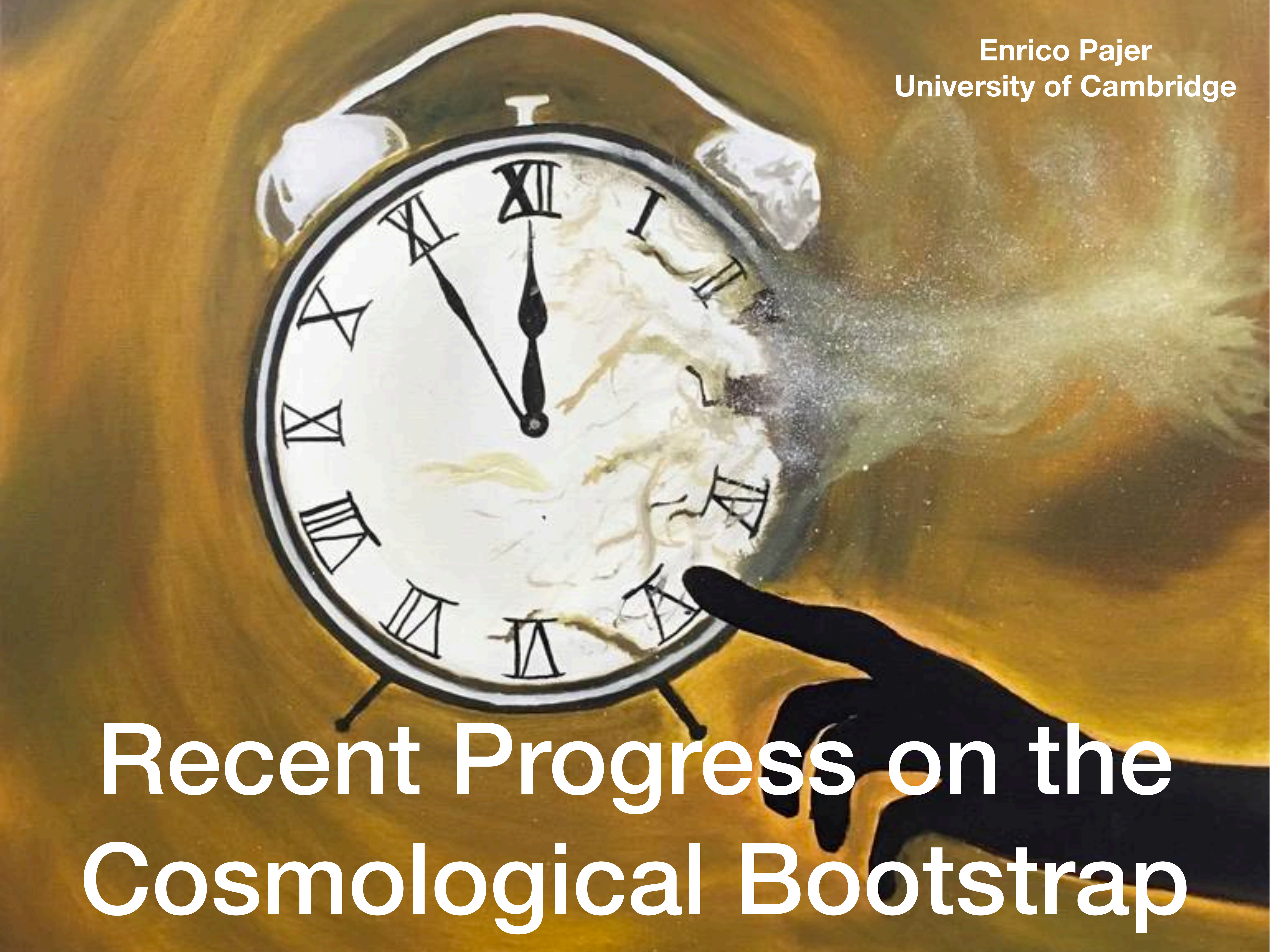
Enrico Pajer
University of Cambridge



Recent Progress on the Cosmological Bootstrap

The Cosmological Bootstrap

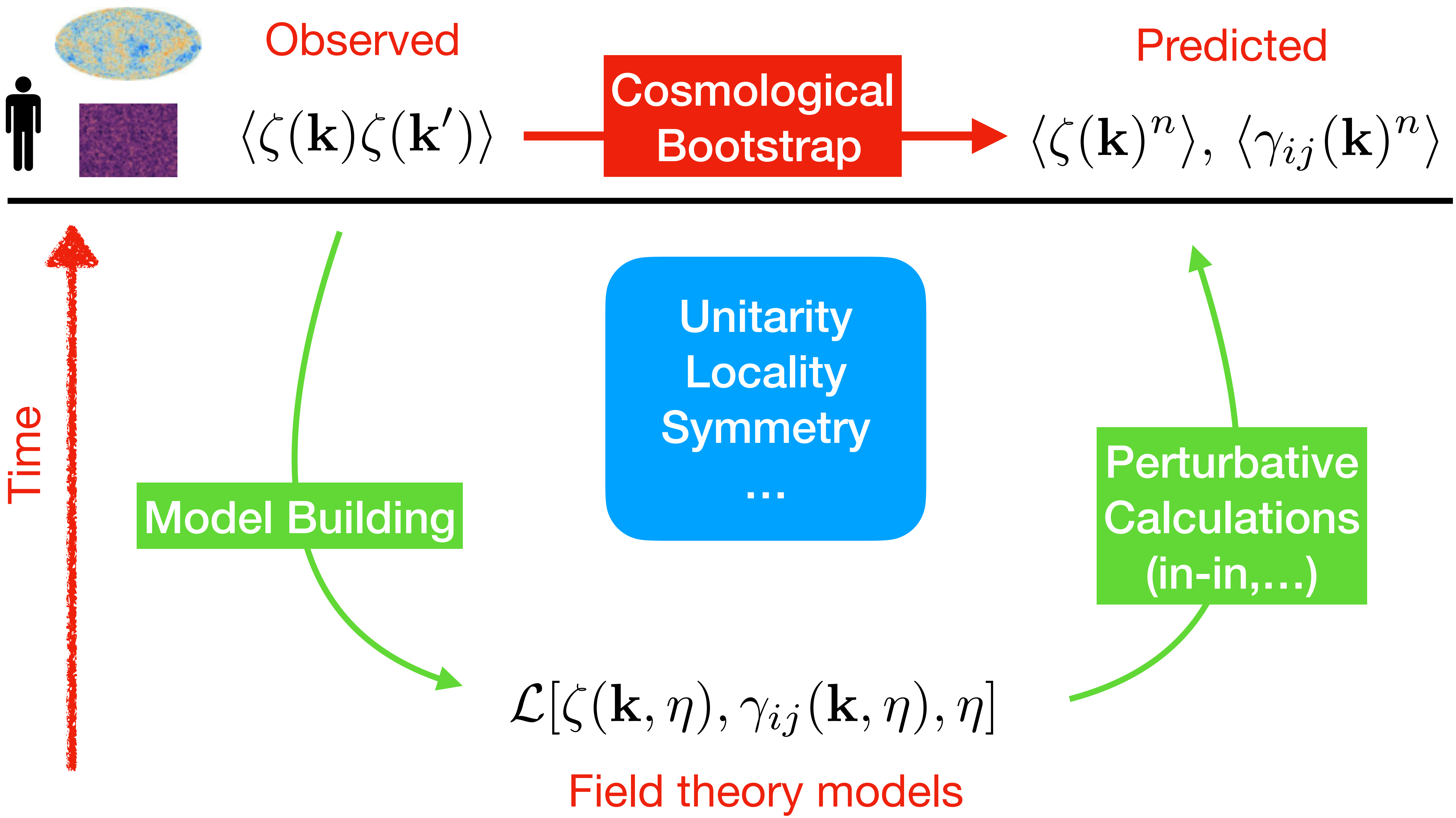


Table of contents


Observables

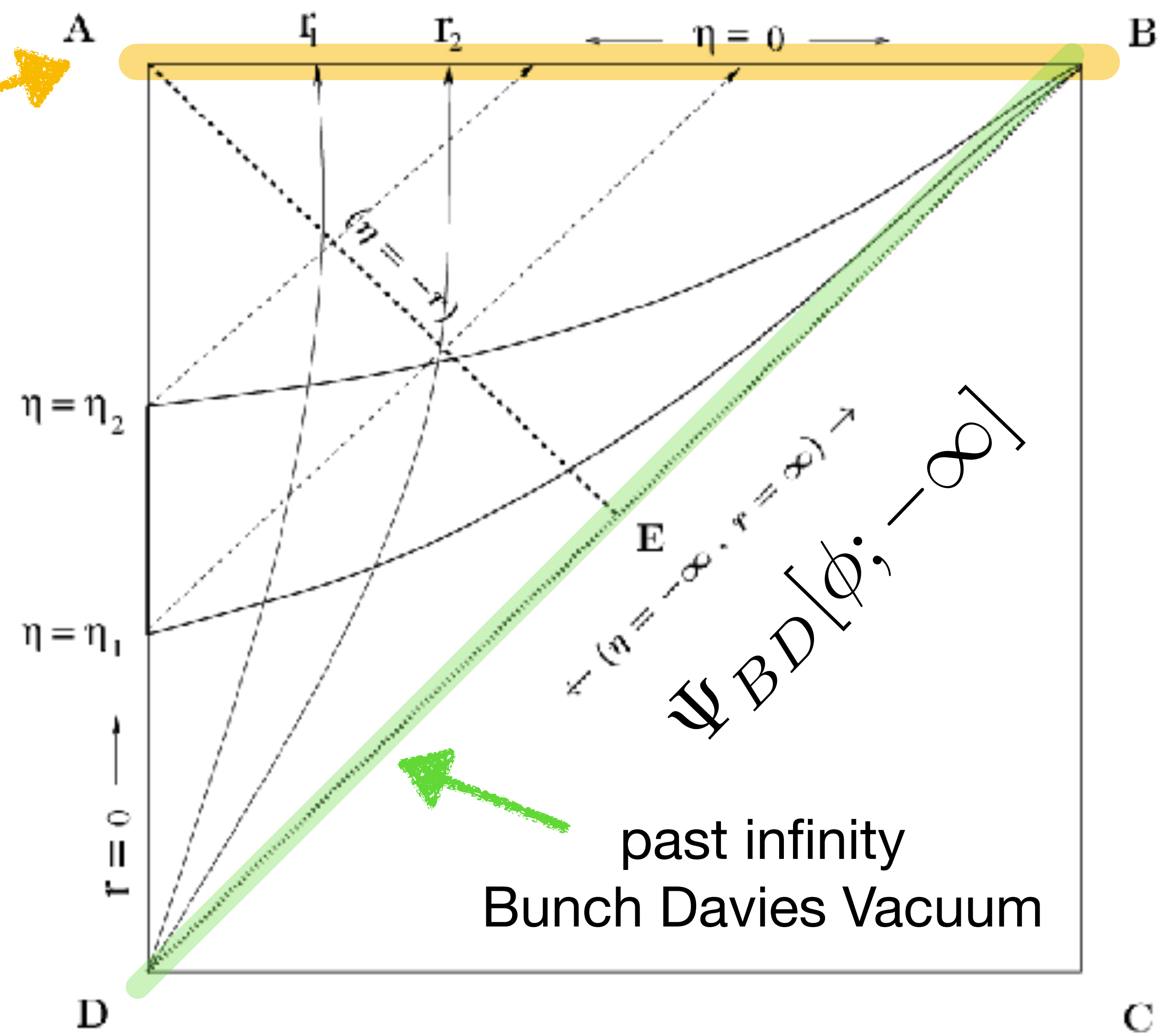
Penrose diagram

- We work in the Poincare' patch (half of dS)

$$ds^2 = -dt^2 + a^2 dx^2 = a^2(-d\eta^2 + dx^2)$$

$$\Psi[\phi; \eta \to 0]$$

The future (conformal) boundary can be thought as the reheating surface after inflation and determines the statistics of *LSS and CMB observations*



The wavefunction

- The *wavefunction of the universe,* $\Psi[\phi] \equiv \langle \phi \,|\, \Psi \rangle$, is a functional of the all fields in the theory (including the metric), at some time:

$$\Psi[\phi, \eta] = \exp\left[\sum_{n=2}^{\infty} \int_{\mathbf{k}} \psi_n(\mathbf{k}_1, \ldots, \mathbf{k}_n; \eta) \prod_a^n \phi(\mathbf{k}_a)\right]$$

- All probabilities can be computed as in QM

$$\langle \mathcal{O} \rangle = \int d\phi \Psi^* \mathcal{O} \Psi$$

- The Ψ_n are closely related to *cosmological correlators*, which determine the statistics of the Cosmic Microwave Background and Large Scale Structures

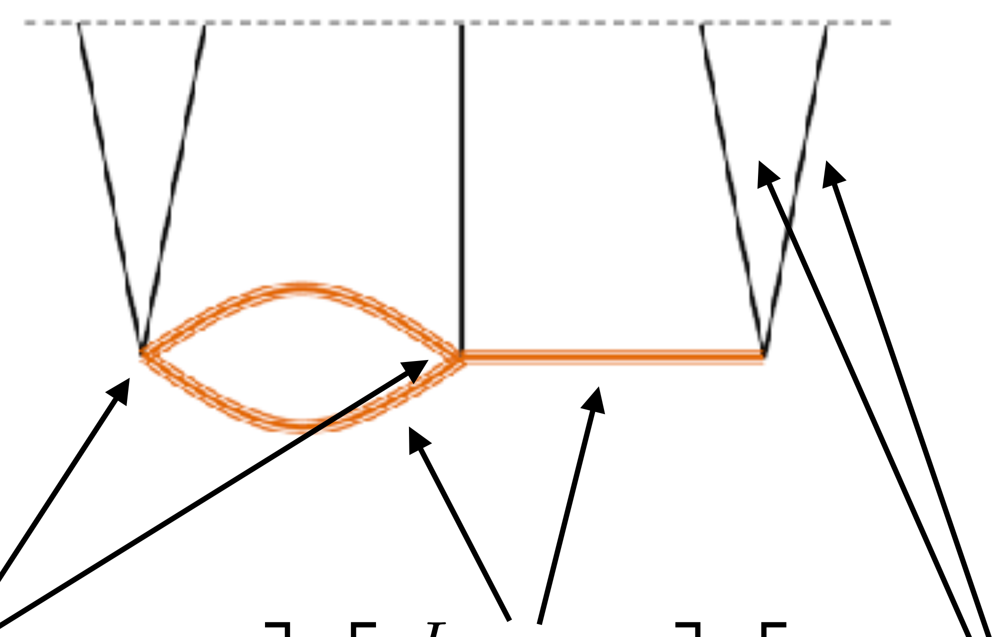
$$\langle \phi\phi \rangle = [2\mathrm{Re}\psi_2(k)]^{-1}\,, \qquad \langle \phi\phi\phi \rangle = -2\mathrm{Re}\psi_3 \left[\prod^3 \mathrm{Re}\psi_2\right]^{-1}$$

Feynman Diagrams

- Given a model, the wavefunction can be computed perturbatively from a path integral

$$\Psi[\phi(\mathbf{x});\eta_0] = \int_{\text{vacuum}}^{\phi(\mathbf{x},\eta_0)} [\mathcal{D}\phi]\, e^{iS[\phi]}$$

with the following Feynman rules (equivalent to in-in):



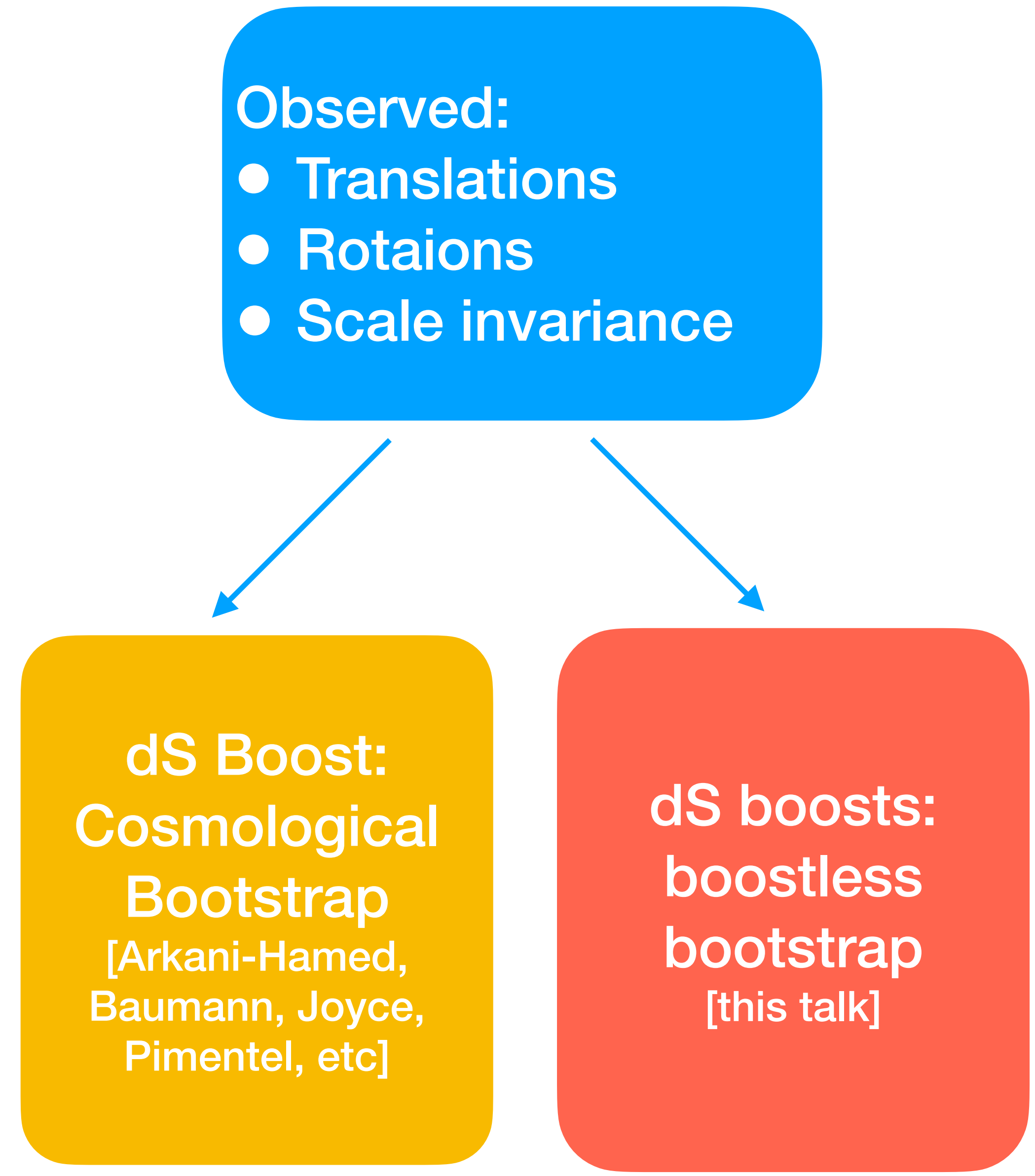
$$\psi_n = \left[\prod_A^V \int d\eta_A\, F_A\right] \left[\prod_m^I G(p)\right] \left[\prod_a^n K(k_a)\right]$$

The Bootstrap

- The wavefunction and the associated correlators have been computed over the past 20 years for a variety of models
- Here, I'll discuss how the wavefunction is constrained by symmetries, causality, locality and unitarity
- Most results are valid to all orders in perturbation theory or non-perturbatively for any FLRW spacetime, including de Sitter and Minkowski
- All constraints in this talk assume the *Bunch-Davies* initial state

Symmetries

- Cosmological perturbations are observed to be statistically *homogeneous* and *isotropic*
- Primordial perturbations are also observed to be approximately scale invariant
- Anything else?
    - Assuming *de Sitter boost*, we can derive general results and connect with CFTs and holography. See beautiful progress by Maldacena, Pimentel, Arkani-Hamed, Baumann, Joyce, etc…
    - In a dS invariant theory, all connected correlators of ζ vanish in single-clock inflation [Green EP '20]. I will not assume boosts



Boost*less* theories

- All cosmological models break Lorentz/de Sitter boosts. *The breaking of boosts can be large and is NOT slow-roll suppressed*, in contrast to the small breaking of dilation

assumed observed symmetries

$$\sum_{a=1}^{n} \vec{k}_a \langle \phi(k_1) \dots \phi(k_n) \rangle = 0$$ translations

$$\sum_{a=1}^{n} k_a^{[i} \partial_{k_a^{j]}} \langle \phi(k_1) \dots \phi(k_n) \rangle = 0$$ rotations

$$\sum_{a=1}^{n} (3 - \Delta + k_a \partial_{k_a}) \langle \phi(k_1) \dots \phi(k_n) \rangle = 0$$ dilations

$$\sum_{a=1}^{n} \left[2\vec{k} \cdot \vec{\partial} \partial_i - k_i \partial^2 + 2(3 - \Delta) \partial_i \right] \langle \phi(k_1) \dots \phi(k_n) \rangle = 0$$ dS boosts

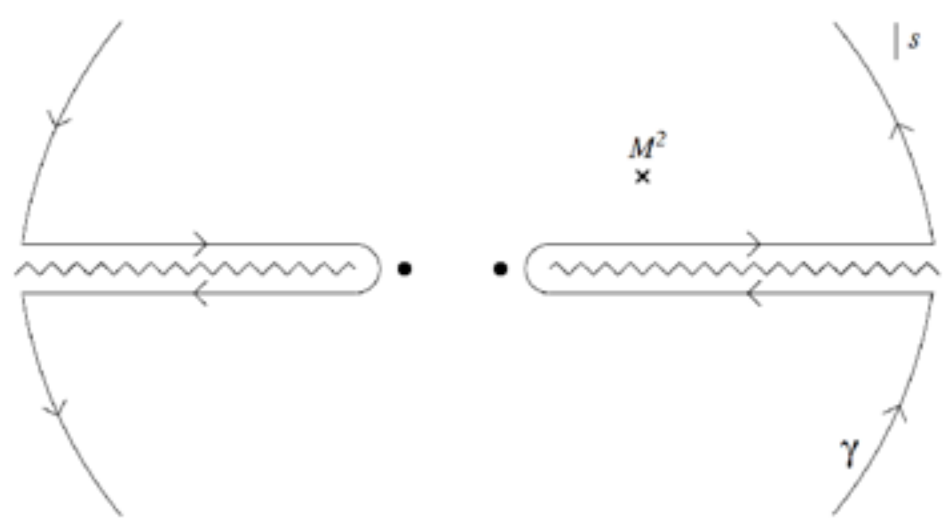
additional

The Analytic Wavefunction

Analyticity and causality

- There is a well-known connection between causality and analyticity, which leads to powerful UV/IR sum rules, analogous to the Kramers Kronig “dispersion relation”.

- Ex: operators commuting outside the light cone implies the 2-to-2 amplitude is analytic in Mandelstam s at fixed t



- What is the analytic structure of wavefunction coefficients?

- Here are some results [Goodhew, Lee, Melville & Pajer ’22]

Off-shell wavefunction

- Analytic in what?! We need to go off-shell.

- Off-shell wavefunction coefficients are the F-transform of amputated (i.e. acted on eq. of motions for all fields), connected in-out Green's functions

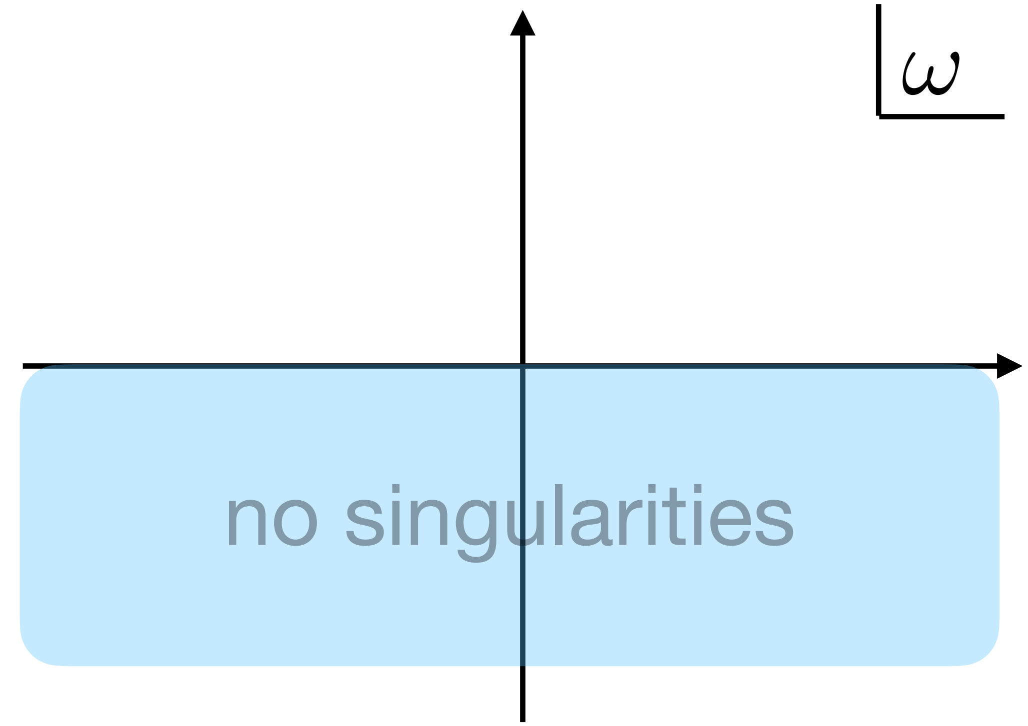
$$\psi_n\left(\{\omega\},\{\mathbf{k}\}\right)=\left[\prod_{j=1}^{n}\int_{-\infty}^{0}d\eta_j\,K_\nu(\omega_a,\eta_a)\right]\,G^{\text{amp.con.}}_{\mathbf{k}_1\ldots\mathbf{k}_n}(t_1,...,t_n)$$

$$\langle\phi(0)=0|\,T\prod_{a}^{n}\hat{\phi}_{\mathbf{k}}(t_a)|\Omega_{\text{in}}\rangle_{con}=G_{\mathbf{k}_1\ldots\mathbf{k}_n}(t_1,...,t_n)\,\delta_D^{(3)}\left(\sum_a^n\mathbf{k}_a\right)$$

- where K are mode functions in any FLRW spacetime with Bunch-Davies initial conditions.

- This is valid non-perturbatively. Reminiscent of LSZ.

Analyticity



$$\psi_n(\omega) \sim \int_{-\infty}^{0} dt\, e^{i\omega t}\, G(t)$$

- Time integral for $\Psi[\phi,\eta_0]$ stops at η_0 because of causality
- Then ψ_n are analytic in ω in the lower-half complex plane because the integral is even more convergent. This is true non-perturbatively
- Sometimes, one can extend to upper-half plane by Hermitian analyticity $\psi_n(\omega^*)=\psi_n^*(\omega)$
- Singularities only on the (negative) real axis

Singularities

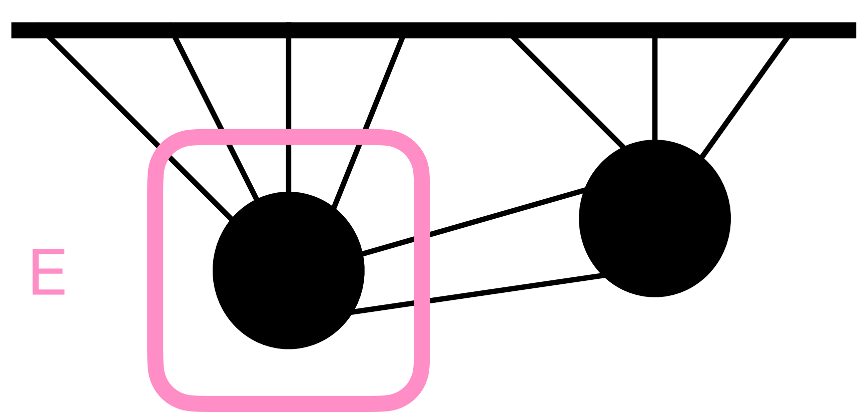
- Locality: at tree-level, the wavefunction has singularities when the "total energy" vanishes. The leading residue is the UV-limit of the flat-space amplitude [Maldacena & Pimentel '11; Raju '12; Arkani-Hamed et al '17-'18; Benincasa '18]

$$\lim_{k_T \to 0} \psi_n \sim \frac{A_n}{k_T^p} \qquad k_T = \sum_a^n |\mathbf{k}_a|$$

- Only other singularities are at *vanishing partial energy*

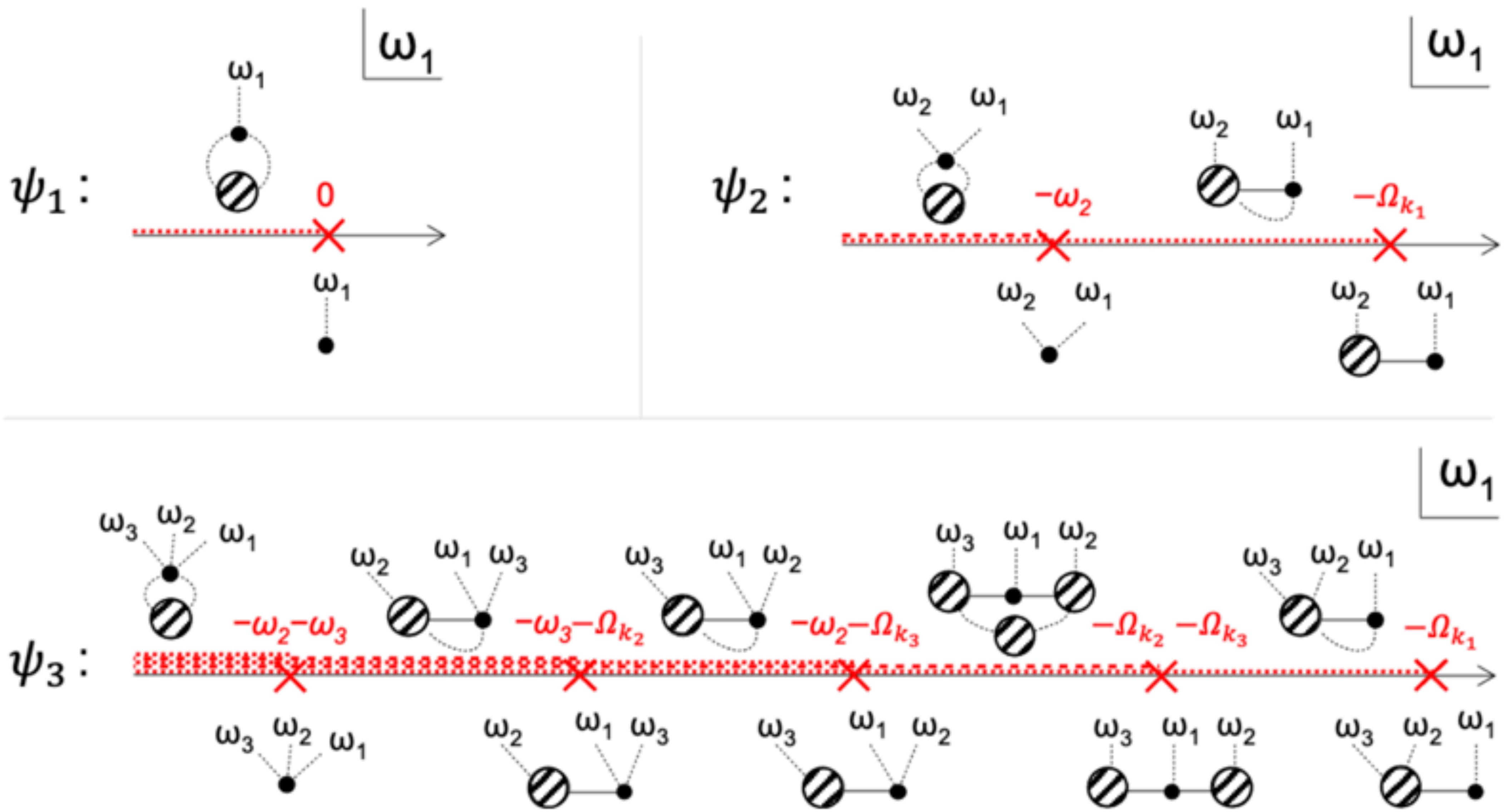
$$\lim_{E \to 0} \psi_n \sim \frac{C_n}{E^p} \qquad E = \sum_a^{\text{int}} |\mathbf{k}_a| + \sum_m^{\text{ext}} |\mathbf{p}_m|$$

- All residues of partial energy singularities are fixed by unitarity! [Jazayeri, EP & Stefanyszyn '21]

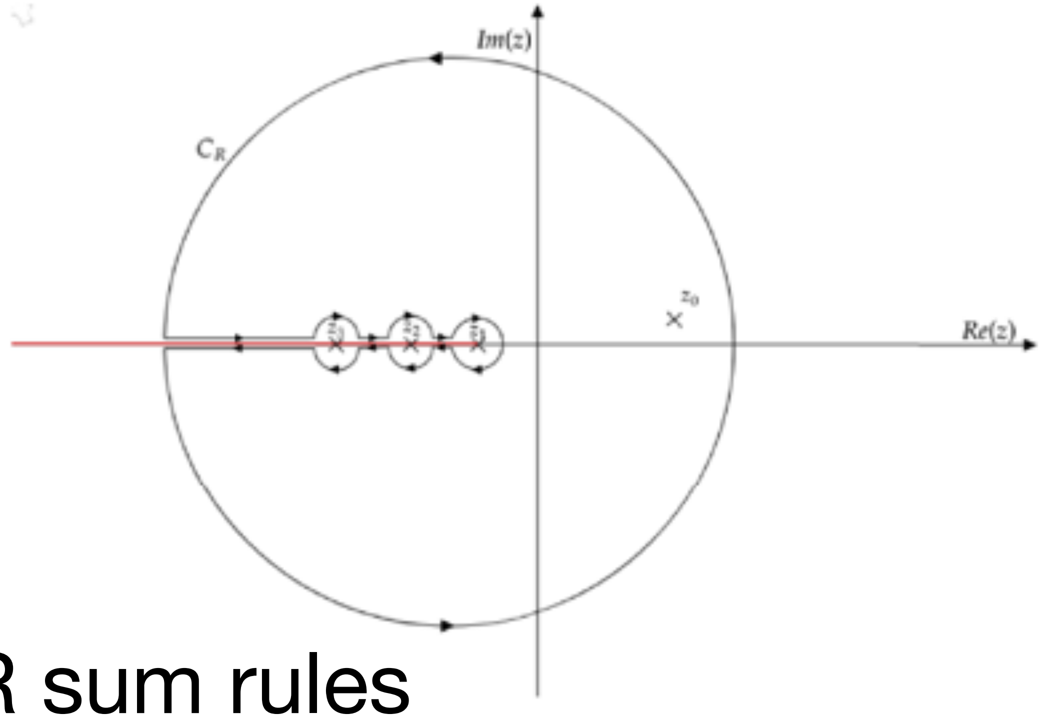


All singularities

- Consider off-shell coefficients as function of a single ω with other kinematics fixed. Singularities occur only on the negative real ω axis where the energy of a perturbative subdiagram vanishes. (There might be additional anomalous thresholds)



UV/IR sum rules

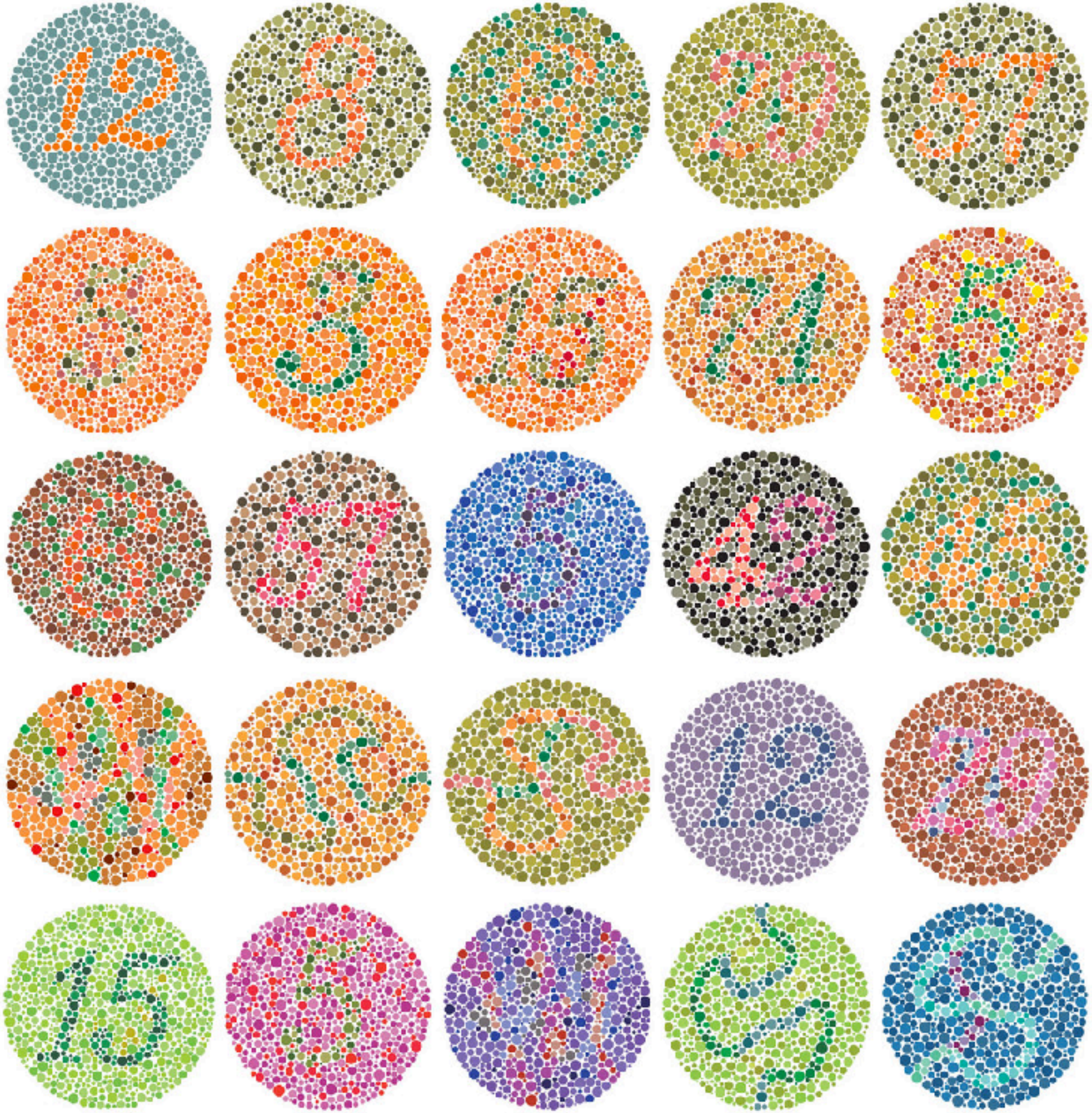


- By Cauchy's theorem we can write UV/IR sum rules

$$\omega_T\psi_{EFT}(\omega_i,\mathbf{k}_j)=\int_{-\infty}^{0}\frac{d\omega}{2\pi i}\frac{\mathrm{disc}(\omega_T\psi_{\rm UV}(\omega,\omega_{i\neq 1},\mathbf{k}_j))}{\omega-\omega_1}+\underset{\infty}{\mathrm{Res}}(\frac{\omega_T\psi_{\rm UV}(\omega,\omega_{i\neq 1},\mathbf{k}_j)}{\omega-\omega_1}).$$

  - The LHS can computed in a low-energy EFT. The RHS depends on the full UV theory.

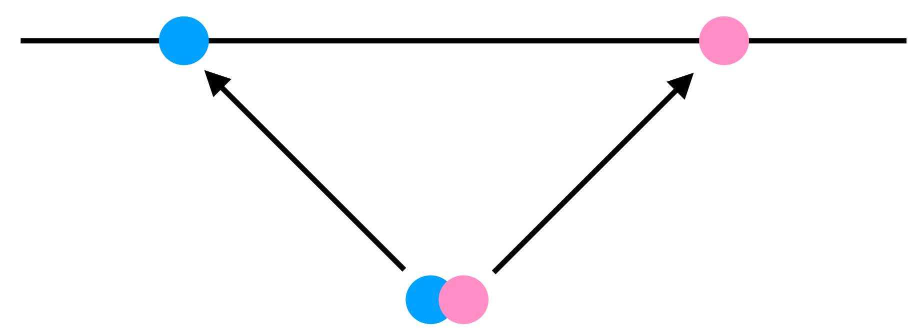
  - This fixes all Wilson coefficients in the EFT, including total derivatives and terms proportional to the eq of motion.



Locality

Locality

- Locality: what happens here cannot affect what happens far away. Operators commute for space-like separation and correlators factorise at large distances (*cluster decomposition*).



- There is no cluster decomposition in dS

- A common sufficient condition is *Manifest Locality*: Lagrangian interactions are products of operators *at the same spacetime point*. No inverse laplacians are allowed.

- The wavefunction of light *scalars and spin-2 fields* ($m^2 < 2H^2$) satisfies *non-perturbatively* the *Manifestly Local Test (MLT)* [Jazayeri, EP & Stefanyszyn '21]

$$\partial_{\omega_1} \psi_n(\omega_1, \ldots, \omega_n; \mathbf{k}_1, \ldots, \mathbf{k}_n)\Big|_{\omega_1=0} = 0$$

manifest locality ->

$$\partial_{\omega_1} \psi_n(\omega_1, \ldots, \omega_n; \mathbf{k}_1, \ldots, \mathbf{k}_n)\Big|_{\substack{\omega_1=k_1=0 \\ \omega_a=|\mathbf{k}_a|}} = 0$$

Derivation

- Two derivations: (*i*) boundary derivation using unitarity and singularities (see paper) and (*ii*) a bulk derivation that uses

$$\psi_n\left(\{\omega\},\{\mathbf{k}\}\right)=\left[\prod_{a=1}^{n}\int_{-\infty}^{0}d\eta_a K(\omega_a\eta_a)\right]\,G^{\mathrm{amp.}}_{\mathbf{k}_1\ldots\mathbf{k}_n}(t_1,...,t_n)$$

- Notice that as $k\to 0$ there is no linear term in k

$$\partial_\omega K_{3/2}(\omega,\eta)|_{\omega=0}=\partial_\omega(1-i\omega\eta)e^{i\omega\eta}|_{\omega=0}=0$$

- The same is true non-perturbatively for $\psi_n(\omega)$

- By manifest locality we can take this on-shell *ω*=*k* and it is still 0 because there cannot be a divergence at k=0

Manifest locality

- The Manifestly Local Test is a *necessary* condition for all manifestly local test*.* All large non-Gaussianities in single field inflation (e.g. EFT of inflation) obey this

$$\mathcal{L} \supset \dot{\phi}^3 + (\partial\phi)^2\dot{\phi} + \dot{\phi}^4 + \ldots$$

- Gravity has non-manifestly-local interactions for backreacting scalars after integrating out lapse and shift

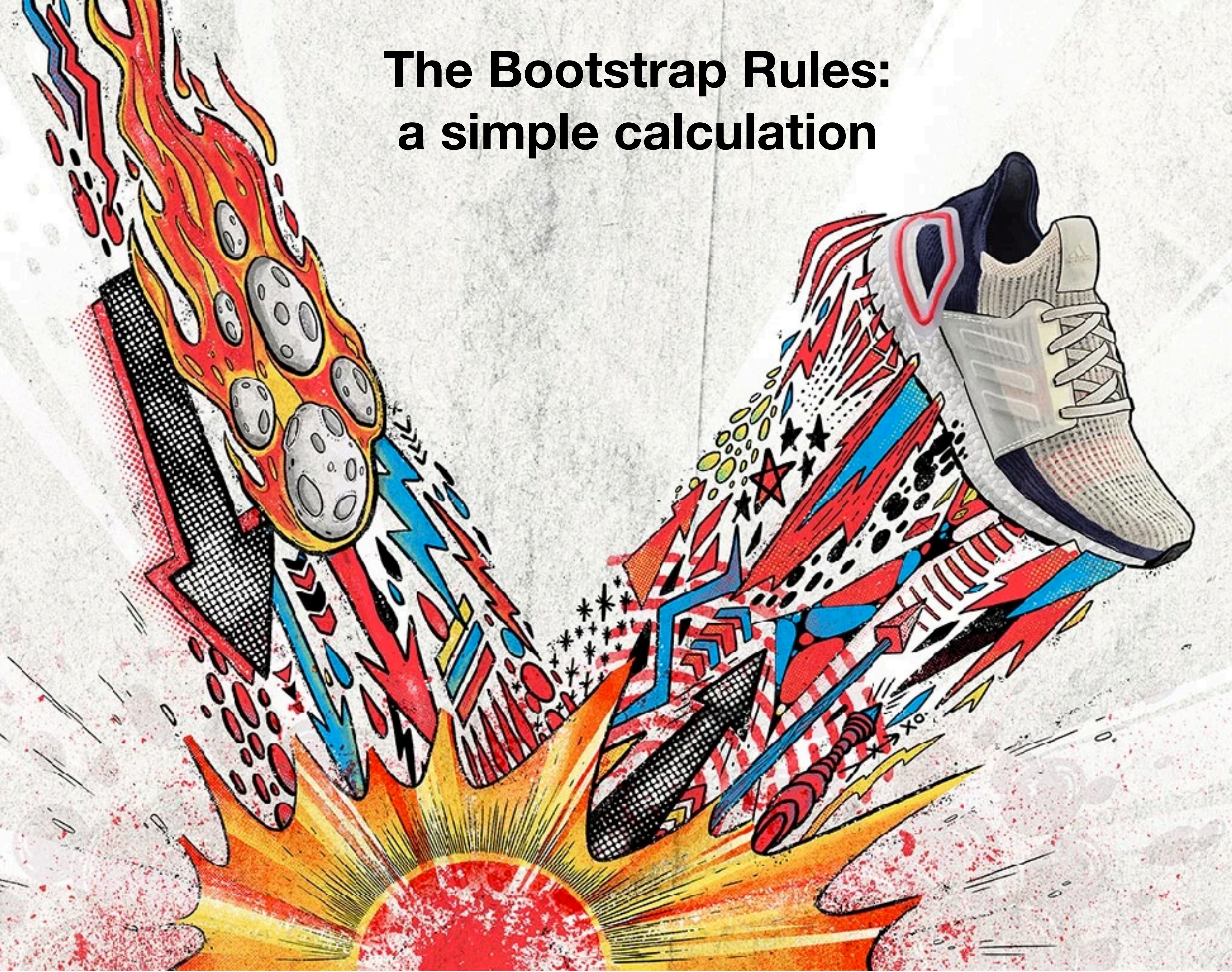
$$\mathcal{L}_{GR} \supset \dot{\zeta}^2\nabla^{-2}\dot{\zeta} + \ldots$$

But the MLT applies more generally to all “soft” interactions

$$\mathcal{L} \supset \int_{\{\mathbf{k}\}} \prod_a^n \phi(\mathbf{k}_a)F(\{\mathbf{k}\}) \;\; \text{s.t.} \;\; \partial_{\mathbf{k}_a}F(\{\mathbf{k}\})\big|_{\mathbf{k}_a=0} = 0$$

In particular, the *MLT applies to gravitons and spectator fields to all orders in perturbation theory.*

The Bootstrap Rules:
a simple calculation



Bootstrap Rules

- Instead of computing bispectra from a model we use a set of Bootstrap Rules based on fundamental principles [EP '20]

- As an example, let's bootstrap the bispectrum (3-point function) of a *scalar.* We can work directly on shell wlog.

- It can only have kT poles by locality!

Scale invariance

Bose symmetry

$$\psi_3 = \frac{\text{Poly}_{3+p}(k_T, e_2, e_3)}{k_T^p}$$

$k_T \equiv k_1 + k_2 + k_3$

$e_2 \equiv k_1 k_2 + k_2 k_3 + k_1 k_3$

$e_3 \equiv k_1 k_2 k_3$

tree level in dS

Bunch Davies vacuum

The calculation

- The Bootstrap Rules reduced the problem to determining the numerical constants C_{mn} via the Manifestly Local Test

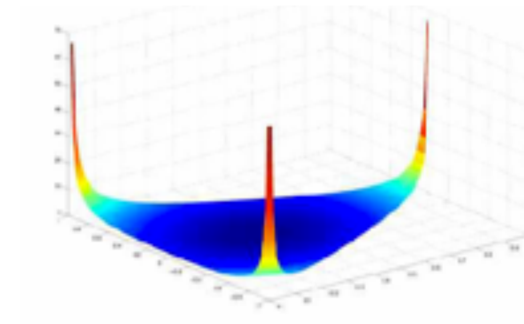
$$\psi_3^{(p)}(k_1,k_2,k_3) = \frac{1}{k_T^p} \sum_{n=0}^{\lfloor \frac{p+3}{3} \rfloor} \sum_{m=0}^{\lfloor \frac{p+3-3n}{2} \rfloor} C_{mn} k_T^{3+p-2m-3n} e_2^m e_3^n,$$

$$\partial_{k_1} \psi_3 \Big|_{k_1=0} = 0$$

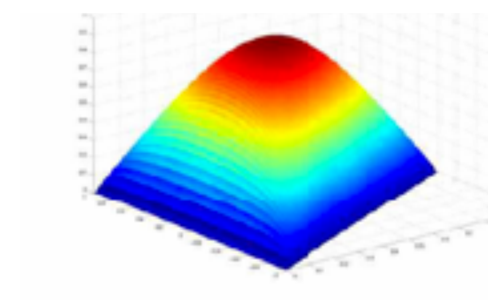
- This yields all manifestly local bispectra for a scalar to *any order in derivatives* in the EFT of inflation

- This gives order by order the shapes of non-Gaussianity that are constraint e.g. by the Cosmic Microwave Background, e.g. the Planck mission

Shapes of non-Gaussianity



$$\psi_3^{(0)} = A_0 \left[4e_3 - e_2 k_T + (3e_3 - 3e_2 k_T + k_T^3)\log(-k_T\eta/\mu)\right]$$
$$\psi_3^{(1)} = 0$$
$$\psi_3^{(2)} = A_2 \left[-k_T^3 + 3k_T e_2 - 11e_3 + \frac{4e_2^2}{k_T} + \frac{4e_2 e_3}{k_T^2}\right]$$
$$\psi_3^{(3)} = A_3 \frac{1}{k_T^3}(k_T^6 - 3k_T^4 e_2 + 11k_T^3 e_3 - 4k_T^2 e_2^2 - 4k_T e_2 e_3 + 12e_3^2) + A_3' \frac{e_3^2}{k_T^3}$$



- $\Psi_3^{(0)}$ contains the famous *local non-Gaussianity,* while $\Psi_3^{(1,2)}$ the so-called *equilateral and orthogonal non-Gaussianities*, the main targets of non-Gaussian searches in the CMB and galaxy surveys!

- In the standard approach the numerical coefficient come from time integrations, here they're fixed algebraically


Unitarity

Unitary time evolution

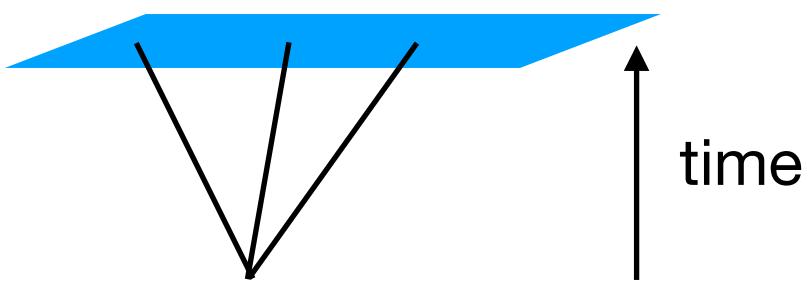
- In Quantum Mechanics we compute probabilities, which must be between 0 and 1 to make sense
- This requires the positive norm of states in the Hilbert space *and* Unitary time evolution, $UU^{\dagger}=1$. Colloquially this is the *conservation of probabilities*
- The consequences of unitarity for particle physics amplitudes were discover over *60 years ago*: the Optical theorem and Cutkosky Cutting Rules.
- In cosmology we don't see the time evolution, so how can we see it's unitary?!

The Cosmological Optical Theorem (COT) [Goodhew, Jazayeri, EP ’20]

- From unitarity, $UU^{\dagger}=1$, we found infinitely many relations.

- The simplest applies to contact n-point functions

$$\psi_n(\{\omega\},\{\mathbf{k}\}) + \psi_n^*(\{-\omega\},\{-\mathbf{k}\}) = 0$$



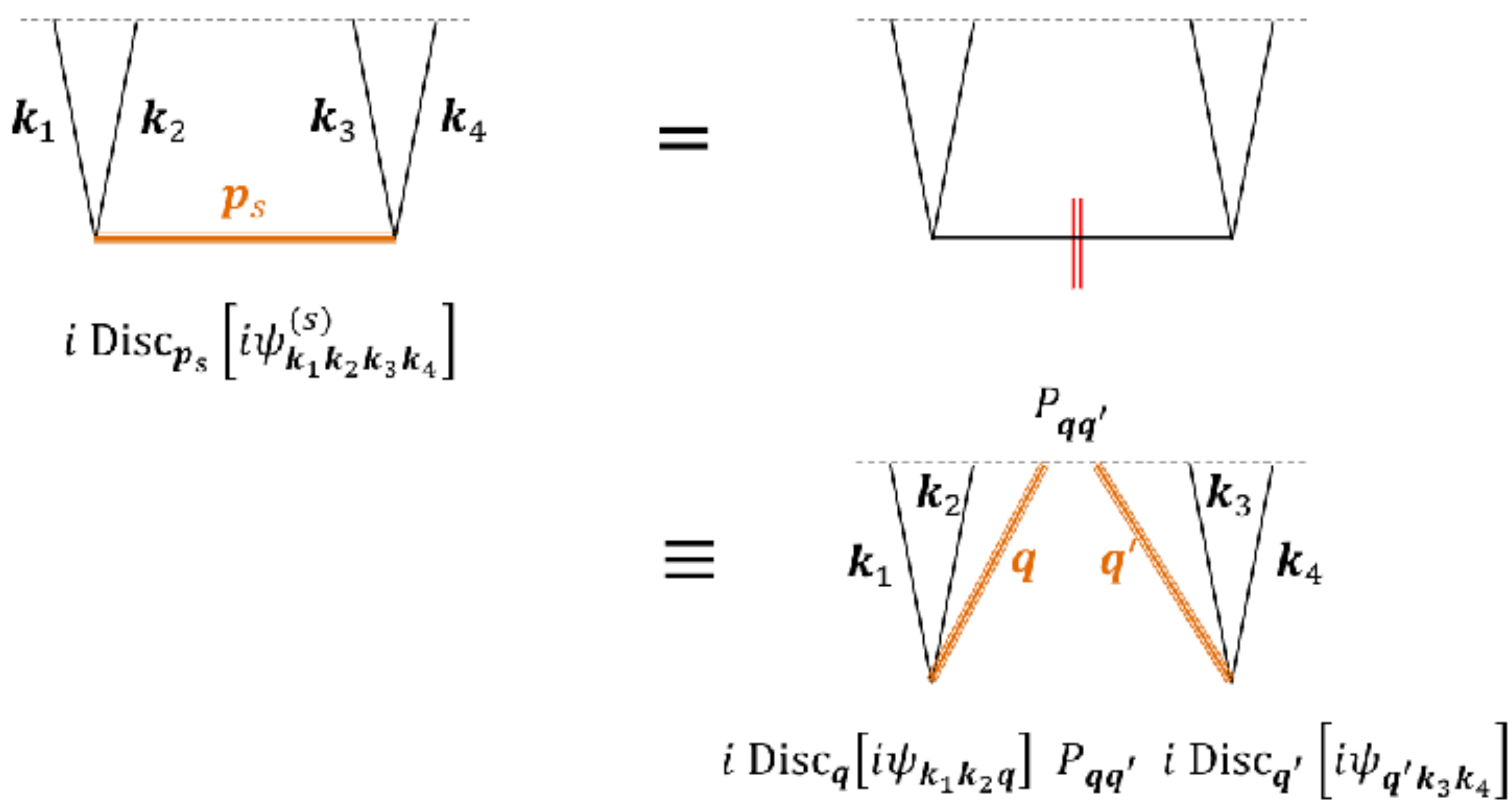
- It follows from unitarity time evolution, but the equation does not involve time! Time “emerges” at boundary as in holography…

- This is a Cosmological Optical Theorem (COT) and can be interpreted as fixing a “discontinuity”

$$\mathrm{Disc}\psi_n = \psi_n(\{\omega\},\{\mathbf{k}\}) + \psi_n^*(\{-\omega\},\{-\mathbf{k}\}) = 0$$

Exchange diagrams

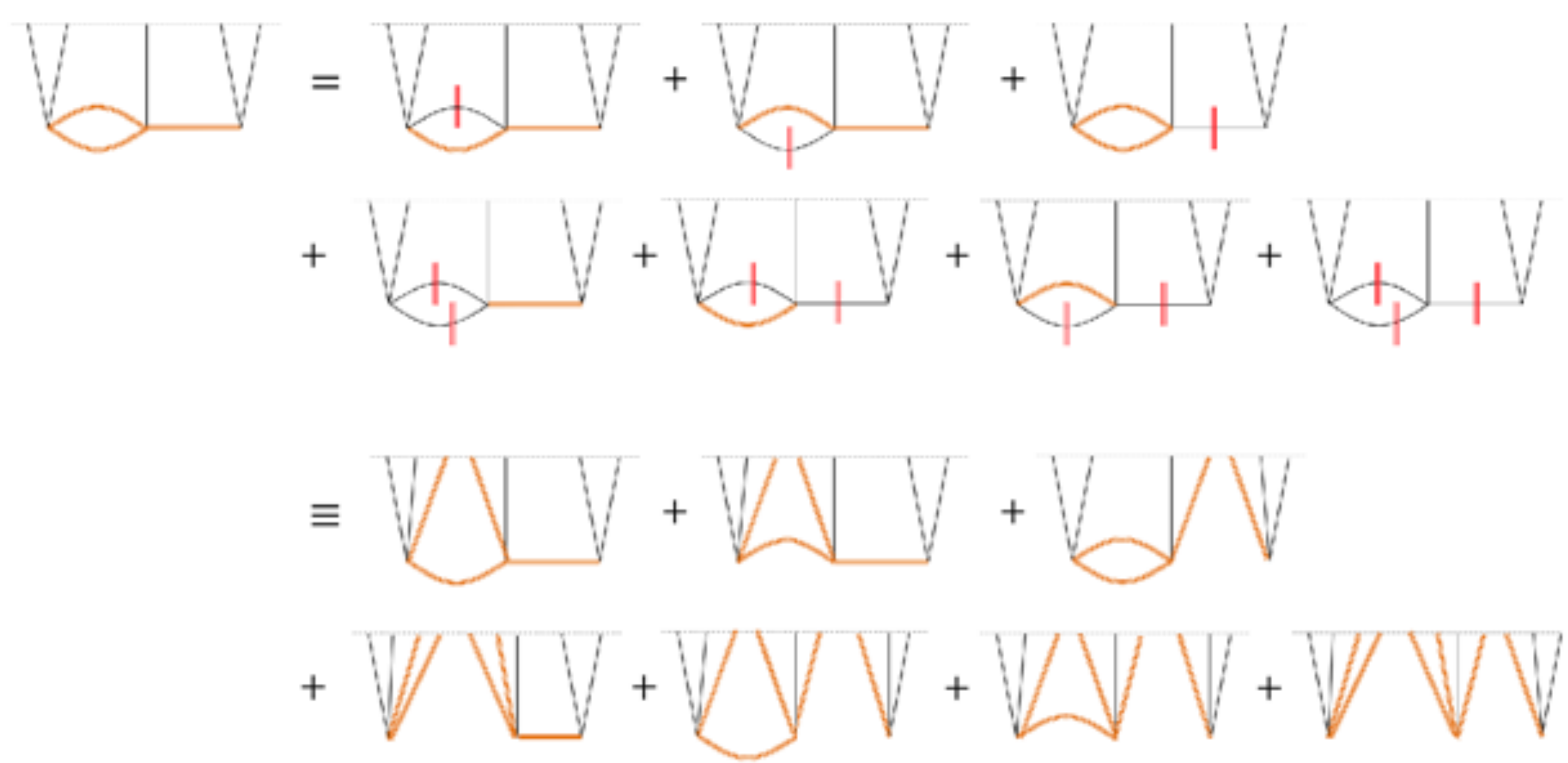
- The next simplest case is a 4-particle exchange diagram (trispectrum). The Cosmo Optical Theorem (COT) is



General diagrams

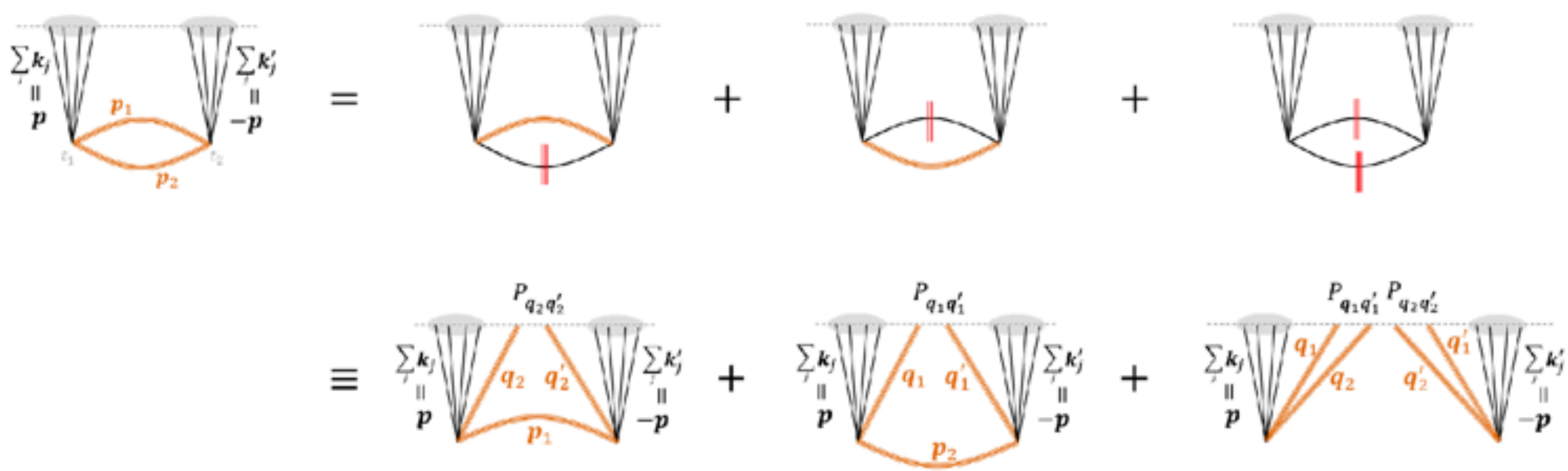
- These relations are valid to *all order in perturbation theory to any number of loops for fields of any mass and spin and arbitrary interactions (around any FLRW admitting a Bunch Davies initial condition)* [Goodhew, Jazayeri & EP '21; Melville & EP '21]

$$i \underset{\substack{\text{internal}\\ \text{lines}}}{\text{disc}} \left[i\,\psi^{(D)} \right] = \sum_{\text{cuts}} \left[\prod_{\substack{\text{cut}\\ \text{momenta}}} \int P \right] \prod_{\text{subdiagrams}} (-i) \underset{\substack{\text{internal \&}\\ \text{cut lines}}}{\text{disc}} \left[i\,\psi^{(\text{subdiagram})} \right] ,$$



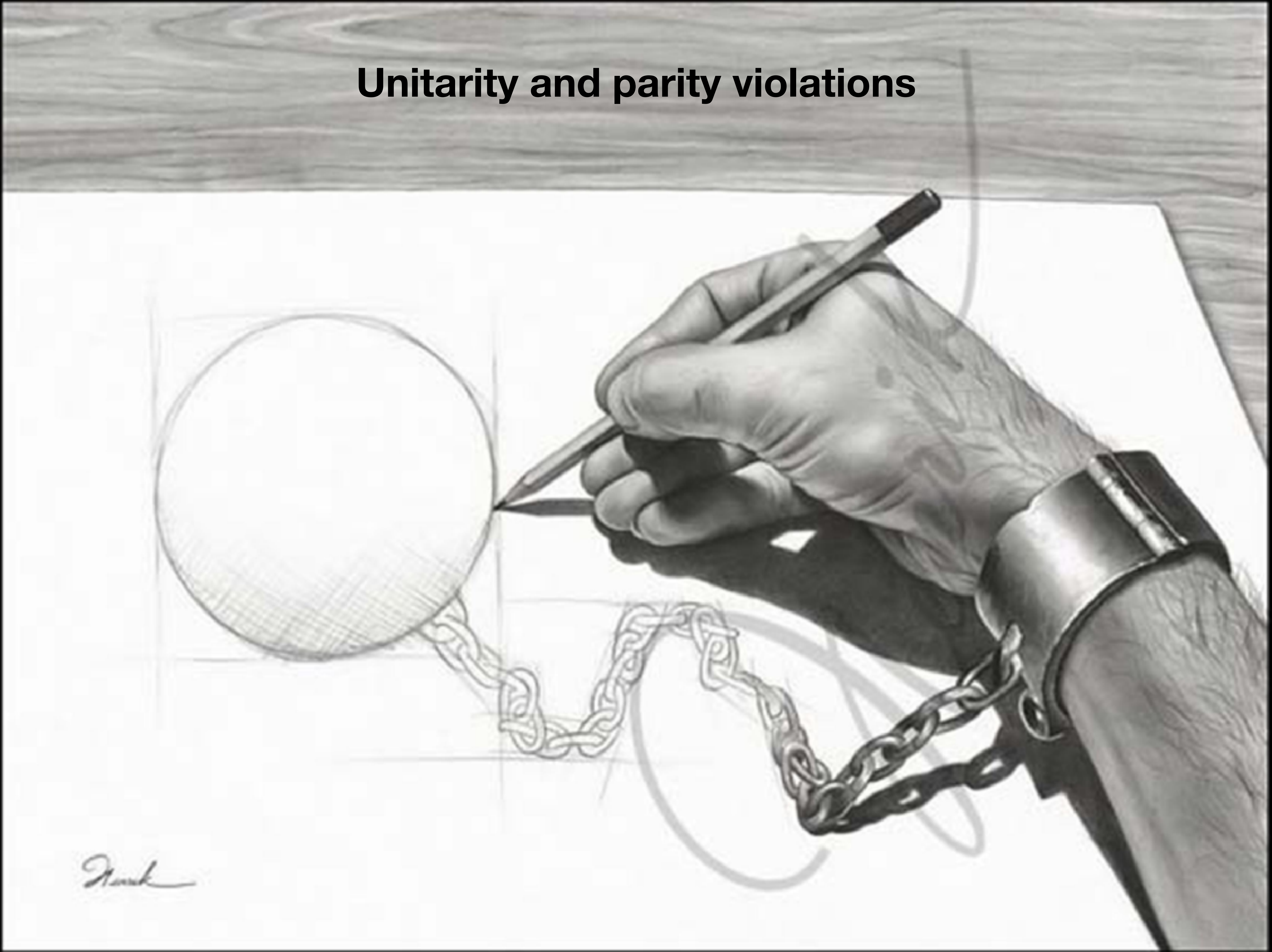
Loop corrections

- Unitarity gives us also *loop corrections*! For example we compute the leading 1-loop corrections for the power spectrum in the EFT of inflation, from tree-level results.



$$i\mathrm{Disc}\left[i\psi^{\text{1-loop}}_{\mathbf{k}_1\mathbf{k}_2}\right] = \frac{H^2}{f_\pi^4}\frac{ik^3}{480\pi}\frac{(1-c_s^2)^2}{c_s^4}\left[(4\tilde{c}_3+9+6c_s^2)^2+15^2\right]$$

Unitarity and parity violations



Contact interactions

- Contact interactions contribute to correlators as

$$\langle \varphi(\mathbf{k}_1)\dots\varphi(\mathbf{k}_n)\rangle = \frac{\int \mathcal{D}\varphi \;\Psi\Psi^* \;\varphi(\mathbf{k}_1)\dots\varphi(\mathbf{k}_n)}{\int \mathcal{D}\varphi \;\Psi\Psi^*} \,,$$

$$B_n^{\text{contact}}(\{k\};\{\mathbf{k}\}) = -\frac{\psi_n'(\{k\};\{\mathbf{k}\}) + \psi_n'^*(\{k\};-\{\mathbf{k}\})}{\prod_{a=1}^{n} 2\ \mathrm{Re}\ \psi_2'(k_a)} \,,$$

which gives the Real or Imaginary part for parity even or odd interactions.

$$\psi_n(\{\omega\},\{\mathbf{k}\})$$

parity

$$\pm\psi_n(\{\omega\},\{-\mathbf{k}\})$$

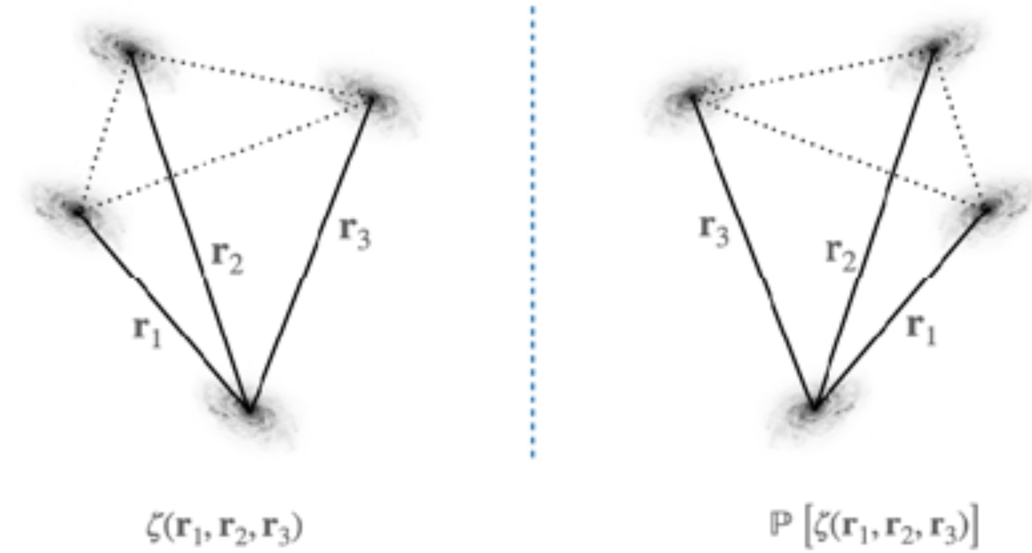
scaling

$$\lambda^{-3}\psi_n(\{\lambda\omega\},\{\lambda\mathbf{k}\})$$

cosmo optical theorem

$$\psi_n^*(\{-\omega\},\{-\mathbf{k}\})$$

No-go for parity odd



- Assuming scale invariance, *unitarity* and a BD initial state, *IR-finite* parity-odd correlators vanish at tree-level for [Cabass, Jazayeri, EP & Stefanyszyn '22]

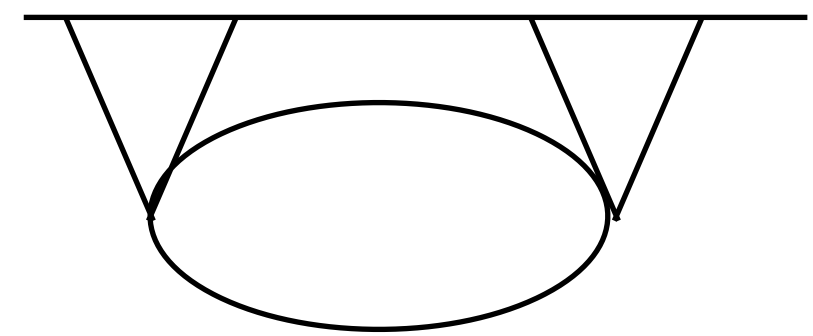
  - Any number of external massless field interacting with conformally coupled scalar fields

  - 4 external massless scalars interacting with any number of massive scalars, or massless fields of any spin.

- If a parity-odd trispectrum were detected (see e.g. [Cahn, Slepian & You '22; Philcox '22]), one would need to relax these assumptions, e.g. break scale invariance, massive or chiral spinning fields

Loop at leading order

- In single field inflation, assuming scale invariance, the parity-odd trispectrum B_4 vanishes.



- Leading contribution is 1-loop!

- 1-loop 1-vertex vanished in dim reg (no momentum flow)

- Calculation is complicated (d-dimensional mode functions, many derivatives), but answer is simple [Lee, McCulloch, EP to appear]

$$B_4^{\rm PO} = i\,\frac{(\mathbf{k}_1 \times \mathbf{k}_2)\cdot \mathbf{k}_3\, \mathrm{Poly}_p(\mathbf{k})}{(k_1 k_2 k_3 k_4)^3 k_T^p}$$

- Is this an observable quantum effect?

Horizons

- There are still basic and very general facts about quantum field theory on cosmological spacetimes that are awaiting to be discovered: *it's a wide open field of research*!
- Questions for the future include:
  - Can we derive "positivity bounds" for cosmology that encode the constraints of a consistent UV completion?
  - Are there measurable non-perturbative quantum gravity effects in cosmological correlators as e.g. in Black Hole physics?
  - Numerically bootstrap fully non-perturbative correlators in dS?
- Because of the ever growing body of cosmological dataset, advancements on the theory side are likely to have important repercussion on the phenomenology and ultimately make a long standing contribution to our understanding of the very early universe.