D-instanton Amplitudes in String Theory

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Lecture 1

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String theory

String theory in any background gives

1. The spectrum of states

– contains some massless states and infinite tower of massive states

2. A formula for the scattering amplitudes involving these states

The original formulation of string theory was perturbative.

The amplitude for any scattering process can be expressed as

$$\sum_{n=0}^{\infty} a_n \, g_s^{2n+\alpha}$$

g_s: string coupling

α: some fixed number for a given scattering process

a_n: coefficients that could depend on the quantum numbers of external states

– can be computed in terms of integrals over the moduli spaces of Riemann surfaces

As in quantum field theories, we expect that string amplitudes will be additional terms that are not captured by the perturbation expansion

– need a non-perturbative formulation

For special backgrounds we have some understanding of the non-perturbative corrections using various dualities

– Matrix model

– S-duality

– AdS/CFT

However we do not yet have a complete non-perturbative formulation of string theory in a general background.

Nevertheless there is one class of non-perturbative corrections that can be studied systematically using (almost) the usual formulation of string theory

– D-instanton corrections

– give additional contribution to an amplitude of the form

$$e^{-C/g_s} \sum_{n=0}^{\infty} b_n \, g_s^{n+\beta}$$

C, β: some constants

b_n's depend on the quantum numbers of external states

– can be computed as integrals over the moduli space of Riemann surfaces with boundaries

However the integrals that contribute to b_n often diverge

Examples of some integrals we shall encounter:

$$\int_0^\infty \frac{dt}{2t}\left(e^t - 1\right)$$

$$\frac{1}{2}\int_0^1 dy\, y^{-2}(1 + 2\omega_1\omega_2 y)$$

$$\int_0^1 dv \int_0^{1/4} dx \left\{\frac{v^{-2} - v^{-1}}{\sin^2(2\pi x)} + 2\,\omega^2\, v^{-1}\right\}$$

The goal of these lectures will be to understand the physical origin of these divergences and extract unambiguous, finite numbers out of them.

$$\int_0^\infty \frac{dt}{2t}\left(e^t - 1\right) \qquad \Rightarrow \qquad \frac{i}{4\pi^2}\, 2\pi\delta(E)$$

E: total energy of all the external states

$$\frac{1}{2}\int_0^1 dy\, y^{-2}(1 + 2\omega_1\omega_2 y) \qquad \Rightarrow \qquad -\frac{1}{2}$$

$$\int_0^1 dv \int_0^{1/4} dx \left\{ \frac{v^{-2} - v^{-1}}{\sin^2(2\pi x)} + 2\,\omega^2\, v^{-1} \right\} \quad \Rightarrow \quad -\frac{1}{2}\,\omega^2 \ln 4$$

As a test of this procedure, we shall verify that when the answer is known from a dual description e.g. matrix model or S-duality, the procedure we shall describe reproduces them correctly.

Given that the string perturbation expansion is expected to be an asymptotic series, does it make sense to compute non-perturbative contribution?

Answer 1:

In many cases the perturbative contribution to specific quantities either vanishes or terminates after a finite order

a) Terms protected by supersymmetry, e.g. R^4 terms in type IIB in D=10, moduli space metric in N=2 supersymmetric theories in D=4, superpotential in N=1 supersymmetric theory in D=4 etc

b) Unitarity violation in c=1 bosonic string theory

c) Barrier penetration in $\hat{c} = 1$ type 0B string theory

Answer 2:

Instantons describe non-trivial saddle points of string theory

Instanton contribution to amplitudes represent the result of the path integral along the steepest descent contour (Lefschetz thimble) of this saddle point

– can be studied independently of the perturbative contribution

– can be used to test dualities between a pair of theories both of which are weakly coupled, e.g. in c $<$ 1 string theories

D-instantons

D-instantons are D-branes with Dirichlet boundary condition along all non-compact directions, including Euclidean time

– describe finite action (C/g_s) classical solutions in string theory

– analogous to instantons in quantum field theory

– give non-perturbative corrections to string amplitudes

$$e^{-C/g_s} \times \text{power series in } g_s$$

World-sheet theory of closed and <u>open</u> strings provide (formal) expressions for the D-instanton contribution to the ampltudes

Integrals over moduli spaces of Riemann surfaces with boundaries generate the series expansion in g_s multiplying e^{-C/g_s}

In these lectures we shall focus on single D-instanton amplitudes for simplicity.

n D-instanton contribution $\propto e^{-n\,C/g_s}$

– more suppressed than single D-instanton contribution.

Systematics of D-instanton induced amplitudes

In the presence of D-instantons the spectrum has both closed strings and open strings with ends on the instanton.

However the open strings describe the modes of the instanton and only exist for limited time

– they are not asymptotic states

The external states in a scattering amplitude will always be closed strings

(or open strings on ordinary time filling D-branes if present)

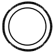
Individual world-sheets with boundaries on the D-instanton do not conserve energy / momentum

– disconnected world-sheets contribute even for generic values of external energy / momentum

For getting leading contribution to the D-instanton amplitude, we

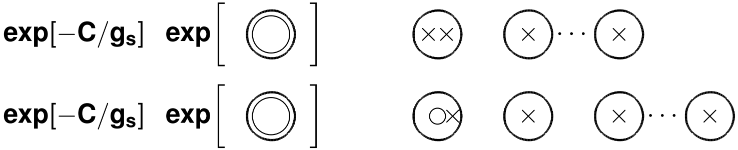
– maximize the number of disks since each disk gives $1/g_s$

– can use as many annuli as we want since annuli $\sim (g_s)^0$

$\exp[-C/g_s]$ $\exp\Big[$  $\Big]$ 

$\times$: closed string vertex operator

At the next order there are more possibilities



etc.

This way we can write down the expression for D-instanton induced amplitude to any order in the string coupling g_s

However, the moduli space integrals diverge from regions of the moduli space where the Riemann surface degenerates

Lecture 2

We have seen that all D-instanton amplitudes have overall normalization factor given by exponential of the annulus amplitude.

Today we shall discuss computation of the annulus amplitude.

We shall pick a specific example but the same procedure can be applied to all cases.

Example: 2D bosonic string theory

World-sheet theory has

1. A scalar X describing time direction

2. A Liouville field χ_L with central charge 25

– describes space direction with a potential

3. b,c ghost system with central charge -26

Ghost number assignment: c: 1, b: -1, matter: 0

Closed string spectrum has a single massless scalar field living on a half line along χ_L

This theory has 'ZZ instanton' with Dirichlet boundary condition on X and χ_L and action $1/g_s$

For D-instantons in 2D bosonic string theory:

$$\textbf{Annulus partition function} = \int_0^\infty \frac{dt}{2t} \left(e^t - 1\right)$$

For D-instantons in type IIB string theory in ten dimensions

$$\textbf{Annulus partition function} = \int_0^\infty \frac{dt}{2t} \left(8 - 8\right)$$

8 from NS sector, -8 from R sector

Naively the answer vanishes

However this gives results for instanton correction that are inconsistent with the prediction of S-duality

In order to make sense of these divergences and extract a finite result we need ‘string field theory’

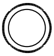
For now it is enough to know that

1. String field theory is a regular quantum field theory with infinite number of fields, one for each mode of the string

2. It is designed so that the perturbative amplitudes reproduce the world-sheet result (formally).

3. String field theory describing the dynamics of open strings on D-instanton is a 0-dimensional field theory since open strings do not carry continuous momentum

Path integral $\Rightarrow$ ordinary integrals

$\mathbf{exp}\Big[$  $\Big] = \mathbf{exp}\left[\int_0^\infty \frac{dt}{2t}\, Z(t)\right]$

t $\propto$ ratio of circumference to the width of the cylinder / annulus

$$Z(t) = \text{Tr}\left\{(-1)^F e^{-t\,L_0} b_0 c_0\right\}$$

Tr is trace over open string states on the D-instanton

F = ghost number + 1

$b_0 c_0$ is needed to remove ghost zero modes

$$Z(t) = \sum_b e^{-t\,h_b} - \sum_f e^{-t\,h_f}$$

h_b, h_f: L_0 eigenvalues of bosonic / fermionic open string states that are annihilated by b_0 (Siegel gauge)

If h_b or $h_f \le 0$, then the integral diverges from large t region.

Strategy for dealing with large t divergence:

1. Use the identities, valid for $h_b, h_f > 0$,

$$\exp\left[\int \frac{dt}{2t}\left(e^{-th_b} - e^{-th_f}\right)\right] = \sqrt{\frac{h_f}{h_b}}$$

$$h_b^{-1/2} = \int \frac{d\psi_b}{\sqrt{2\pi}} e^{-\frac{1}{2}h_b\psi_b^2}, \qquad \psi_b : \text{grassmann even}$$

$$h_f = \int du_f\, dv_f\, e^{-h_f u_f v_f}, \qquad u_f, v_f : \text{grassmann odd}$$

2. Interpret the modes ψ_b, u_f, v_f as open string fields (D=0) and the exponent as open string field theory action in Siegel gauge

3. Modes with $h_b < 0$ are tachyonic modes and integration over them can be carried out along the steepest descent contour

4. Modes with $h_b = 0$ and $h_f = 0$ represent respectively the bosonic and fermionic zero modes

– need to be treated carefully.

Origin of zero modes

1. Bosonic zero modes ψ_b^0 can arise from the freedom of translating the instanton along flat directions e.g. Euclidean time

Remedy: Change variables from ψ_b^0 to D-instanton position y.

If $\psi_b^0 = K_1\, y$ then $d\psi_b^0 = K_1\, dy$

Integration over y has to be done at the end and produces a factor of $\int dy e^{iEy} = 2\pi\delta(E)$, with E being the total energy of external states

In superstring theory similar treatment is needed for the fermion zero modes associated with broken supersymmetry.

For 2D bosonic string theory

$$\mathbf{Z}(\mathbf{t}) = \left(\mathbf{e}^{\mathbf{t}} - \mathbf{1}\right)$$

$\mathbf{e}^{\mathbf{t}} \Rightarrow$ a mode with $\mathbf{h_b} = -\mathbf{1} \quad \Rightarrow \quad$ produces $\sqrt{\mathbf{1}/\mathbf{h_b}} = \mathbf{i}$

The bosonic translation zero mode should give +1

Why do we have $-\mathbf{1}$?

2. We have fermion zero modes coming from ghost sector

$$\mathbf{c_1 c_{-1}}|\mathbf{0}\rangle, \qquad |\mathbf{0}\rangle$$

They are results of wrongly fixing the U(1) ‘gauge symmetry’ on the instanton

Consider the gauge invariant open string field theory on a Dp-brane

– has a U(1) gauge field.

Action:

$$\int d^{p+1}x \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \left(\frac{1}{\sqrt{2}}\, \partial^{\mu} A_{\mu} - \phi\right)^{2}\right]$$

ϕ**: mode associated with the state** $c_0 e^{ik.X}(0)|0\rangle$

– not present in the Siegel gauge but is present in the gauge invariant theory

Gauge transformation:

$$\delta A_{\mu} = \sqrt{2}\, \partial_{\mu} \theta(x), \quad \delta\phi = \Box\theta(x)$$

$$\mathbf{S} = \int \mathbf{d^{p+1}x} \left[\frac{\mathbf{1}}{\mathbf{4}} \mathbf{F}_{\mu\nu} \mathbf{F}^{\mu\nu} + \left(\frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \, \partial^\mu \mathbf{A}_\mu - \phi \right)^{\mathbf{2}} \right]$$

$$\delta \mathbf{A}_\mu = \sqrt{\mathbf{2}} \, \partial_\mu \theta(\mathbf{x}), \quad \delta\phi = \Box \theta(\mathbf{x})$$

Siegel gauge $\phi = 0$ leads to gauge fixed action including ghosts:

$$\int \mathbf{d^{p+1}x} \left[-\frac{\mathbf{1}}{\mathbf{2}} \mathbf{A}^\mu \Box \mathbf{A}_\mu - \mathbf{u} \Box \mathbf{v} \right], \qquad \mathbf{u}, \mathbf{v} : \text{ ghosts}$$

On D-instanton, there is no $\mathbf{A}_\mu$ and all fields are x independent

$\Rightarrow \mathbf{u} \Box \mathbf{v} = \mathbf{0}$

$\Rightarrow$ leads to ghost zero modes

– arise since we are attempting to gauge fix a rigid symmetry with parameter θ under which $\delta\phi = \mathbf{0}$

Remedy: Undo the gauge fixing by using a gauge invariant form of the path integral

1. Integrate over ϕ and drop the integration over the ghosts

$$\Rightarrow \int \mathbf{d}\phi\, \mathbf{e}^{-\phi^{2}} = \sqrt{\pi}$$

2. Divide by the volume of the gauge group

$$\Rightarrow \int \mathbf{d}\theta$$

– can be found by carefully comparing the string field theory gauge transformation laws with $\psi \to \mathbf{e}^{\mathbf{i}\alpha}\psi$ where α has period 2π.

ψ: any state of the open string with one end on the instanton

If $\theta = \mathbf{K_2}\,\alpha$ then $\int \mathbf{d}\theta = \mathbf{K_2}\,2\pi$

Exponential of the annulus diagram is:

$$\exp\left[\int_0^\infty \frac{dt}{2t}\,Z(t)\right] = \exp\left[\int_0^\infty \frac{dt}{2t}\,(e^t - 1)\right]$$

e^t is from a tachyon with $h_b = -1$

-1 is the result of

1. A bosonic zero mode associated with translation along X

2. Two fermionic zero modes from the ghost

Final result for the annulus diagram:

$$i\,\sqrt{\pi}\,\frac{1}{\sqrt{2\pi}}\,\frac{K_1}{2\pi K_2}\;2\pi\delta(E)$$

We shall find K_1, K_2 but for this we need more details of open string field theory.

Lecture 3

Open (bosonic) string field theory

H: Full vector space of open string states in matter + ghost sector

An off-shell 'classical' open string field $|\psi\rangle$ is an arbitrary element of H of ghost number 1.

Let $|\phi_r^{(n)}\rangle$ be a set of basis states in H of ghost number n.

Then $|\psi\rangle = \sum_r \psi_r |\phi_r^{(1)}\rangle$

ψ_r are the dynamical variables over which we do (path) integration.

Action:

$$\mathbf{S} = \frac{1}{2}\langle\psi|\mathbf{Q_B}|\psi\rangle + \textbf{interaction terms}$$

$$\mathbf{Q_B} = \oint_0 \mathbf{dz}\ [\mathbf{c(z)\, T_m(z)} + \mathbf{b(z)c(z)}\partial\mathbf{c(z)}]$$

$\mathbf{T_m(z)}$: matter stress tensor

$\mathbf{Q_B^2 = 0}$

The action S is invariant under gauge transformation:

$$\delta|\psi\rangle = \mathbf{Q_B}|\lambda\rangle + \cdots$$

$|\lambda\rangle$: arbitrary state in H of ghost number 0.

If we expand $|\lambda\rangle$ as $\sum_r \lambda_r|\phi_r^{(0)}\rangle$, then λ_r are the 'gauge transformation' parameters

Siegel gauge: $b_0|\psi\rangle = 0$.

$$S = \frac{1}{2}\langle\psi|c_0 L_0|\psi\rangle + \cdots$$

The gauge fixing leads to Faddeev-Popov ghosts

Result: The full action including the ghosts has the form

$$S = \frac{1}{2}\langle\widetilde{\psi}|c_0 L_0|\widetilde{\psi}\rangle + \cdots$$

with $|\widetilde{\psi}\rangle$ having arbitrary ghost number subject to $b_0|\widetilde{\psi}\rangle = 0$

Components of $|\widetilde{\psi}\rangle$ with ghost number other than 1 are the Faddeev-Popov ghosts

Propagator $\propto (L_0)^{-1}$

$|\varphi_r^{(n)}\rangle$: A basis of states of ghost number n, satisfying $b_0|\varphi_r^{(n)}\rangle = 0$

In order that the gauge fixed action $\frac{1}{2}\langle\widetilde{\psi}|c_0 L_0|\widetilde{\psi}\rangle$ has the form

$$-\frac{1}{2}h_b\psi_b^2 + h_f u_f v_f$$

we need to normalize the basis states as

$$\langle\varphi_r^{(1)}|c_0|\varphi_s^{(1)}\rangle = \delta_{rs}, \qquad \langle\varphi_r^{(2)}|c_0|\varphi_s^{(0)}\rangle = \delta_{rs}$$

etc.

Also the basis states for out of Siegel gauge states for classical string field must be chosen as $c_0|\varphi_r^{(0)}\rangle$ so that gauge transformation produces an L_0

If $|\psi\rangle = \sum_r \chi_r|\varphi_r^{(1)}\rangle + \sum_r \phi_r c_0|\varphi_r^{(0)}\rangle$ and $|\lambda\rangle = \sum_r \lambda_r|\varphi_r^{(0)}\rangle$ then

$$\delta|\psi\rangle = Q_B|\lambda\rangle \qquad \Rightarrow \qquad \delta\phi_r = h_r\lambda_r$$

h_r: L_0 eigenvalue of $|\varphi_r\rangle$

Classical string field:

$$|\psi\rangle = \chi\, c_1|0\rangle + \psi_b^0\, c_1\alpha_{-1}|0\rangle + i\,\phi\, c_0|0\rangle + \cdots$$

Gauge transformation parameters:

$$|\lambda\rangle = i\,\theta|0\rangle + \cdots$$

Siegel gauge field:

$$|\widetilde{\psi}\rangle = \chi\, c_1|0\rangle + \psi_b^0\, c_1\alpha_{-1}|0\rangle + u|0\rangle + v\, c_1 c_{-1}|0\rangle + \cdots$$

χ**: tachyon corresponding to the h $= -1$ state**

α_{-1}**: oscillator of X satisfying** $[\alpha_1, \alpha_{-1}] = 1$

Factors of i ensure that χ and θ are real

If $|0\rangle$ had carried L_0 eigenvalue h, e.g. by carrying momentum k, then gauge transformation law would give $\delta\phi = h\,\lambda$ and the Siegel gauge $\phi = 0$ would give a Faddeev-Popov determinant h

– would be reproduced by the ghost action huv

For h=0 this procedure breaks down.

Go back to the original gauge invariant formulation:

$$\int \frac{d\chi}{\sqrt{2\pi}} \int \frac{d\psi_b^0}{\sqrt{2\pi}} \int d\phi \, e^{-S} / \int d\theta$$

$$S = -\frac{1}{2}\chi^2 + \phi^2$$

Note: Comparison with the world-sheet result fixes the normalization of the path integral measure over the open string fields.

We could have replaced $d\phi$ by $d\phi/\sqrt{2\pi}$ but then $d\theta$ will also be replaced by $d\theta/\sqrt{2\pi}$ so that the Faddeev-Popov determinant remains L_0.

Relation between ψ_b^0 and y:

1. The dependence of an amplitude on the D-instanton position y must be of the form

$$e^{-i\omega\, y}$$

where ω is the total energy carried by all the closed string states

y insertion in an amplitude should product a $-i\omega$ factor

Compare this with the result of the ψ_b^0 insertion

State multiplying ψ_b^0 in string field expansion

$$c_1\alpha_{-1}|0\rangle = c(0) i\sqrt{2}\partial X(0)|0\rangle$$

X is normalized so that

$$\partial X(z)\partial X(w) = -\frac{1}{2(z-w)^2} + \text{non-singular}$$

$\Rightarrow$ vertex operator for ψ_b^0:

$$\text{unintegrated}: \; c(z) i\sqrt{2}\partial X(z), \qquad \text{integrated}: \; i\sqrt{2}\partial X(z)$$

The disk amplitude with one insertion of ψ_b^0 and n closed string vertex operators $V_1, \cdots V_n$ of energy $\omega_1, \cdots \omega_n$ is

$$A \propto g_o \left\langle \int dz\, i\, \sqrt{2} \partial X(z) \prod_{k=1}^{n} V_k(z_k) \right\rangle$$

$g_o = (g_s/2\pi^2)^{1/2}$: open string coupling constant

Using OPE

$$\partial X(z) V_k(z_k) = -\frac{i\omega_k}{2(z - z_k)} V_k(z_k) + \text{non-singular}$$

we get

$$A = i\pi\, \sqrt{2}\, g_o\, \omega\, \left\langle \prod_{k=1}^{n} V_k(z_k) \right\rangle, \qquad \omega \equiv \sum_k \omega_k$$

$\Rightarrow \psi_b^0$ insertion in an amplitude produces a factor of $i\pi\, \sqrt{2}\, g_o\, \omega$

Since y insertion produces a factor of $i\omega$, we have

$$\psi_b = y/(\pi\sqrt{2}\, g_o)$$

Relation between θ and α:

α: rigid gauge transformation parameter

An open string stretched between the original D-instanton and a second spectator D-instanton picks up a phase $e^{i\alpha}$.

This gives infinitesimal transformation law:

$$\delta\xi = i\,\alpha\,\xi$$

ξ: Any state of the open string stretched between the pair of D-instantons

We can compare this with the known gauge transformation law of ξ in open string field theory:

$$\delta\xi = g_o\,K\,\theta\,\xi$$

K: three point function of normalized vertex operators of θ, ξ, ξ^c

In the expansion of the string field, θ multiplies i $|0\rangle$

$\Rightarrow$ **θ vertex operator is i $\times$ identity**

$\Rightarrow$ **three point function of θ, ξ, ξ^c reduces to i $\times$ two point function of ξ, ξ^c**

i as long as ξ, ξ^c are normalized

Compare $\delta\xi = \mathbf{i}\, \mathbf{g_o}\, \theta\xi$ with $\delta\xi = \mathbf{i}\alpha\xi$

This gives $\theta = \alpha/\mathbf{g_o}$ and therefore

$$\int \mathbf{d}\theta = \int \mathbf{d}\alpha/\mathbf{g_o} = \mathbf{2}\pi/\mathbf{g_o}$$

Summary of string field theory results

In the path integral representation the tachyon of $h_b = -1$ contributes $\sqrt{1/h_b} = i$

Contribution from translational zero mode:

$$\int \frac{d\psi_b^0}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \int \frac{dy}{\pi\, g_o \sqrt{2}}$$

y: D-instanton location along time direction, couples via $e^{iyE_{total}}$

$\int dy$ produces $2\pi\delta\,(E_{total})$ at the end

Ghost zero mode integral is replaced by

$$\int d\phi\, e^{-\phi^2} \Big/ \int d\theta = \frac{\sqrt{\pi}}{2\pi/g_o}$$

Net normalization $i/(4\pi^2)$ agrees with a dual matrix model result.

When the result is known from a dual description, this procedure produces the correct result in all cases that have been studied.

1. **2D bosonic string theory**

2. **c<1 bosonic string theory**

3. **Type IIB in D=10**

4. **Type IIA / IIB on CY_3**

5. **$\hat{c} = 1$ type 0B string theory**

6. **IIA/IIB on CY_3 orientifolds**

7. **Sine-Liouville deformation of c=1 bosonic string theory**

More on string field theory (SFT)

Perturbative amplitudes in string field theory are given by sum of Feynman diagrams

Open string propagator $\propto (L_0)^{-1}$

SFT is designed so that formally the sum of Feynman diagrams reproduce the world-sheet expression after using

$$(L_0)^{-1} = \int_0^\infty dt\, e^{-L_0 t}, \qquad (L_0 + \bar{L}_0)^{-1} = \int_0^\infty dt\, e^{-(L_0+\bar{L}_0)t}$$

t: Schwinger parameter

t's become the moduli of Riemann surfaces after change of variables

$$\textbf{SFT} \quad \Rightarrow \quad (L_0)^{-1} = \int_0^\infty dt\, e^{-L_0 t} \quad \Leftarrow \quad \textbf{world-sheet}$$

1. This is an identity for $L_0 > 0$

2. For $L_0 < 0$ the rhs diverges from $t \to \infty$ end but the lhs is finite and we can use lhs as the correct expression

3. For $L_0 = 0$ both sides diverge

However, on the lhs we sit on the pole of a propagator and insights from QFT can be used to make sense of this.

This is the essence of why string field theory is useful for dealing with divergences in the integrals over the moduli spaces of Riemann surfaces

Lecture 4

In the case under study we have both open and closed string fields.

The external states in a Feynman diagram are closed strings

The internal propagators are of closed and open strings.

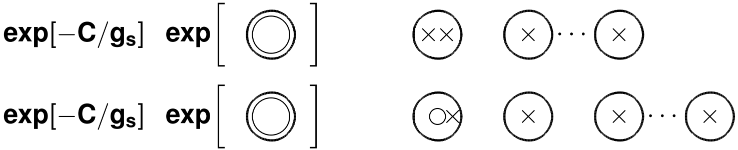
Closed string propagator $\propto (\mathbf{L_0} + \bar{\mathbf{L}}_0)^{-1} = \int_0^\infty \mathbf{dt}\,\mathbf{exp}\left[-\mathbf{t}(\mathbf{L_0} + \bar{\mathbf{L}}_0)\right]$

$\mathbf{L_0} + \bar{\mathbf{L}}_0$ **eigenvalues** $\propto \mathbf{k^2} + \mathbf{m^2}$**,** **m: mass of the string mode**

Since closed strings carry momentum k, it is possible to make the $\mathbf{L_0} + \bar{\mathbf{L}}_0$ **eigenvalue** $\mathbf{k^2} + \mathbf{m^2}$ **positive by analytic continuation in k.**

– we can integrate out the closed strings and consider an effective string field theory in which the internal states are only open strings.

At the next order we need to compute



etc.

Define:

$g_s\, f(\omega_1, \omega_2)$: Ratio of disk two point function to product of two disk one point functions

$g_s\, g(\omega)$: Ratio of annulus one point function to disk one point function

$g_s\, C$: Partition function for disk with two holes and torus with one hole.

Order g_s contribution to the n-point amplitude:

$$\mathbf{g_s \times \text{leading order contribution} \times \left[\sum_{j<k} f(\omega_j, \omega_k) + \sum_j g(\omega_j) + C\right]}$$

f, g and C have divergences.

$$\mathbf{f = f_{fimite} + f_{div}, \qquad g = g_{finite} + g_{div}, \qquad C = C_{finite} + C_{div}}$$

$$\mathbf{f_{div}(\omega_1, \omega_2) = \frac{1}{2}\int_0^1 dy\, y^{-2}(1 + 2\omega_1\omega_2 y) \equiv A_f + B_f\, \omega_1\omega_2}$$

$$\mathbf{g_{div}(\omega) = \int_0^1 dv \int_0^{1/4} dx \left\{\frac{v^{-2} - v^{-1}}{\sin^2(2\pi x)} + 2\,\omega^2\, v^{-1}\right\} \equiv A_g + B_g\omega^2}$$

n-point function at order g_s:

$= \mathbf{g_s \times}$ **leading order contribution**

$$\mathbf{\times \left[\frac{n(n-1)}{2} A_f + n\, A_g + C + \left\{B_g - \frac{B_f}{2}\right\} \sum_j \omega_j^2 + \text{finite}\right]}$$

We again need to make use of string field theory

Strategy:

1. Express the amplitudes as sum over SFT Feynman diagrams

– automatically replaces the tachyon contribution by $1/h$ where h is the L_0 eigenvalue

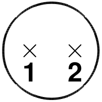
2. Remove the zero mode contribution to the propagators since they are to be integrated at the end or removed altogether.

3. Add the propagator of the field ϕ that was not present in the world-sheet formulation but should be present.

4. Account for corrections to the jacobian factors for change of variable from ψ_b^0 to y and θ to α

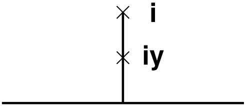
We shall first describe the analysis of $\mathbf{f_{div}}(\omega_1, \omega_2)$.

– related to the divergent part of disk / UHP two point function:



$$\mathbf{f_{div}}(\omega_1, \omega_2) = \frac{1}{2} \int_0^1 \mathbf{dy}\, \mathbf{y}^{-2}(1 + 2\omega_1\omega_2\mathbf{y})$$

On the UHP, closed string vertex operators are located at i and iy



$$\mathbf{f}_{\mathbf{div}}(\omega_1, \omega_2) = \frac{1}{2} \int_0^1 \mathbf{dy}\, \mathbf{y}^{-2}(1 + 2\omega_1\omega_2 \mathbf{y})$$

Feynman diagrams:

(a) **(b)**

Thick lines: Closed strings **Thin lines: open strings**

The open-closed interaction vertices are UHP two point functions

To compute these amplitudes we need the two point open-closed interaction term for off-shell external states.

Need to choose a 'local coordinate' w_i at the location of each vertex operator.

If the UHP coordinate z is related to w as $z = f(w)$ then we insert the vertex operator $f \circ V(w)$ – conformal transform of V by f

e.g. for dimension h primeries, $f \circ V(w) = f'(w)^h V(f(w))$

Since only the open strings are off-shell, we need a choice of local coordinates at the open string puncture.

C-O interaction vertex

Put C at i, O at 0

Choose local coordinate at O to be

$$\mathbf{w} = \lambda\,\mathbf{z} \quad \Rightarrow \quad \mathbf{f}(\mathbf{w}) = \mathbf{w}/\lambda \quad \Rightarrow \quad \mathbf{f} \circ \mathbf{V}(\mathbf{w}) = \lambda^{-\mathbf{h_V}}\mathbf{V}(\mathbf{z})$$

λ**: an arbitrary constant, taken to be large for convenience**

$\Rightarrow$ **the two point function of a closed string state C and open string state O is**

$$\langle \mathbf{V_C}(\mathbf{i})\mathbf{V_O}(\mathbf{0})\rangle_{\mathbf{UHP}}\,\lambda^{-\mathbf{h_O}}$$

(a) (b)

We need to find the relation between y and the Schwinger parameter $\mathbf{q = e^{-t}}$ for diagram (a).

Diagram (a) corresponds to two UHP's sewed via

$$\mathbf{ww' = -q \quad \Rightarrow \quad \lambda^2 zz' = -q, \quad q \equiv e^{-t}, \quad t : \text{Schwinger parameter}}$$

On the sewed surface the punctures are located at

$$\mathbf{z = i, \qquad z' = i \quad \Rightarrow \quad z = i\, q/\lambda^2 \equiv i\, y}$$

This gives $\mathbf{y = q/\lambda^2}$.

$$\mathbf{y} = \mathbf{q}/\lambda^2$$

$$0 \leq q \leq 1 \qquad \Rightarrow \quad 0 \leq y \leq 1/\lambda^2$$

The region $1/\lambda^2 < y \leq 1$ comes from diagram (b) and gives finite result.

Analyze f_{div} using this:

$$\frac{1}{2}\int_0^1 dy\, y^{-2}(1+2\omega_1\omega_2 y) = \frac{1}{2}\left\{\int_0^{1/\lambda^2} + \int_{1/\lambda^2}^1\right\} dy\, y^{-2}(1+2\omega_1\omega_2 y)$$

$$= \frac{1}{2}\int_0^1 dq\{\lambda^2 q^{-2} + 2\omega_1\omega_2 q^{-1}\} + \frac{1}{2}\int_{1/\lambda^2}^1 dy y^{-2}(1+2\omega_1\omega_2 y)$$

$$\Rightarrow -\frac{1}{2}\lambda^2 + \frac{1}{2}\int_{1/\lambda^2}^1 dy y^{-2}(1+2\omega_1\omega_2 y) = -\frac{1}{2} + 2\,\omega_1\omega_2 \ln\lambda$$

$-\omega_1, \omega_2$ are energies of incoming and outgoing C.

For the choice of local coordinates we have made, the C-ϕ vertex vanishes.

$\Rightarrow$ **no need to include ϕ exchange contribution.**

Final result:

$$f_{div}(\omega_1, \omega_2) = -\frac{1}{2} + 2\omega_1\omega_2 \ln\lambda \equiv A_f + B_f\omega_1\omega_2$$

$$A_f = -\frac{1}{2}, \qquad B_f = \ln\lambda^2$$

Note: If we had chosen a different local coordinate for the C-O vertex, the result will be different

– compensated by ϕ exchange diagram for A_f.

For B_f some part may also cancel against contribution to $2\,B_g$.

We now turn to the divergent part of the annulus one point function:

– four types of contributions

1. g_{feynman} from the Feynman diagrams with zero mode contribution to the propagators removed

2. g_{ϕ} with one or more ϕ propagators

3. Correction to the relation between ψ_b^0 and y

$$\psi_b^0 = K_1 y \left[1 + g_s \int d\omega C(\omega)\, F(\omega)\right]$$

F: computable function

Then the path integral gets an additional Jacobian factor while changing variables from ψ_b^0 to y

$$\left[1 + g_s \int d\omega C(\omega)\, F(\omega)\right] \simeq \exp\left[g_s \int d\omega C(\omega)\, F(\omega)\right]$$

$\Rightarrow$ **new contribution $g_{jac}(\omega)$**

4. There is a similar correction to the $\theta - \alpha$ relation

$$\theta = K_2\, \alpha \left[1 + g_s \int d\omega C(\omega)\, G(\omega)\right]$$

$\Rightarrow g_{gauge}(\omega)$

Results:

$$\mathbf{g}_{\mathbf{feynman}}(\omega) = -\frac{\mathbf{2}}{\pi}\int_{(\mathbf{2}\widetilde{\lambda})^{-\mathbf{1}}}^{\mathbf{1}} \mathbf{d}\beta\,(\mathbf{1}+\beta^{\mathbf{2}})^{-\mathbf{1}}\,\widetilde{\lambda}^{\mathbf{2}}\,\mathbf{f}(\beta)^{\mathbf{2}} + \frac{\widetilde{\lambda}}{\mathbf{4}\pi} + \frac{\mathbf{1}}{\mathbf{2}}\omega^{\mathbf{2}}\,\mathbf{ln}\,\frac{\alpha^{\mathbf{2}}\widetilde{\lambda}^{\mathbf{2}}}{\mathbf{4}}$$

$$\mathbf{g}_{\phi}(\omega) = \frac{\mathbf{2}}{\pi}\int_{(\mathbf{2}\widetilde{\lambda})^{-\mathbf{1}}}^{\mathbf{1}} \mathbf{d}\beta\,(\mathbf{1}+\beta^{\mathbf{2}})^{-\mathbf{1}}\,\widetilde{\lambda}^{\mathbf{2}}\,\mathbf{f}(\beta)^{\mathbf{2}} + \frac{\widetilde{\lambda}}{\mathbf{4}\pi}$$

$$\mathbf{g}_{\mathbf{jac}}(\omega) = -\frac{\widetilde{\lambda}}{\pi} - \omega^{\mathbf{2}}\mathbf{ln}\frac{\widetilde{\lambda}^{\mathbf{2}}}{\lambda^{\mathbf{2}}}$$

$$\mathbf{g}_{\mathbf{gauge}}(\omega) = \frac{\widetilde{\lambda}}{\mathbf{2}\pi}$$

Total

$$\mathbf{g}_{\mathbf{div}}(\omega) = \frac{\mathbf{1}}{\mathbf{2}}\,\omega^{\mathbf{2}}\,\mathbf{ln}\,\frac{\lambda^{\mathbf{2}}}{\mathbf{4}} \equiv \mathbf{A}_{\mathbf{g}} + \mathbf{B}_{\mathbf{g}}\omega^{\mathbf{2}}$$

$$\Rightarrow \qquad \mathbf{A}_{\mathbf{g}} = \mathbf{0}, \qquad \mathbf{B}_{\mathbf{g}} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{ln}\,\frac{\lambda^{\mathbf{2}}}{\mathbf{4}}$$

Recall $\mathbf{B}_{\mathbf{f}} = \mathbf{ln}\lambda^{\mathbf{2}}$

$\Rightarrow \mathbf{B}_{\mathbf{f}} - \mathbf{2B}_{\mathbf{g}} = \mathbf{ln}\,\mathbf{4}$ **is independent of** λ

Unitarity

Based on our understanding of D-instanton amplitudes, one can also analyze unitarity of these amplitudes

Result: The only source of unitarity violation is in the imaginary part of the exponential of the annulus partition function

– related to the tachyonic modes on the instanton

Conclusion

World-sheet theory, aided by string field theory, provides a fully systematic procedure for computing D-instanton contribution to an amplitude

Besides being of practical use, this can be used to gain deeper understanding of string theory, e.g,

– testing duality conjectures

– role of resurgence

etc.