda(x),\, A_\mu(x) \to A_\mu(x) + \partial_\mu\Lambda(x)$

– Several contact terms.

– Expanding to quadratic order:

$$\mathcal{L}_{(2)} = \frac{c_1\mu^3}{6} + \frac{\mu c_1}{2}\left[(\dot\pi + A^0)^2 - \frac{1}{2}(\partial_i\pi - A_i)^2 + \mu\left(\dot\pi + A^0\right)\right] + \frac{2c_2}{\mu}\left[-\pi\Box\ddot\pi + 2A^0\Box\dot\pi + A^0\Box A^0\right]$$

$$+\frac{2c_3}{\mu}\left[-\pi\Box\ddot\pi + 2A^0\Box_{c_s}\dot\pi - A^i\partial_i\ddot\pi + (\dot A^0)^2 + \dot A^0\partial_i A^i\right] +$$

$$+\frac{(b+d)}{2\mu}\left[(\partial_i A^0)^2 + (\partial_0 A_i)^2 + 2\dot A^0(\partial_i A_i)\right] - \frac{b}{4\mu}(\partial_i A_j - \partial_j A_i)^2\,,$$

JJ calculation

– Noether current:

$$J^0_N = -\frac{\mu^2 c_1}{2} - \mu c_1 \dot{\pi} - \frac{4c_2}{\mu}\Box\dot{\pi} - \frac{4c_3}{\mu}\Box_{c_s}\dot{\pi}\,,$$

$$J^i_N = \frac{\mu c_1}{2}\partial_i \pi - \frac{2c_3}{\mu}\partial_i\ddot{\pi}\,,$$

– We compute the correlation functions of the Noether currents, using

$$\mathcal{L}_{(2),A=0} = \frac{\mu c_1}{2}\pi\Box_{c_s}\pi - \frac{2(c_2+c_3)}{\mu}\pi\Box\ddot{\pi}$$

– and add the contact terms, as prescribed by the path integral formula:

$$\frac{1}{Z}\int \mathcal{D}\phi\; e^{i\int_{\mathbb{R}^3} \mathrm{d}^3x\,\mathcal{L}\left(\phi_0, A^{(0)}_\mu\right)}\;\frac{\delta^2\mathcal{L}\left(\phi_0, A^{(0)}_\mu\right)}{\delta A^{(0)}_\mu(x)\,\delta A^{(0)}_\nu(x)}\Bigg|_{A^{(0)}_\mu=0}$$

JJ conservation

– We notice that it is true that

$$k_\mu \langle J^\mu(-k) J^\nu(k) \rangle = 0$$

– without any contact terms.

- Proof: consider $\mathcal{K} = \int \mathcal{D}\phi \; e^{i \int \mathrm{d}^d x \; \mathcal{L}(\phi, A_\mu)}$ and change variables $\phi' = e^{-i\alpha(x)}\phi$, and use $\mathcal{D}\phi' = \mathcal{D}\phi$, to get $\mathcal{K} = \int \mathcal{D}\phi \; e^{i \int \mathrm{d}^d x \; \mathcal{L}(\phi'(\phi), A_\mu)}$

- Gauge invariance $\mathcal{L}\left(\phi'(\phi), A_\mu - \partial_\mu \alpha\right) = \mathcal{L}\left(\phi, A_\mu\right)$

$$\Rightarrow \mathcal{L}\left(\phi'(\phi), A_\mu\right) = \mathcal{L}\left(\phi, A_\mu + \partial_\mu \alpha\right)$$

- So:

$$\mathcal{K} = \int \mathcal{D}\phi \; e^{i \int \mathrm{d}^d x \; \mathcal{L}(\phi, A_\mu + \partial_\mu \alpha)} = \int \mathcal{D}\phi \; e^{i \int \mathrm{d}^d x \; \mathcal{L}(\phi, A_\mu)} \left(1 + i \int \mathrm{d}^d x \; \partial_\mu \alpha(x) \frac{\delta S}{\delta A_\mu(x)}\right)$$

JJ conservation

– So:

$$0 = \int \mathcal{D}\phi \, e^{i \int \mathrm{d}^d x \, \mathcal{L}(\phi, A_\mu)} \int \mathrm{d}^d x \, \partial_\mu \alpha(x) \frac{\delta S}{\delta A_\mu(x)}$$

$$= - \int \mathrm{d}^d x \, \alpha(x) \, \partial_{x^\mu} \int \mathcal{D}\phi \, e^{i \int \mathrm{d}^d x' \, \mathcal{L}(\phi(x'), A_\nu(x'))} \frac{\delta S}{\delta A_\mu(x)}$$

$$= i \int \mathrm{d}^d x \, \alpha(x) \, \partial_{x^\mu} \frac{\delta}{\delta A_\mu(x)} \int \mathcal{D}\phi \, e^{i \int \mathrm{d}^d x' \, \mathcal{L}(\phi(x'), A_\nu(x'))} \, .$$

$$\Rightarrow \quad 0 = \partial_{x^\mu} \frac{\delta}{\delta A_\mu(x)} \int \mathcal{D}\phi \, e^{i \int \mathrm{d}^d x' \, \mathcal{L}(\phi(x'), A_\nu(x'))}$$

– Take a second derivative:

$$\Rightarrow \quad 0 = \partial_{x^\mu} \frac{\delta^2}{\delta A_\mu(x) \delta A_\nu(y)} \int \mathcal{D}\phi \, e^{i \int \mathrm{d}^d x' \, \mathcal{L}(\phi(x'), A_\rho(x'))} \Bigg|_{A_\sigma = 0}$$

– This is our functional form. But notice that it includes the contact terms.

JJ conservation

– So: $k_\mu \langle J^\mu(-k) J^\nu(k) \rangle = 0$

– $\implies$ it has 2 tensorial structures (non relativistic theory):

$$i\langle J^\mu(-k) J^\nu(k) \rangle = \mathsf{A} \left(k^\mu k^\nu - \eta^{\mu\nu} k^2 \right) + \mathsf{B} \left(k^i k^j - \delta^{ij} \boldsymbol{k}^2 \right)$$

– Wordking in d=3:

$$\mathsf{A} = -\frac{\mu c_1}{2\left(\omega^2 - c_s^2 \boldsymbol{k}^2\right)} + \frac{c_2}{\mu} \frac{\left(\omega^2 - \boldsymbol{k}^2\right) \boldsymbol{k}^2}{\left(\omega^2 - c_s^2 \boldsymbol{k}^2\right)^2} - \frac{c_3}{\mu} \frac{\omega^2 \boldsymbol{k}^2}{\left(\omega^2 - c_s^2 \boldsymbol{k}^2\right)^2} + \frac{b}{\mu} + \frac{d}{\mu} ,$$

$$\mathsf{B} = \frac{\mu c_1}{4\left(\omega^2 - c_s^2 \boldsymbol{k}^2\right)} + \frac{c_2}{\mu} \frac{\left(\omega^2 - \boldsymbol{k}^2\right)^2}{\left(\omega^2 - c_s^2 \boldsymbol{k}^2\right)^2} - \frac{c_3}{\mu} \frac{\omega^2(\omega^2 - \boldsymbol{k}^2)}{\left(\omega^2 - c_s^2 \boldsymbol{k}^2\right)^2} - \frac{d}{\mu} .$$

Positivity bounds from JJ

– There is a rich kinematical structure. Consider:

$$\tilde{f}(\omega) = \tilde{G}^{\mu\nu}(k) V_\mu(k) V_\nu(k) \Big|_{k=(\omega, \boldsymbol{k}_0 + \omega \boldsymbol{\xi})},$$

– Take $\boldsymbol{k}_0 = \boldsymbol{0}$ (as it does not change the result)

– Take the most general $V(\omega) = \alpha(\omega)\hat{K} + \beta(\omega)\hat{E} + \gamma(\omega)\hat{F}$

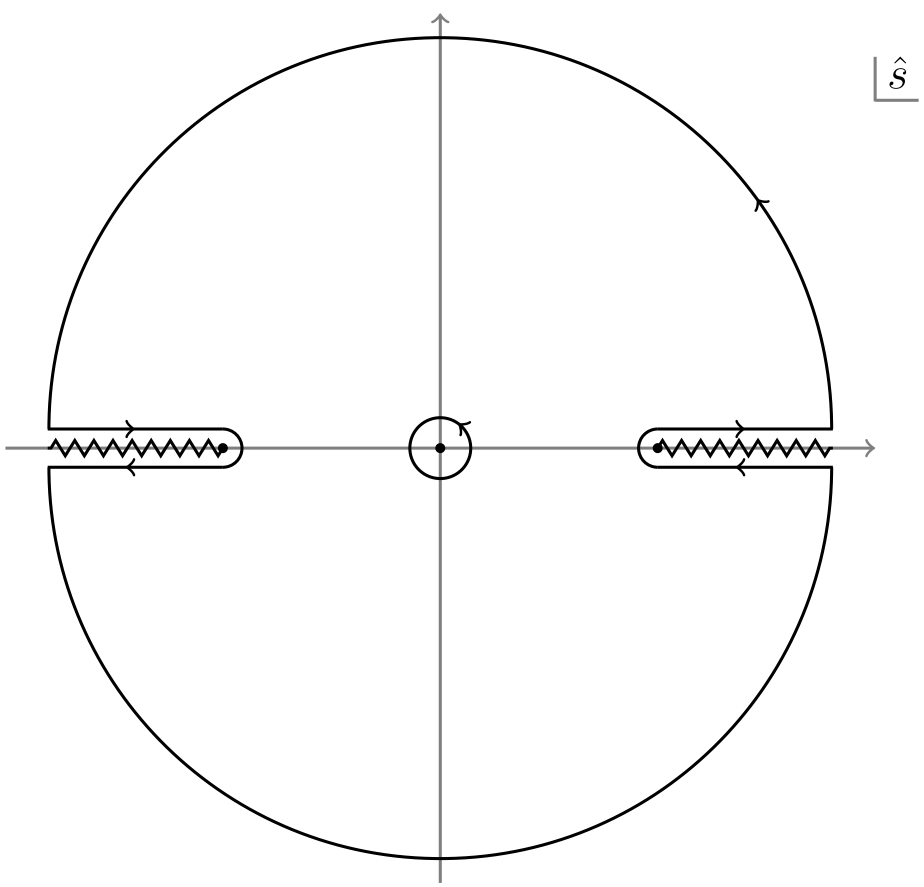
– (expanded in a base)

– Get: $\tilde{f}(\omega) = \mathsf{A}\omega^2(1-\xi^2)(\beta^2+\gamma^2) - \mathsf{B}\xi^2\omega^2\gamma^2$

– Contour argument:

– $\displaystyle \oint d\omega \frac{\tilde{f}(\omega)}{\omega^3} = i\pi \tilde{f}''(0)$

– $\tilde{f}''(0) \geq 0$



Positivity bounds from JJ

– $\tilde{f}''(0) \geq 0$:

$$c_2 \frac{\xi^2(1-\xi^2)}{(1-\xi^2/2)^2}\beta^2 - c_3 \frac{\xi^2}{(1-\xi^2/2)^2}\beta^2 + b(\beta^2+\gamma^2) + d\left(\beta^2 + \frac{\gamma^2}{1-\xi^2}\right) \geq 0$$

– . $\xi \to 1$ with $\gamma \neq 0$ we obtain $\boxed{d \geq 0}$.

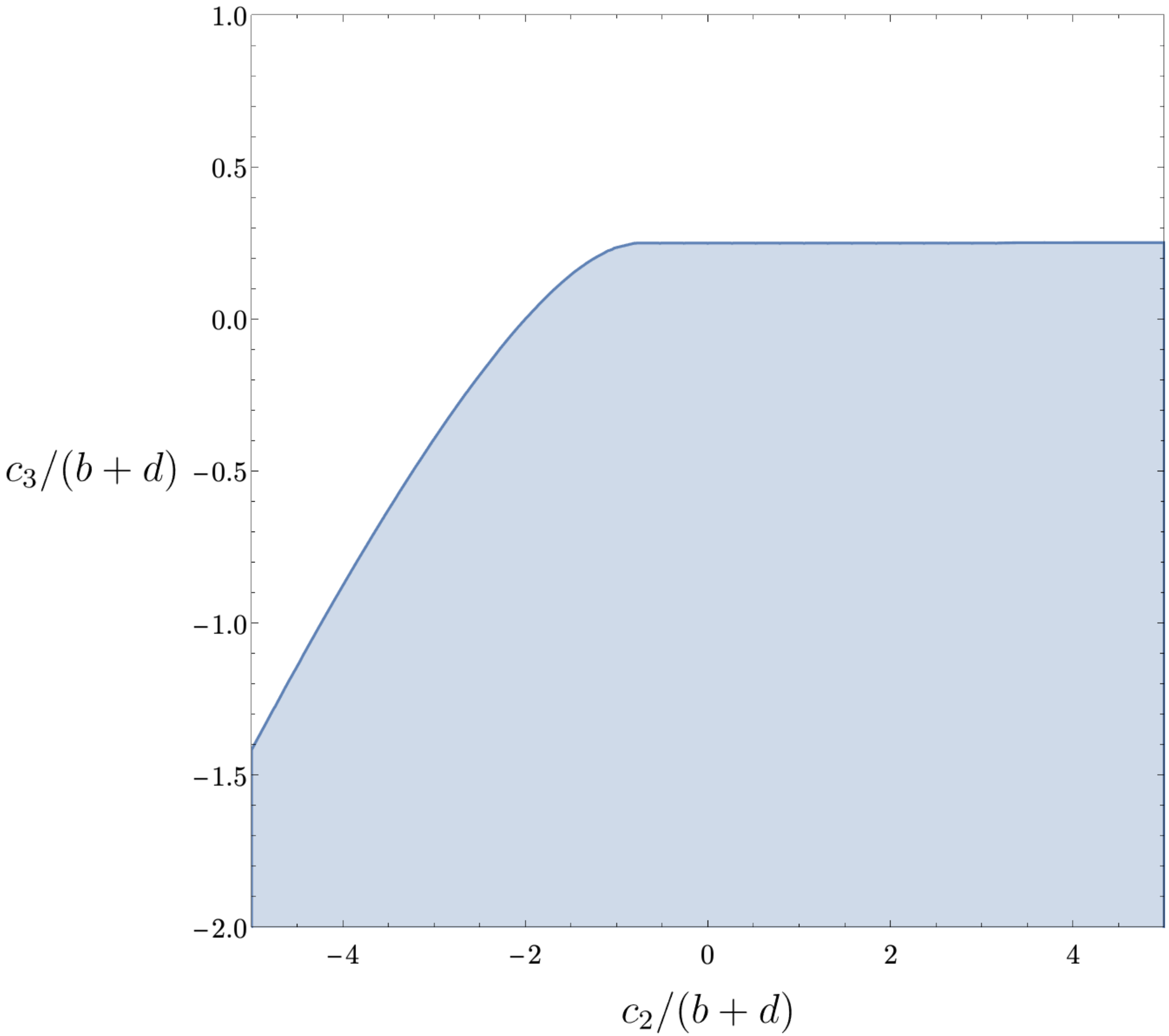
– . letting $\xi \to 0$ we get $\boxed{b + d \geq 0}$

– .Look at terms in γ^2 : most stringent if for $\gamma = 0$:

$$\boxed{\frac{c_2}{b+d}(1-\xi^2) - \frac{c_3}{b+d} \geq -\frac{(1-\xi^2/2)^2}{\xi^2}\,.} \qquad \xi \in [0,1)$$

Positivity bounds from JJ

– bound : $$\frac{c_2}{b+d}(1-\xi^2) - \frac{c_3}{b+d} \geq -\frac{(1-\xi^2/2)^2}{\xi^2}\,. \qquad \xi \in [0,1)$$



TT calculation

– We need to go to NNLO. It is possible to classify all the operators, and at quadratic order, there are only 3 independent ones:

$$S = \int \mathrm{d}^3x \sqrt{-\hat{g}} \left(\frac{c_1}{6} - c_2 \hat{R} + c_3 \hat{R}^{\mu\nu} \hat{\partial}_\mu \chi \hat{\partial}_\nu \chi + c_4 \hat{R}^2 + c_5 \hat{R}_{\mu\nu} \hat{R}^{\mu\nu} + c_6 \hat{R}^0_\mu \hat{R}^{\mu 0} \right)$$

$$\hat{R}^0_\mu \equiv \hat{R}^\lambda{}_\mu \partial_\lambda \chi$$

– We consider $\langle T^{\mu\nu}(-k) T^{\rho\sigma}(k) \rangle$, again, defined through path integral

– Conservation constraints the form: $i\langle T^{\mu\nu}(-k)T^{\rho\sigma}(k)\rangle_{\mathsf{subl.}} = \mathsf{C}(k)\,\Pi^{\mu\nu\rho\sigma}(k) + \mathsf{D}(k)\,\tilde{\Pi}^{\mu\nu\rho\sigma}(k)$

with

$$\Pi^{\mu\nu\rho\sigma} = \frac{1}{2}\left(\pi^{\mu\rho}\pi^{\nu\sigma} + \pi^{\mu\sigma}\pi^{\nu\rho}\right) - \frac{1}{d-1}\pi^{\mu\nu}\pi^{\rho\sigma}\,,$$

$$\tilde{\Pi}^{\mu\nu\rho\sigma} = \frac{1}{4}\left(\pi^{\mu\rho}\tilde{\pi}^{\nu\sigma} + \pi^{\mu\sigma}\tilde{\pi}^{\nu\rho} + \pi^{\nu\sigma}\tilde{\pi}^{\mu\rho} + \pi^{\nu\rho}\tilde{\pi}^{\mu\sigma}\right) - \frac{1}{d-2}\tilde{\pi}^{\mu\nu}\tilde{\pi}^{\rho\sigma}\,,$$

where

$$\pi^{\mu\nu} \equiv \eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\,,$$

$$\tilde{\pi}^{\mu\nu} = \delta^{mn} - \frac{k^m k^n}{\boldsymbol{k}^2}\,.$$

TT conservation

– Similar to current:

$$0 = -i\nabla_{x^\mu} \int \mathcal{D}\phi \; e^{i \int \mathrm{d}^d x' \, \sqrt{-g} \, \mathcal{L}(\phi(x'), g_{\rho\sigma}(x'))} \left(\frac{1}{\sqrt{-g(x)}} \frac{\delta S}{\delta g_{\mu\nu}(x)} \right) =$$

$$= \nabla_{x^\mu} \left(\frac{1}{\sqrt{-g(x)}} \frac{\delta}{\delta g_{\mu\nu}(x)} \int \mathcal{D}\phi \; e^{i \int \mathrm{d}^d x' \, \sqrt{-g} \, \mathcal{L}(\phi(x'), g_{\rho\sigma}(x'))} \right) .$$

– when act with second derivative, we hit the Christoffell:

$$0 = \frac{1}{\sqrt{-g(y)}} \frac{\delta}{\delta g_{\rho\sigma}(y)} \nabla_{x^\mu} \left(\frac{1}{\sqrt{-g(x)}} \frac{\delta}{\delta g_{\mu\nu}(x)} \int \mathcal{D}\phi \; e^{i \int \mathrm{d}^d x' \, \sqrt{-g} \, \mathcal{L}\left(\phi(x'), g_{\alpha\beta}(x')\right)} \right)$$

$$= \nabla_{x^\mu} \left(\frac{1}{\sqrt{(-g(x)(-g(y))}} \frac{\delta^2}{\delta g_{\mu\nu}(x) \delta g_{\rho\sigma}(y)} \int \mathcal{D}\phi \; e^{i \int \mathrm{d}^d x' \, \sqrt{-g} \, \mathcal{L}\left(\phi(x'), g_{\alpha\beta}(x')\right)} \right)$$

$$+ \frac{1}{\sqrt{-g(y)}} \frac{\delta}{\delta g_{\rho\sigma}(y)} \left(\frac{1}{\sqrt{-g(x)}} \Gamma^{\nu}_{\theta\gamma}(x) \right) \left(\frac{\delta}{\delta g_{\theta\gamma}} \int \mathcal{D}\phi \; e^{i \int \mathrm{d}^d x' \, \sqrt{-g} \, \mathcal{L}\left(\phi(x'), g_{\alpha\beta}(x')\right)} \right)$$

TT conservation

– At $g_{\mu\nu} = \eta_{\mu\nu}$,

$$0 = \partial_{x^\mu} \left(\frac{\delta^2}{\delta g_{\mu\nu}(x) \delta g_{\rho\sigma}(y)} \int \mathcal{D}\phi \; e^{i \int \mathrm{d}^d x' \sqrt{-g} \, \mathcal{L}\left(\phi(x'), g_{\alpha\beta}(x')\right)} \right) \Bigg|_{g_{\alpha\beta}=\eta_{\alpha\beta}}$$

$$+ \frac{1}{\sqrt{-g(x)}} \frac{\delta \Gamma^{\nu}_{\theta\gamma}(x)}{\delta g_{\rho\sigma}(y)} \Bigg|_{g_{\mu\nu}=\eta_{\mu\nu}} \cdot \left(\frac{\delta}{\delta g_{\theta\gamma}} \int \mathcal{D}\phi \; e^{i \int \mathrm{d}^d x' \sqrt{-g} \, \mathcal{L}\left(\phi(x'), g_{\alpha\beta}(x')\right)} \right) \Bigg|_{g_{\mu\nu}=\eta_{\mu\nu}}$$

– The second term is proportion to $\delta^{(d)}(x-y)$ and to the vev of the stress tensor. (for us it is proportional to c_1)

TT calculation

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– We consider $\langle T^{\mu\nu}(-k) T^{\rho\sigma}(k) \rangle$, again, defined through path integral

– Conservation constraints the form: $i\langle T^{\mu\nu}(-k) T^{\rho\sigma}(k) \rangle_{\mathsf{subl.}} = \mathsf{C}(k)\, \Pi^{\mu\nu\rho\sigma}(k) + \mathsf{D}(k)\, \tilde{\Pi}^{\mu\nu\rho\sigma}(k)$

with

$$\Pi^{\mu\nu\rho\sigma} = \frac{1}{2}\left(\pi^{\mu\rho}\pi^{\nu\sigma} + \pi^{\mu\sigma}\pi^{\nu\rho}\right) - \frac{1}{d-1}\pi^{\mu\nu}\pi^{\rho\sigma}\,,$$

$$\tilde{\Pi}^{\mu\nu\rho\sigma} = \frac{1}{4}\left(\pi^{\mu\rho}\tilde{\pi}^{\nu\sigma} + \pi^{\mu\sigma}\tilde{\pi}^{\nu\rho} + \pi^{\nu\sigma}\tilde{\pi}^{\mu\rho} + \pi^{\nu\rho}\tilde{\pi}^{\mu\sigma}\right) - \frac{1}{d-2}\tilde{\pi}^{\mu\nu}\tilde{\pi}^{\rho\sigma}\,,$$

where

$$\pi^{\mu\nu} \equiv \eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\,,$$

$$\tilde{\pi}^{\mu\nu} = \delta^{mn} - \frac{k^m k^n}{\boldsymbol{k}^2}\,.$$

TT calculation

–.

$$\mathsf{C} = -\frac{\mu}{2}\frac{\omega^2(\omega^2-\boldsymbol{k}^2)^2}{(\omega^2-c_s^2\boldsymbol{k}^2)^2}(c_2+c_3)+\frac{1}{\mu}\frac{\boldsymbol{k}^4(\omega^2-\boldsymbol{k}^2)^2}{(\omega^2-c_s^2\boldsymbol{k}^2)^2}c_4+\frac{1}{2\mu}\frac{(\omega^2-\boldsymbol{k}^2)^2\left(\omega^2(\omega^2-\boldsymbol{k}^2)+\boldsymbol{k}^4\right)}{(\omega^2-c_s^2\boldsymbol{k}^2)^2}c_5$$

$$+\frac{1}{4\mu}\frac{\boldsymbol{k}^2\omega^2(\omega^2-\boldsymbol{k}^2)^2}{(\omega^2-c_s^2\boldsymbol{k}^2)^2}c_6-\frac{1}{2\mu}\frac{(c_2+c_3)^2}{c_1}\frac{\boldsymbol{k}^4\omega^2(\omega^2-\boldsymbol{k}^2)^2}{(\omega^2-c_s^2\boldsymbol{k}^2)^3}\,, \quad (7$$

$$\mathsf{D} = -\frac{\mu}{4}\frac{\boldsymbol{k}^4(\omega^2-\boldsymbol{k}^2)}{(\omega^2-c_s^2\boldsymbol{k}^2)^2}(c_2+c_3)-\frac{1}{\mu}\frac{\boldsymbol{k}^4(\omega^2-\boldsymbol{k}^2)^2}{(\omega^2-c_s^2\boldsymbol{k}^2)^2}\left(2c_4+\frac{3}{4}c_5\right)+\frac{1}{8\mu}\frac{\boldsymbol{k}^6(\omega^2-\boldsymbol{k}^2)}{(\omega^2-c_s^2\boldsymbol{k}^2)^2}c_6$$

$$+\frac{1}{\mu}\frac{(c_2+c_3)^2}{c_1}\frac{\boldsymbol{k}^4\omega^2(\omega^2-\boldsymbol{k}^2)^2}{(\omega^2-c_s^2\boldsymbol{k}^2)^3}\,. \quad (7$$

–Contract with general symmetric 2-tensor: $\langle T^{\mu\nu}T^{\rho\sigma}\rangle A_{\mu\nu}A_{\rho\sigma}$

$$A_{\mu\nu} = \alpha\hat{K}_\mu\hat{K}_\nu+\beta\hat{E}_\mu\hat{E}_\nu+\gamma\hat{F}_\mu\hat{F}_\nu+\tilde{\alpha}\left(\hat{K}_\mu\hat{E}_\nu+\hat{K}_\nu\hat{E}_\mu\right)+\tilde{\beta}\left(\hat{K}_\mu\hat{F}_\nu+\hat{K}_\nu\hat{F}_\mu\right)+\tilde{\gamma}\left(\hat{E}_\mu\hat{F}_\nu+\hat{E}_\nu\hat{F}_\mu\right)$$

–We get the bound:

$$i\langle T^{\mu\nu}T^{\rho\sigma}\rangle_{\mathsf{subl.}}A_{\mu\nu}A_{\rho\sigma} = \frac{\mathsf{C}}{2}\left[(\beta-\gamma)^2+4\tilde{\gamma}^2\right]+\mathsf{D}\tilde{\gamma}^2$$

TT positivity

–Explicitly

$$4\xi^4\delta^2 c_4 + 2\left[(2-\xi^2)^2\tilde{\gamma}^2 + (1-\xi^2+\xi^4)\delta^2\right] c_5 + \xi^2\left(\frac{(2-\xi^2)^2}{1-\xi^2}\tilde{\gamma}^2 + \delta^2\right) c_6$$

$$\geq \frac{4\xi^4\delta^2}{2-\xi^2}\frac{(c_2+c_3)^2}{c_1}$$

–Not hard to show that the most stringent bounds are:

$$\boxed{c_5 \geq 0} \text{ and } \boxed{c_6 \geq 0},$$

$$\boxed{4c_4 + 2c_5 + c_6 \geq 4(c_2+c_3)^2/c_1}.$$

Summary of the bounds

–By working at NLO and NNLO, we obtained:

$$
\begin{aligned}
c_1 &\geq 0 \quad \text{(for healthy fluctuations)}, \\
\frac{c_2}{b+d}(1-\xi^2) - \frac{c_3}{b+d} &\geq -\frac{(1-\xi^2/2)^2}{\xi^2}\,, \\
d &\geq 0\,, \\
b+d &\geq 0\,, \\
4c_4 + 2c_5 + c_6 &\geq 4(c_2+c_3)^2/c_1\,, \\
c_5 &\geq 0\,, \\
c_6 &\geq 0\,.
\end{aligned}
$$

Loop corrections?

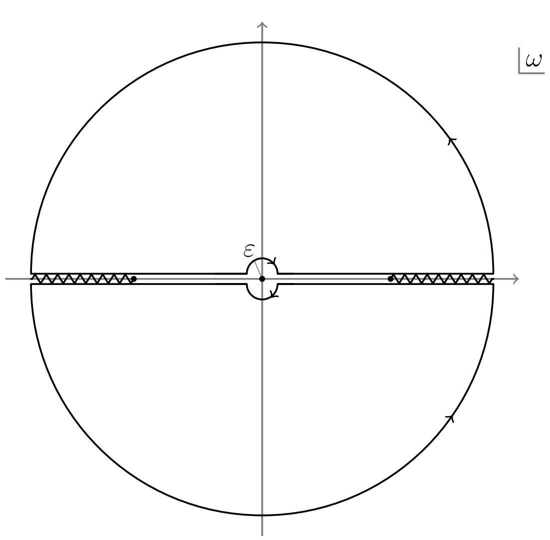
– So far, we worked at tree-level. In this particular case, up to NNLO d=3, there are no-loop corrections. In fact, in canonical normalization:

$$\mathcal{L} = \frac{1}{2}\left[\dot{\pi}_c^{\,2} - \frac{1}{2}(\partial_i\pi_c)^2\right] + \frac{1}{c_1^{1/2}\mu^{3/2}}\dot{\pi}_c^3 + \frac{1}{c_1\mu^3}\dot{\pi}_c^4 + \frac{c_{2;3}}{c_1\mu^2}\partial^2\pi_c\partial^2\pi_c + \frac{c_{2;3}}{c_1^{3/2}\mu^3}\partial^2\pi_c\partial^2\pi_c\dot{\pi}$$

$$+ \frac{c_{4;5;6}}{c_1\mu^4}\partial^3\pi_c\partial^3\pi_c + \ldots$$

– and combinations of $c_{2;3}$ and $c_{4;5;6}$ do not have the right μ-dependence to make these coefficient run (it will happen at higher order).

– In general, however, no problem: one can do the loop with this contour, and use a finite radius:



Conclusions

– We have constructed a method to derive robust bound on coefficients of operators where Boosts are spontaneously broken.

– Method based on 2-point functions of conserved current and stress tensor.

  – proved that they have the right analytic properties and also controlled UV behavior thanks to CFT UV assumption

  – then argument similar to S-matrix derived.

- Many applications:

  – Light in Material

  – QCD at finite μ

  – Inflation

- Limitations:

  – need to go to high order to ensure convergence

  – presence of the contact terms

- …Perhaps, we just started… perhaps…