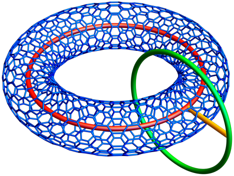
String nets at finite temperature

Julien Vidal
Laboratoire de Physique Théorique de la Matière Condensée
CNRS, Sorbonne University, Paris



in collaboration with:

- Jean-Noël Fuchs and Anna Ritz-Zwilling (LPTMC, Paris)
- Steven H. Simon (Rudolf Peierls Centre for Theoretical Physics, Oxford)

J. Vidal, Phys. Rev. B **105**, L041110 (2022) / arXiv:2108.13425

A. Ritz-Zwilling, J.-N. Fuchs, S. H. Simon, and J. Vidal, arXiv:2309.00343

Topological quantum order in condensed matter in four dates

- 1989 : High-T_c superconductors, FQHE (X.-G. Wen, F. Wilczek, A. Zee)
- 1991 : Quantum antiferromagnets (Read, Sachdev)
- 1997 : Fault-tolerant quantum computation (A. Kitaev, J. Preskill)
- 2005 : String-net condensation (M. Levin, X.-G. Wen)

Topological quantum order in condensed matter in four dates

- 1989 : High-T_c superconductors, FQHE (X.-G. Wen, F. Wilczek, A. Zee)
- 1991 : Quantum antiferromagnets (Read, Sachdev)
- 1997 : Fault-tolerant quantum computation (A. Kitaev, J. Preskill)
- 2005 : String-net condensation (M. Levin, X.-G. Wen)

Concepts coming from 70's and 80's

- Lattice gauge theories (Wegner, Wilson, Kogut)
- Resonating valence bond states (Anderson, Baskaran)
- Conformal field theories (Pasquier, Verlinde, Moore, Seiberg)
- Topological quantum field theories (Witten)
- Knot theory (Kauffman, Jones)

Topological quantum order in condensed matter in four dates

- 1989 : High-T_c superconductors, FQHE (X.-G. Wen, F. Wilczek, A. Zee)
- 1991 : Quantum antiferromagnets (Read, Sachdev)
- 1997 : Fault-tolerant quantum computation (A. Kitaev, J. Preskill)
- 2005 : String-net condensation (M. Levin, X.-G. Wen)

Concepts coming from 70's and 80's

- Lattice gauge theories (Wegner, Wilson, Kogut)
- Resonating valence bond states (Anderson, Baskaran)
- Conformal field theories (Pasquier, Verlinde, Moore, Seiberg)
- Topological quantum field theories (Witten)
- Knot theory (Kauffman, Jones)

In this talk:

Two-dimensional gapped quantum systems with:

- Topology-dependent ground-state degeneracy
- Anyonic excitations

Outline

Outline

The Levin-Wen model in a nutshell

The Levin-Wen (string-net) model

- Lattice model defined on a trivalent graph (e.g., honeycomb lattice) with: N_v vertices and N_p plaquettes

M. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005)
C.-H. Lin, M. Levin, and F. J. Burnell, Phys. Rev. B **103**, 195155 (2021)

The Levin-Wen model in a nutshell

The Levin-Wen (string-net) model

- Lattice model defined on a trivalent graph (e.g., honeycomb lattice) with: $N_{\rm v}$ vertices and $N_{\rm p}$ plaquettes

M. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005)
C.-H. Lin, M. Levin, and F. J. Burnell, Phys. Rev. B **103**, 195155 (2021)

Input (microscopic degrees of freedom): Unitary Fusion Category $\mathcal{C}$

- Set of objects (strings, labels, charges, superselection sectors, particles...)
- Fusion rules: $a \times b = \sum_c N_{ab}^c c \rightarrow$ Quantum dimensions d_a's

The Levin-Wen model in a nutshell

The Levin-Wen (string-net) model

- Lattice model defined on a trivalent graph (e.g., honeycomb lattice) with: N_{v} vertices and N_{p} plaquettes

M. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005)
C.-H. Lin, M. Levin, and F. J. Burnell, Phys. Rev. B **103**, 195155 (2021)

Input (microscopic degrees of freedom): Unitary Fusion Category $\mathcal{C}$

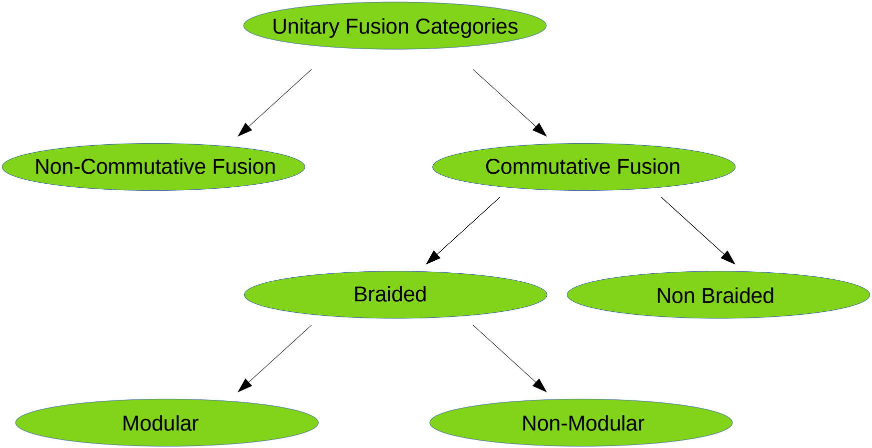
- Set of objects (strings, labels, charges, superselection sectors, particles...)
- Fusion rules: $a \times b = \sum_c N_{ab}^c c \to$ Quantum dimensions d_a's

Output (emergent excitations): $\mathcal{Z}(\mathcal{C})$ (Drinfeld center of $\mathcal{C}$)

- Anyon theory (other set of objects, other fusion rules)
  $\sim$ Toolbox to generate all achiral topological phases (no gapless edge modes)

The Levin-Wen model in a nutshell

A possible tree of UFCs



The Levin-Wen model in a nutshell

Vertex constraints and Hilbert space

- Degrees of freedom are the objects of $\mathcal{C}$
- Degrees of freedom are defined on the links of the trivalent graph
- Hilbert space $\mathcal{H}$: set of configurations respecting fusion rules at each vertex

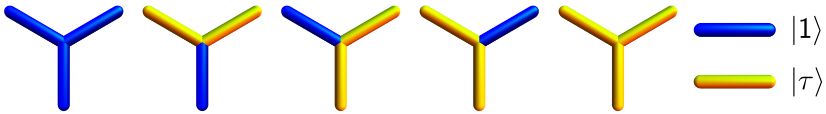
The Levin-Wen model in a nutshell

Vertex constraints and Hilbert space

- Degrees of freedom are the objects of $\mathcal{C}$
- Degrees of freedom are defined on the links of the trivalent graph
- Hilbert space $\mathcal{H}$: set of configurations respecting fusion rules at each vertex

Ex: Fibonacci category

- Two objects: $\{1, \tau\}$
- Fusion rules: $1 \times 1 = 1$, $1 \times \tau = \tau$, $\tau \times \tau = 1 + \tau$



Hilbert space dimension for any trivalent graph with N_v vertices

- dim $\mathcal{H} = (1+\varphi^2)^{\frac{N_\mathrm{v}}{2}} + (1+\varphi^{-2})^{\frac{N_\mathrm{v}}{2}}$, $\quad \varphi = \frac{1+\sqrt{5}}{2}$ (golden ratio)

The Levin-Wen model in a nutshell

The Hamiltonian

$$H = -\sum_p P_p$$

- H is a sum of local commuting projectors: $[P_p, P_{p'}] = 0$
- P_p: projector acting on the links of the plaquette p
- Energy spectrum: $E_q = -N_{\mathrm{p}} + q$

$\sim$ Think about the toric code **without charge excitations!**

M. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005)

C.-H. Lin, M. Levin, and F. J. Burnell, Phys. Rev. B **103**, 195155 (2021)

Outline

Spectrum degeneracies

On a genus-g compact orientable surface (sphere, torus,...)

- Ground-state degeneracy: $\mathcal{D}_0 = \sum_{i \in \mathcal{Z}(\mathcal{C})} \left(\frac{d_i}{D}\right)^{\chi}$ =Turaev-Viro invariant
- d_i: quantum dimension of the object $i \in \mathcal{Z}(\mathcal{C})$
- D: total quantum dimension of $\mathcal{Z}(\mathcal{C})$ ($D^2 = \sum_i d_i^2$)
- Euler-Poincaré characteristic: $\chi = 2 - 2g$
- $g = 0$: $\mathcal{D}_0 = 1$
- $g = 1$: $\mathcal{D}_0 =$ Number of objects in $\mathcal{Z}(\mathcal{C})$
- $g \geqslant 2$: $\mathcal{D}_0$ depends (non trivially) on $\mathcal{Z}(\mathcal{C})$

Z. Kádár, A. Marzuoli, and M. Rasetti, Adv. Math. Phys. **2010**, 671039 (2010)

F. J. Burnell and S. H. Simon, Ann. Phys. **325**, 2550 (2010)

V. G. Turaev and O. Y. Viro, Topology **31**, 865 (1992)

G. Moore and N. Seiberg, Comm. Math. Phys. **123**, 177 (1989)

E. Verlinde, Nucl. Phys. B **300**, 360 (1988)

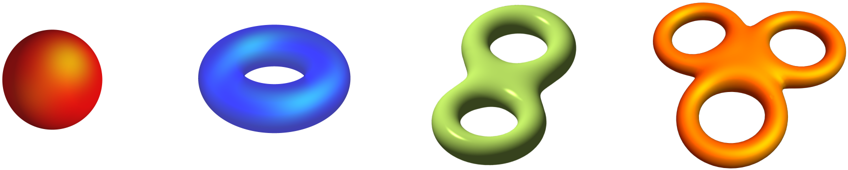
Spectrum degeneracies

Ex: Fibonacci category (commutative, braided, modular)

- Objects of $\mathcal{C}$: $\{1, \tau\}$
- $d_1 = 1$, $d_\tau = \frac{1+\sqrt{5}}{2}$
- Objects of $\mathcal{Z}(\mathcal{C})$: $\{(1,1), (1,\tau), (\tau,1), (\tau,\tau)\}$
- $d_{(i,j)} = d_i \, d_j$

Y. Hu, S. D. Stirling, and Y.-S. Wu, Phys. Rev. B **85**, 075107 (2012)
M. D. Schulz, S. Dusuel, K. P. Schmidt, and J. Vidal, Phys. Rev. Lett. **110**, 147203 (2013)

$g = 0$ $\quad g = 1$ $\quad g = 2$ $\quad g = 3$



$\mathcal{D}_0 = 1$ $\quad \mathcal{D}_0 = 4$ $\quad \mathcal{D}_0 = 25$ $\quad \mathcal{D}_0 = 225$

Spectrum degeneracies

Ex: Haagerup subfactor category $\mathcal{H}_3$ (non-commutative)

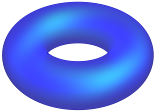
- Objects of $\mathcal{C}$: $\{1, \alpha, \alpha^*, \rho, \rho\alpha, \rho\alpha^*\}$
- $d_1 = d_\alpha = d_{\alpha^*} = 1, \; d_\rho = d_{\rho\alpha} = d_{\rho\alpha^*} = \frac{3+\sqrt{13}}{2}$
- Objects of $\mathcal{Z}(\mathcal{C})$: $\{0, \mu^1, \mu^2, \mu^3, \mu^4, \mu^5, \mu^6, \pi^1, \pi^2, \sigma^1, \sigma^2\,\sigma^3\}$
- $d_0 = 1, \; d_\mu = 3d_\rho, \; d_\pi = 3d_\rho + 1, \; d_\sigma = 3d_\rho + 2$

M. Asaeda and U. Haagerup, Comm. Math. Phys. **202**, 1 (1999)
S.-M. Hong, E. Rowell, and Z. Wang, Commun. Contemp. Math. **10**, 1049 (2008)

$g = 0$

$\mathcal{D}_0 = 1$

$g = 1$

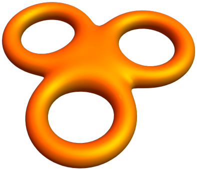


$\mathcal{D}_0 = 12$

$g = 2$

$\mathcal{D}_0 = 1401$

$g = 3$



$\mathcal{D}_0 = 1\,603\,329$

Spectrum degeneracies

Excitations of the Levin-Wen model

- Excitations are the nontrivial objects of $\mathcal{Z}(\mathcal{C})$
- General construction of $\mathcal{Z}(\mathcal{C})$ via the Ocneanu tube algebra
- Some objects only violate the plaquette constraints (fluxons)
- Other objects violate the vertex constraints (not considered here !)

T. Lan and X.-G. Wen, Phys. Rev. B **90**, 115119 (2014)

Spectrum degeneracies

Excitations of the Levin-Wen model

- Excitations are the nontrivial objects of $\mathcal{Z}(\mathcal{C})$
- General construction of $\mathcal{Z}(\mathcal{C})$ via the Ocneanu tube algebra
- Some objects only violate the plaquette constraints (fluxons)
- Other objects violate the vertex constraints (not considered here !)

T. Lan and X.-G. Wen, Phys. Rev. B **90**, 115119 (2014)

Main results (valid for an arbitrary UFC $\mathcal{C}$)

- Number of states with q fluxons on a genus-g surface with N_p plaquettes:

$$\boxed{\mathcal{D}_q = \binom{N_\mathrm{p}}{q} \sum_{i \in \mathcal{Z}(\mathcal{C})} \left(\frac{d_i}{D}\right)^{\chi - q} \left(n_i - \frac{d_i}{D}\right)^q}$$

- $n_i \in \mathbb{N}$: internal multiplicity of the particle i given by the tube algebra $0 \leqslant n_i \leqslant d_i$ ($n_i = 0$ if i is not a fluxon)

J. Vidal, Phys. Rev. B **105**, L041110 (2022)

A. Ritz-Zwilling, J.-N. Fuchs, S. H. Simon, and J. Vidal, arXiv:2309.00343

Outline

Partition function

Energy spectrum

- Levin-Wen Hamiltonian: $H = -\sum_p P_p$
- Ground-state energy: $E_0 = -N_{\mathrm{p}}$
- $q-$fluxon state energy: $E_q = -N_{\mathrm{p}} + q$

Partition function

Energy spectrum

- Levin-Wen Hamiltonian: $H = -\sum_p P_p$
- Ground-state energy: $E_0 = -N_\mathrm{p}$
- $q-$fluxon state energy: $E_q = -N_\mathrm{p} + q$

Exact finite-size and finite-temperature partition function

- Partition function: $Z = \mathrm{Tr}\left(\mathrm{e}^{-\beta H}\right) = \sum_{q=0}^{N_\mathrm{p}} \mathcal{D}_q \, \mathrm{e}^{-\beta E_q}$ $\qquad (\beta = 1/T)$

$$\boxed{Z = \sum_{i \in \mathcal{Z}(\mathcal{C})} \left(\frac{d_i}{D}\right)^{\chi} \left(D \frac{n_i}{d_i} + \mathrm{e}^{\beta} - 1\right)^{N_\mathrm{p}}}$$

- Z depends on the fusion rules: $d_a \times d_b = \sum_c N_c^{ab} d_c$
- Z depends on the surface topology: $\chi = 2 - 2g$
- Z depends on the number of plaquettes: N_p

A. Ritz-Zwilling, J.-N. Fuchs, S. H. Simon, and J. Vidal, in preparation

Partition function

Energy spectrum

- Levin-Wen Hamiltonian: $H = -\sum_p P_p$
- Ground-state energy: $E_0 = -N_\mathrm{p}$
- $q-$fluxon state energy: $E_q = -N_\mathrm{p} + q$

Exact finite-size and finite-temperature partition function

- Partition function: $Z = \mathrm{Tr}\left(\mathrm{e}^{-\beta H}\right) = \sum_{q=0}^{N_\mathrm{p}} \mathcal{D}_q \, \mathrm{e}^{-\beta E_q}$ $\quad (\beta = 1/T)$

$$\boxed{Z = \sum_{i \in \mathcal{Z}(\mathcal{C})} \left(\frac{d_i}{D}\right)^{\chi} \left(D \frac{n_i}{d_i} + \mathrm{e}^{\beta} - 1\right)^{N_\mathrm{p}}}$$

Driven by "**pure fluxons**" ($n_i = d_i$) in the thermodynamical limit ($N_\mathrm{p} \to \infty$)

A. Ritz-Zwilling, J.-N. Fuchs, S. H. Simon, and J. Vidal, in preparation

Partition function

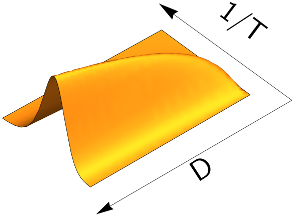
Specific heat and absence of thermal phase transition

- Specific heat per plaquette : $c = \frac{\beta^2}{N_\mathrm{p}} \frac{\partial^2 \ln Z}{\partial \beta^2}$
- Thermodynamical limit ($N_\mathrm{p} \to \infty$): $c = \frac{\mathrm{e}^\beta \, \beta^2 (D-1)}{(D-1+\mathrm{e}^\beta)^2}$
- Only depends on the total quantum dimension D

No finite-temperature phase transition in the Levin-Wen model !

J. Vidal, Phys. Rev. B **105**, L041110 (2022)

A. Ritz-Zwilling, J.-N. Fuchs, S. H. Simon, and J. Vidal, in preparation



Partition function

Specific heat and absence of thermal phase transition

- Specific heat per plaquette : $c = \dfrac{\beta^2}{N_\mathrm{p}} \dfrac{\partial^2 \ln Z}{\partial \beta^2}$
- Thermodynamical limit ($N_\mathrm{p} \to \infty$): $c = \dfrac{\mathrm{e}^\beta\, \beta^2 (D-1)}{(D-1+\mathrm{e}^\beta)^2}$
- Only depends on the total quantum dimension D

No finite-temperature phase transition in the Levin-Wen model !

J. Vidal, Phys. Rev. B **105**, L041110 (2022)

A. Ritz-Zwilling, J.-N. Fuchs, S. H. Simon, and J. Vidal, in preparation

Absence of topological order at $T > 0$ for any <u>two-dimensional</u> Hamiltonian which is a sum of local commuting projectors !

M. Hastings, Phys. Rev. Lett. **107**, 210501 (2011)

Outlook

Summary

- Exact degeneracies of the energy spectrum valid for:
  - Any graph
  - Any compact oriented surface
  - Any topology
  - Any UFC with/without fusion multiplicities ($N_{ab}^{c} \geqslant 1$)
- Exact partition function $\rightarrow$ No finite-temperature phase transition

A. Ritz-Zwilling, J.-N. Fuchs, S. H. Simon, and J. Vidal, arXiv:2309.00343

Outlook

Summary

- Exact degeneracies of the energy spectrum valid for:
  - Any graph
  - Any compact oriented surface
  - Any topology
  - Any UFC with/without fusion multiplicities ($N_{ab}^{c} \geqslant 1$)
- Exact partition function $\rightarrow$ No finite-temperature phase transition

A. Ritz-Zwilling, J.-N. Fuchs, S. H. Simon, and J. Vidal, arXiv:2309.00343

Soon available...

- Wegner-Wilson loops at finite T
- Topological entropy at finite T

A. Ritz-Zwilling, J.-N. Fuchs, S. H. Simon, and J. Vidal, in preparation

To go beyond...

Forthcoming conference in Les Houches (french Alps): 1-12 April 2024

Topological order: Anyons and Fractons

Website: https://topoanyons.sciencesconf.org/