Weak universality in spin models and lattice gauge theories

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Fractionalization and Emergent Gauge Fields in Quantum Matter (ICTP)

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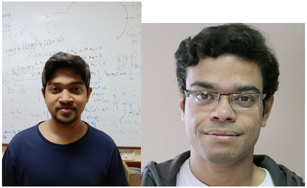
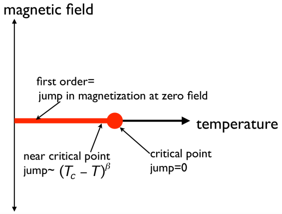
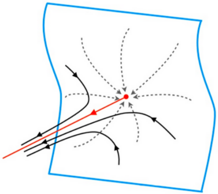


Indrajit Sau (IACS, Kolkata), Debasish Banerjee (SINP, Kolkata)

- Udupa, Sur, Nandy, Sen, Sen; arXiv:2307.11161v3
- Sau, Sen, Banerjee; Phys. Rev. Lett. 130, 071901 (2023)

Universality near critical points



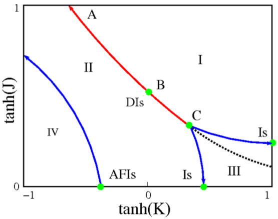


- Long-distance properties of critical points described by fixed points of renormalization group flows
- Different microscopic systems may ultimately flow to the same fixed point (universality)
- Critical exponents act as fingerprints (e.g., $O \sim (T_c - T)^\beta$ as $T \to T_c$ from below, $\chi \sim |T - T_c|^{-\gamma}$ etc)
- Typically fixed by nature of order parameter+dimensionality

Continuosly varying critical exponents

$H = -J\sum_{\langle ij\rangle}(\sigma_i\sigma_j + \tau_i\tau_j) - K\sum_{\langle ij\rangle}\sigma_i\sigma_j\tau_i\tau_j$

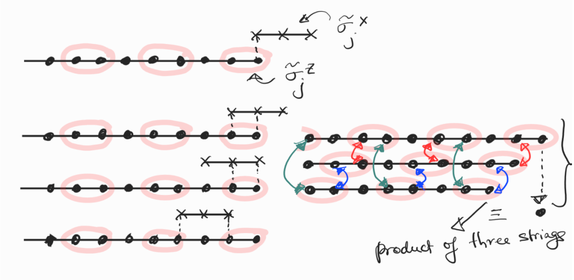
(Ashkin-Teller model on 2D square lattice [Phys. Rev. 64, 178 (1943)]) Kadanoff and Brown, Annals of Physics 121, 318 (1979), Cardy, J. Phys A: Math. Gen. 20, L891 (1987), Delfino and Grinza, Nucl. Phys. B 682, 521 (2004), $\cdots$

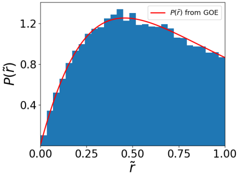


- **Region I** Ferro–$\langle\sigma\tau\rangle \neq 0$ with $\langle\sigma\rangle = \pm\langle\tau\rangle$ (Breaks $\mathbb{Z}_2 \times \mathbb{Z}_2$)
- **Region II** Para–$\langle\sigma\rangle = \langle\tau\rangle = 0$
- Red line is self-dual with $\exp(-2K) = \sinh(2J)$
- BKT $\rightarrow$ decoupled Ising $\rightarrow$ 4-state Potts ($\nu \in (\infty, 2/3)$) $\gamma/\nu = 7/4$, $\beta/\nu = 1/8$, $\eta = 1/4$ “weak universality”, Suzuki, Prog. Theor. Phys. 51, 1992 (1974)

Self-dual critical points

- $H_p = -\sum_j [\sigma^z_j \sigma^z_{j+1} \cdots \sigma^z_{j+p-1} + h\sigma^x_j]$ Turban, J. Phys. C 15, L65 (1982)





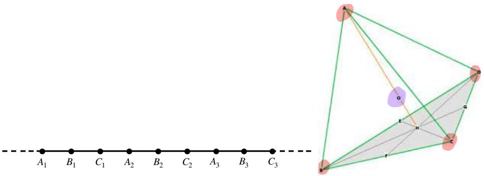
Question: Nature of $h_c = 1$ for the non-integrable model $H_3 = -\sum_j [\sigma^z_j \sigma^z_{j+1} \sigma^z_{j+2} + h\sigma^x_j]$?

Penson, Jullien, Pfeuty, Phys. Rev. B 26, 6334 (1982), Kolb, Penson, J. Phys. A 19, L779 (1986), Alcaraz, Barber, J. Phys. A 20, 179 (1987)

Order parameter

$H_3 = -\sum_{j=1}^{L}[\sigma_j^z \sigma_{j+1}^z \sigma_{j+2}^z + h\sigma^x]$

Consider PBC and L a multiple of 3



For $h \to 0$, the ground states are

$\cdots +++(+++)++++++\cdots \Rightarrow$ (+1,+1,+1)

$\cdots +--(+--)+--+--\cdots \Rightarrow$ (+1,-1,-1)

$\cdots -+-(-+-)-+--+-\cdots \Rightarrow$ (-1,+1,-1)

$\cdots --+(--+)--+--+\cdots \Rightarrow$ (-1,-1,+1)

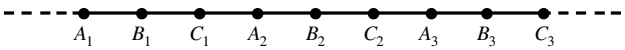
Define $m_{A,B,C} = \frac{3}{L} \sum_{n=1}^{L/3} \sigma^z_{A,B,C}$ and

$m = \sqrt{\langle m_A^2 \rangle + \langle m_B^2 \rangle + \langle m_C^2 \rangle}$

Symmetries

$H_3 = -\sum_{j=1}^{L} [\sigma_j^z \sigma_{j+1}^z \sigma_{j+2}^z + h\sigma^x]$

Consider PBC and L a multiple of 3



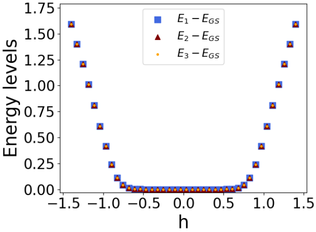
- $D_1 = \prod_{j=1}^{L/3} \sigma_{A_j}^x \sigma_{B_j}^x$, $D_2 = \prod_{j=1}^{L/3} \sigma_{B_j}^x \sigma_{C_j}^x$, $D_3 = \prod_{j=1}^{L/3} \sigma_{C_j}^x \sigma_{A_j}^x$
- $[D_i, H_3] = 0$ and $D_1 D_2 D_3 = I$. H_3 model has $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry
- $(1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1)$ have equal number of states
- $T(D_1 + \omega D_2 + \omega^2 D_3)T^{-1} = \exp(-i2\pi/3)(D_1 + \omega D_2 + \omega^2 D_3)$
- $|\psi_k\rangle$, $|\psi_{k-2\pi/3}\rangle = (D_1 + \omega D_2 + \omega^2 D_3)|\psi_k\rangle$ and $|\psi_{k+2\pi/3}\rangle = (D_1 + \omega^{-1} D_2 + \omega^{-2} D_3)|\psi_k\rangle$ are degenerate. Not true for $(1,1,1)$
- In the entire spectrum, three-fourths of the states have an exact three-fold deg.

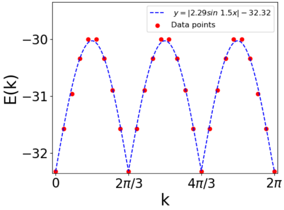
Data for the critical point from ED

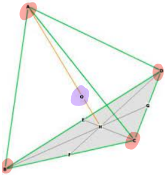
Exponent	Method used	Three-spin	Two-spin
z	Δ scaling with L at h_c	1.0267 (14)	1.0026 (3)
β/ν	m scaling with L at h_c	0.1291 (18)	0.1337 (64)
γ/ν	χ scaling with L at h_c	1.7976 (34)	1.7936 (20)
ν	$\frac{d}{dh}(\Delta L)$ scaling with L at h_c	0.7538 (45)	1.0335 (42)
c	EE Scaling at h_c	1.0644 (72)	0.5096 (13)
	Energy scaling at h_c	0.9585 (15)	0.5034 (68)

- Self-dual point seems consistent with AT criticality
- $\nu \approx 0.75$ consistent with earlier studies. Critical point intermediate between 4-state Potts and decoupled Ising?

Heuristic arguments for four-state Potts criticality

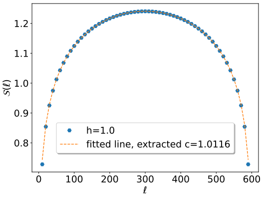


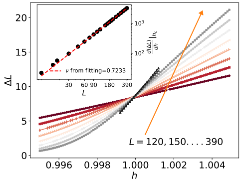


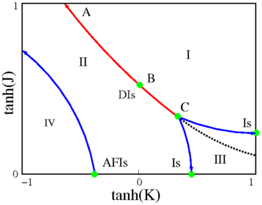


- Four-state Potts CFT have several relevant and marginal operators with following degeneracies and conformal dimensions:
  $3 \times \left(\frac{1}{16}, \frac{1}{16}\right)$, $1 \times \left(\frac{1}{4}, \frac{1}{4}\right)$, $3 \times \left(\frac{9}{16}, \frac{9}{16}\right)$, and $3 \times (1, 1)$
  Dijkgraaf, Verlinde, Verlinde, Commun. Math. Phys. 115, 649 (1988)
- For a critical spin chain (with PBC and $L \gg 1$), $E_\alpha = A + B\frac{2\pi}{L}\left(\Delta_\alpha - \frac{c}{12}\right) + \cdots$
  Affleck, Gepner, Schulz, Ziman, J. Phys. A 22, 511 (1989), Zou, Milsted, Vidal, Phys. Rev. Lett. 121, 230402 (2018)
- Four-state Potts CFT has S_4 symmetry

DMRG data on long open chains (I)

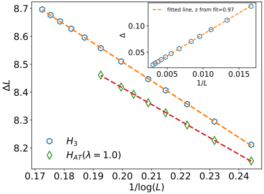


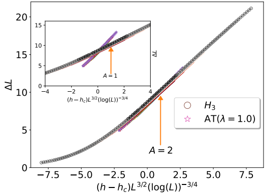


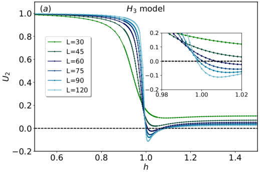


- $S(l) = \frac{c}{6} \log\left(\frac{L}{\pi} \sin\left(\frac{\pi l}{L}\right)\right) + c'$ [Calabrese, Cardy, J. Stat. Mech. P06002 (2004)] gives $c \approx 1$ for a long chain with $L = 600$ at $h_c = 1$ for the H_3 model
- Assuming $\frac{d(\Delta L)}{dh}|_{h_c} \sim L^{1/\nu}$ gives $\nu \approx 0.72$ which is marginally lower compared to $\nu \approx 0.75$ from ED. Note that this result assumes "standard" scaling $\Delta L = \mathcal{F}((h - h_c)L^{1/\nu})$
- Four-state Potts CFT, however, is multicritical. Conventional scaling may not follow due to interference of multiple critical lines

DMRG data on long open chains (II)



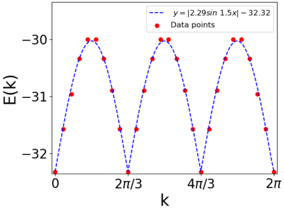


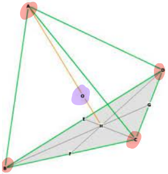


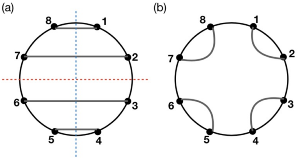
- Important additive and multiplicative log corrections known for 2D classical four-state Potts criticality [Cardy, Nauenberg, Scalapino, Phys. Rev. B 22, 2560 (1980), Salas and Sokal, J. Stat. Phys. 88, 567 (1997)]
- $\frac{\Delta|_{h_c} L}{|h_c|} = a^* + \frac{b}{\log(L)} + \cdots$. $a^* \approx 9.86(9.61)$ for H_3 model (AT model at $\lambda = 1$)
- $\Delta L = \mathcal{F}(A(h-h_c)L^{3/2}(\log L)^{-3/4})$ which implies $\frac{\Delta L}{dh}|_{h_c} \sim L^{3/2}(\log L)^{-3/4}$. This mimics $L^{1/\nu}$ with $\nu \approx 0.72$ for the accessible sizes from DMRG ($L \leq 390$)
- Pseudo-first-order behavior in $U_2 = \frac{5}{2} - \frac{3}{2}\frac{\langle m^4 \rangle}{\langle m^2 \rangle^2}$ [Jin, Sen, Sandvik, PRL (2012)]

- $H_{AT} = -h\sum_{j=1}(\sigma_j^x + \tau_j^x + \lambda\sigma_j^x\tau_j^x) - \sum_{j=1}(\sigma_j^z\sigma_{j+1}^z + \tau_j^z\tau_{j+1}^z + \lambda\sigma_j^z\sigma_{j+1}^z\tau_j^z\tau_{j+1}^z)$ (Quantum AT model) $\lambda = 0(1)$ decoupled Ising (four-state Potts)

Conclusions (I)



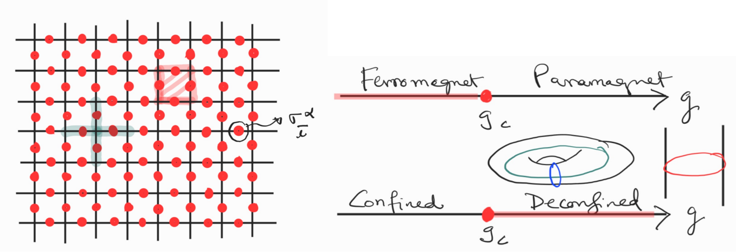




- Self-dual point of the three-spin model seems to be in four-state Potts universality class
- Addition of other terms to this model to see how the nature of criticality changes?
- Quantum many-body scars+anomalous infinite temperature autocorrelations [not discussed here]

Udupa, Sur, Nandy, Sen, Sen; arXiv:2307.11161v3

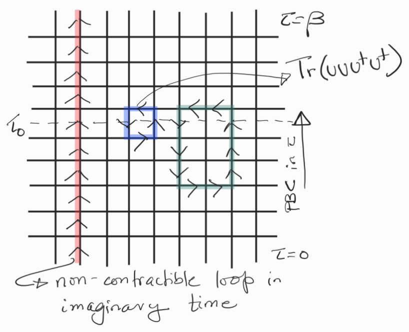
Confinement-deconfinement transitions



- $H_{Z_2} = -\left[\sum_l \sigma_l^x + g \sum_\Box (\prod_{l\in\Box} \sigma_l^z)\right]$; $\prod_{l\in+} \sigma_l^x = 1$ for all $+$
- Dual of 2D TFIM on square lattice

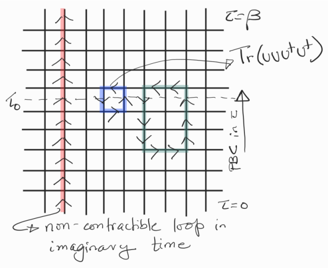
  Kogut, Rev. Mod. Phys. 51, 659 (1979)
- Gauge-invariant states must satisfy appropriate Gauss law

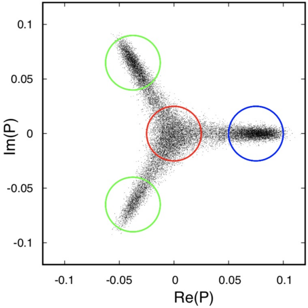
Polyakov loop and center symmetry (I)



- Consider LGTs without dynamical matter
- Wilson loop $W_c = \mathrm{Tr} \prod_{\{n,\mu\} \in C} U_\mu(n)$ (gauge-invariant)
- Gauge symmetry (local)
  $U_\mu(n) \to U'_\mu(n) = \Omega(n) U_\mu(n) \Omega^\dagger(n+\mu)$
- Polyakov loop $P(\vec{n}) = \mathrm{Tr} \prod_{l=0}^{N_\tau - 1} U_4(\vec{n}, l)$

Polyakov loop and center symmetry (II)





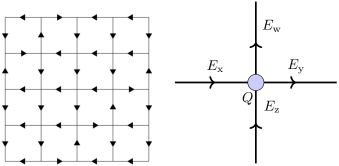
- Center symmetry (global) $U_4(\vec{n},\tau_0) \to zU_4(\vec{n},\tau_0)$ for all $\vec{n}$
- E.g., $z = \exp(i2\pi k/3)I_{3\times 3}$ with $k = 0, 1, 2$ for $SU(3)$ LGT
- Action invariant but $P \to zP$ under this symmetry
- Center symm spontaneously broken in deconfined phase [Celik, Engels, Satz, Phys. Lett. 125 B, 411 (1983), Kovacs, PoS LATTICE2021:238, $\cdots$]

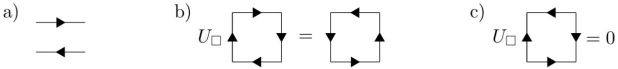
Svetitsky-Yaffe Conjecture

Argument of Svetitsky and Yaffe (Nucl. Phys. B 210, 423(1982)) to figure out universality class of thermal deconfinement transitions in pure gauge theories assuming a continuous transition

- Theory with gauge symmetry group G in $d+1$ dimensions $\Leftrightarrow$ spin model in d dimensions with center of G as global symmetry group
- SY showed that effective model of Polyakov loops has only short-range interactions
- Rest of the arguments appeal to universality
- E.g., $SU(2)$ in $3+1$ dim $\Leftrightarrow$ Ising in 3 dim
- For abelian gauge theories, center of G is G itself
- E.g, $U(1)$ in $2+1$ dim $\Leftrightarrow$ $U(1)$ in 2 dim (BKT transition)

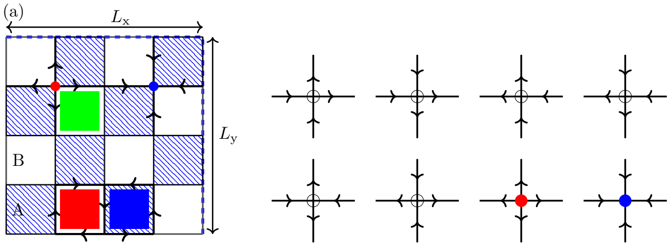
Quantum Link $U(1)$ Gauge Theory ($s = 1/2$)



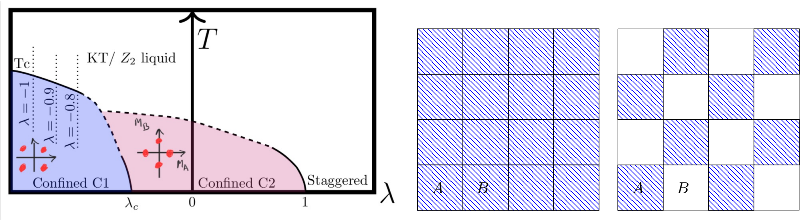


- Two in–two out rule (spin ice constraint)
- $H = -J \sum_\square (U_\square + U_\square^\dagger) + \lambda J (U_\square + U_\square^\dagger)^2$ (RK type Model) [Chandrasekharan, Wiese, Nucl. Phys. B 492, 455 (1997), Hermele, Fisher, Balents, Phys. Rev. B 69, 064404 (2004)]
- $G_r = \sum_\mu (E_{r,\mu} - E_{r,r-\mu})$, $[G_r, H] = 0$ for all r
- $V = \prod_r \exp(-i\theta_r G_r)$ and $\hat{H} = VHV^\dagger = H$ where $\theta_r \in [0, 2\pi)$ [$U(1)$ local invariance]

Reducing $U(1)$ LGT to $\mathbb{Z}_2$ LGT



- $Z = \mathrm{Tr}\ [e^{-\beta H}\mathbb{P}_{\mathbb{G}}]$; $\mathbb{P}_G = \prod_r \frac{1}{8}\{6\delta(G_r) + \delta(G_r - 2) + \delta(G_r + 2)\}$ alongwith $\sum_r q_r = 0$ versus $\mathbb{P}_G = \prod_r \delta(G_r)$ for the original theory
- For the $\mathbb{Z}_2$ theory, $V = \prod_r \exp(-i\theta_r G_r)$ and $\hat{H} = VHV^\dagger = H$ where $\theta_r = 0, \pi$ [$\mathbb{Z}_2$ local invariance]
- Only six states satisfy the $q_r = 0$, and two for $q_r = \pm 2$.
- T controls the density of the $q_r = \pm 2$. Energy gap $O(|\lambda|)$
- Annealed disorder: impurities in thermal equilibirum.



- Susceptibility

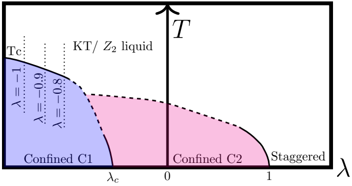
$$\chi_{\text{tot}} = \frac{1}{V}\langle M^2 \rangle\,,\ \chi_{\text{conn}} = \frac{\beta}{V}\sum_X (\langle M_X^2\rangle - \langle |M_X|\rangle^2),$$

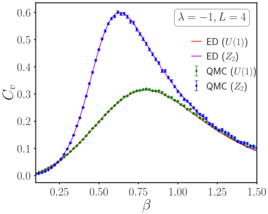
where $M^2 = \sum_{X=A,B}(M_X^2)$, $V = L^2$, and $\beta = \epsilon L_T$.

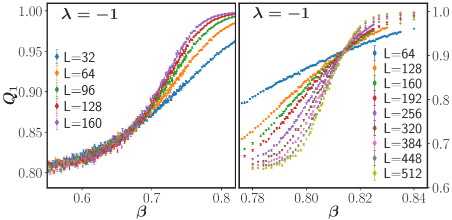
- Three different Binder cumulants:

$$Q_1 = \frac{1}{2}\sum_X \frac{\langle |M_X|\rangle^2}{\langle M_X^2\rangle};\ \ Q_{2a} = 2 - \frac{\langle M^4\rangle}{\langle M^2\rangle^2};\ Q_{2b} = \frac{3}{2} - \frac{1}{4}\sum_X \frac{\langle M_X^4\rangle}{\langle M_X^2\rangle^2}.$$

- Cluster algorithm generalizing Banerjee, Jiang, Widmer, Wiese. J. Stat. Mech. (2013) P12010.

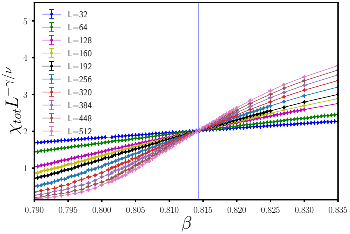


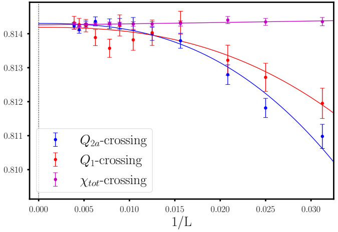




- Following the SY conjecture: the $U(1)$ LGT shows a BKT transition.
- The presence of $q_r = \pm 2$ changes the critical behaviour

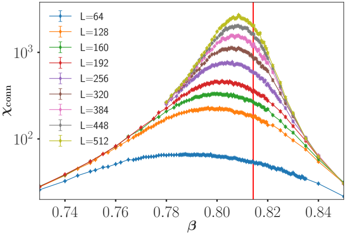
Estimating the critical coupling

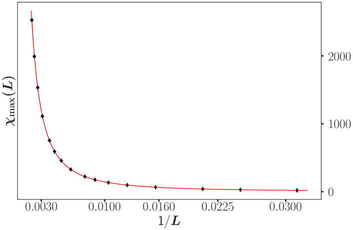




- Crossing points of $\chi_{tot} \cdot L^{-\frac{\gamma}{\nu}}$, Q_1, Q_{2a}, Q_{2b} to estimate $T_c = 1/\beta_c$.
- Fix $\frac{\gamma}{\nu} = \frac{7}{4}$, value for 2d Ising model.
- All observables give consistent estimates of β_c for $L > 100a$.

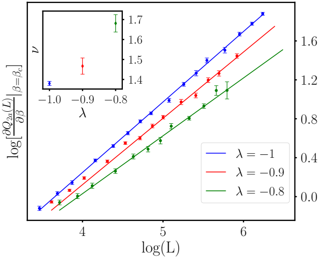
Estimating the critical exponent η





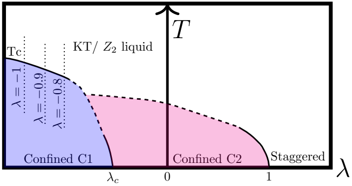
- Scaling of the peak of χ_{conn} to compute η:
  $\chi_{\text{conn,max}}(L) = b \cdot L^{\gamma/\nu} = bL^{2-\eta}$; $\chi^2/DOF \sim 1.3$.
- Extracted from three different bare couplings, λ.
- Independent validation of the assumption $\frac{\gamma}{\nu} = \frac{7}{4}$.

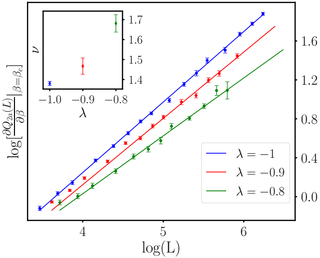
Weak Universality: floating ν



- For a dimensionless phenomenological coupling $R(\beta, L)$:
  $\left.\frac{\partial R(L)}{\partial \beta}\right|_{\beta_c} = aL^{1/\nu}(1 + bL^{-\omega})$
- Slope of log-log plot of the derivative vs lattice size gives $1/\nu$.
- Consistent values of ν obtained from Q_1, Q_{2a}, Q_{2b}, all > 1.

Conclusions (II)

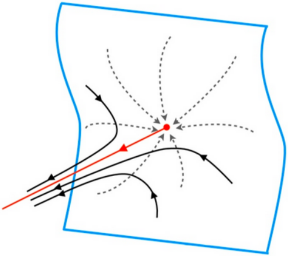




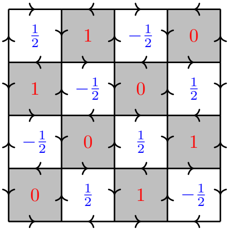
- $U(1)$ quantum link model with annealed $Q = \pm 2$ impurity charges in (2+1)-d seems to display weak universality
- Interesting extension of the Svetitsky-Yaffe conjecture for thermal deconfinement transitions
- Identification of marginal operator in effective field theory?

Sau, Sen, Banerjee; Phys. Rev. Lett. 130, 071901 (2023)+ongoing work

(Supp) Perturbative argument



- $S_{\text{critical}} + \int d^d r [g_0 O(\vec{r})]$ where $\langle O(\vec{r}_1) O(\vec{r}_2) \rangle \sim 1/|r_1 - r_2|^{2y_0}$
- Perturbation relevant (irrelevant) if $y_0 < d$ ($y_0 > d$)
- Perturbation marginal if $y_0 = d$
- For 2D Ising model, $O(\vec{r}) \sim \sigma(\vec{r})$ has $y_0 = 1/8$ and $O(\vec{r}) \sim \sigma(\vec{r})\sigma(\vec{r} + \hat{a}) = \epsilon(\vec{r})$ has $y_0 = 1$
- $S \sim S_1 + S_2 + \tau \int d^2 r (\epsilon_1(\vec{r}) + \epsilon_2(\vec{r})) + \rho \int d^2 r \epsilon_1(\vec{r}) \epsilon_2(\vec{r})$



- Cluster Algorithm for simulating the dualized version of the model.

  Banerjee, Jiang, Widmer, Wiese. J. Stat. Mech. (2013) P12010.

- Pure Gauge Theory in $(2 + 1)$-d maps to a height model in 3d.
- The computation is done on a Euclidean system with $L \times L \times \beta$, where the β is varied, and $L \to \infty$ for thermodynamic limit.
- Two-component order parameter (M_A, M_B) capture the ordering of the two sublattices as well as track center symmetry.

(Supp) Precision of estimations

L_T	β_c	η	$\nu(Q_1)$	$\nu(Q_{2a})$	$\nu(Q_{2b})$
		$\lambda = -1.0$			
24	0.814279(14)	0.2472(9)	1.35(2)	1.38(1)	1.38(2)
16	0.813783(15)	0.2479(9)	1.32(4)	1.34(2)	1.34(4)
8	0.811129(14)	0.2489(8)	1.33(3)	1.31(2)	1.34(3)
4	0.801059(12)	0.2509(8)	1.29(1)	1.31(1)	1.29(2)
2	0.767685(10)	0.2497(7)	1.19(1)	1.20(1)	1.20(1)
		$\lambda = -0.9$			
24	0.885292(17)	0.2550 (18)	1.45(3)	1.47(4)	1.45(3)
		$\lambda = -0.8$			
24	0.968196(26)	0.2511 (10)	1.64(9)	1.68(4)	1.64(8)

Estimates of β_c, η, ν for different values of L_T, λ. For 2d Ising, $\eta = \frac{1}{4}$, and $\nu = 1$.