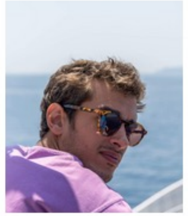
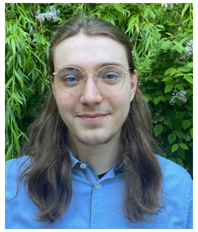
Quantum speed limits to operator growth

Nicoletta Carabba

University of Luxembourg

Supervisor: Prof. Adolfo del Campo

Group at Luxembourg

Dr. Federico Balducci | Dr. Mithun Thudiyangal | Dr. Jing Yang | Nicoletta Carabba | Léonce Dupays | Apollo Matsoukas | Naim Mackel

Adolfo del Campo (PI)

UNIVERSITÉ DU LUXEMBOURG

Collaborator Aurelia Chenu (PI)
Quantum Dynamics and Control Group

General questions

- What is the fundamental timescale of a physical process?

➡ **Quantum speed limit (QSL)**

- How complex is a given quantum evolution?

➡ **Complexity:**

  - for states ➞ Circuit complexity
  - for operators ➞ Operator growth

Outline

- Algebraic QSLs on operator flows
  - Correlation functions
  - Dynamical susceptibilities
  - Quantum metrology: Fisher Information

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Quantum 6, 884 (2022).

- Dispersion bound on Krylov complexity

Communications Physics **5**, Article number: 207 (2022)

- Geometric QSL
  - Saturation
  - Hamiltonian flows (Wegner, Toda)
  - Krylov complexity

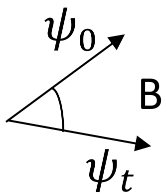
arXiv > quant-ph > arXiv:2301.04372

QSLs on state evolution

- Meaning of energy-time Heisenberg uncertainty principle? $\Delta E \Delta t \geq \frac{\hbar}{2}$

- **MT** first QSL [Mandelstamm, Tamm, 1945] : $\psi_0 \to \psi_t = e^{iH\frac{t}{\hbar}}\psi_0 = \psi_\perp$

Extensions: arbitrary angles, mixed states, driven Hamiltonians, open dynamics



Bures angle

$$t \geq \hbar \frac{\arccos |\langle \psi_t | \psi_0 \rangle|}{\Delta H}$$

$$\Delta H = \sqrt{\langle H^2 \rangle - \langle H \rangle^2}$$

- **ML** second QSL [Margolous, Levitin, 1998] :

$$t_\perp \geq \frac{\hbar}{2} \frac{\pi}{\langle H \rangle - E_0}$$

QSLs on operator flows

- QSLs for states: MT [Mandelstamm, Tamm, 1945] and ML [Margolous, Levitin, 1998]

$$t \geq \hbar \frac{\arccos |\langle \psi_t | \psi_0 \rangle|}{\Delta H} \qquad t_\perp \geq \frac{\hbar}{2} \frac{\pi}{\langle H \rangle - E_0}$$

- Generalization to operators in the Heisenberg picture: $\partial_t O = i[H, O]$

$$t \geq \frac{\sqrt{2(1 - \mathrm{Re}\langle O_0 | O_t \rangle)}}{\Delta \mathbb{L}} \qquad t \geq \frac{1 - \mathrm{Re}\langle O_0 | O_t \rangle}{\alpha \langle |\mathbb{L}| \rangle}$$

$\alpha \approx 0.724$

$$\mathbb{L} = \frac{1}{\hbar}[H, \cdot] \qquad |O\rangle = \sum_{i,j} O_{ij} |i\rangle |j\rangle \qquad \|O\| = 1 \qquad \langle A | B \rangle = \mathrm{Tr}(A^\dagger B)$$

Autocorrelation functions (I)

Bounds on the symmetric (real) and anti-symmetric (imaginary) parts of $C(t) = \mathrm{Tr}(O_t^\dagger O_0 \rho)$

$$\mathrm{Re}\, C(t) \geq 1 - \frac{1}{2} \langle \dot{O}_t^2 \rangle_0 t^2$$

$$\mathrm{Re}\, C(t) \geq 1 - \frac{\alpha}{\hbar} \langle O_0 \{H - E_0, O_0\} \rangle_0 t$$

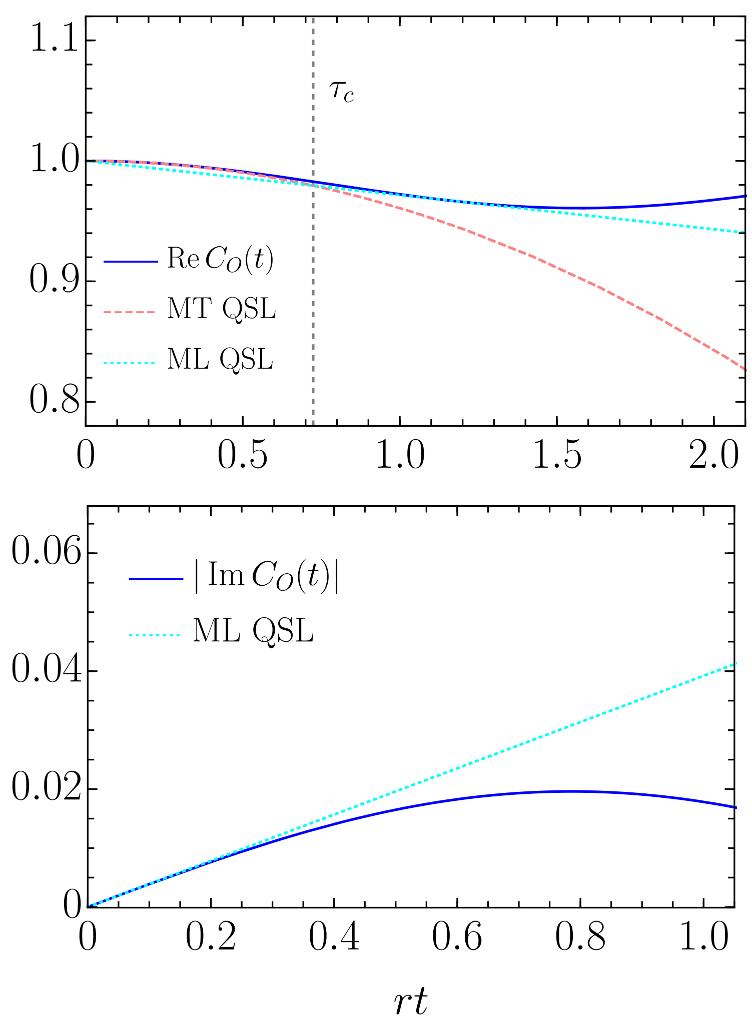
→ MT to ML crossover at time

$$\tau_c = \frac{2\alpha}{\hbar} \frac{\langle O_0 \{H - E_0, O_0\} \rangle_0}{\langle \dot{O}_t^2 \rangle_0}$$

$$|\,\mathrm{Im}\, C(t)| \leq \langle O_0 \{H - E_0, O_0\} \rangle_0 \frac{t}{\hbar}$$

→ Bounds on dynamical susceptibilities through linear response [Kubo, 1957]

Autocorrelation functions (II)



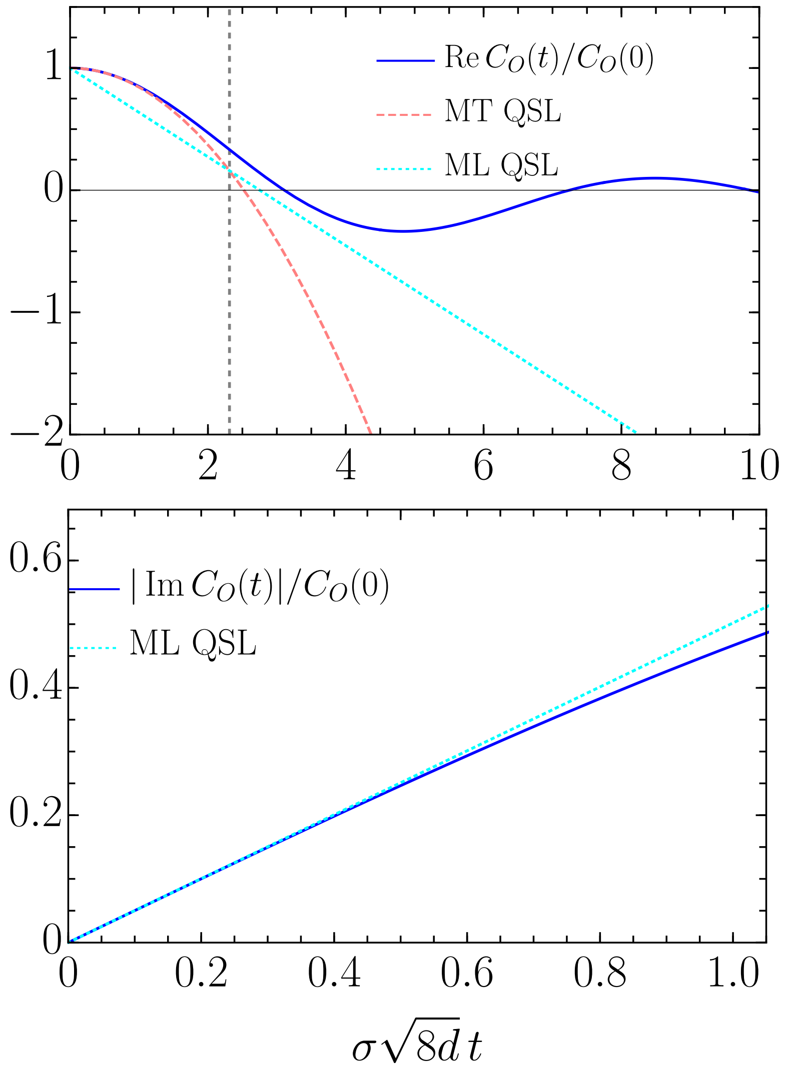
← 2-level system

$$H = k1 + \overrightarrow{r} \cdot \overrightarrow{\sigma}$$

at thermal equilibrium

$H, O_0 \in$ Gaussian Orthogonal Ensamble (GOE) with

$\sigma = 1, \qquad d = 200$ →



Dynamical susceptibilities

- Linear response theory [Kubo, 1957] : $H = H_0 + \lambda V f(t)$

$$\langle A\rangle_t - \langle A\rangle_0 = \lambda \int_{-\infty}^{\infty} \chi_{AV}(t-s) f(s) ds, \quad \chi_{AV}(t) = -i\theta(t)\langle [A_I(t), V_I(0)]\rangle_0$$

- From Heisenberg uncertainty relation:

$$|\chi_{AV}(t)| \leq \frac{2}{\hbar}\theta(t)\Delta_0 A \Delta_0 V$$

- From Bogoliobov inequality:

$$T_B \equiv \frac{\langle A^2\rangle_0 \langle [V,[H_0,V]]\rangle_0}{4k_B(\Delta_0 A \Delta_0 V)^2}$$

$$|\chi_{AV}(t)| \leq \frac{2\theta(t)}{\hbar}\sqrt{\frac{T}{T_B}}\Delta_0 V \Delta_0 A$$

- From QSL: $\tau_{QSL} = \hbar \langle V\{H_0 - E_0, V\}\rangle_0^{-1/3}$, $|\chi_{VV}(t)| \leq \frac{t}{\tau_{QSL}^3} 2\theta(t)\hbar$

tighter at early times

Experimentally testable bounds

- Particles in an electric field: $H(t) = H_0 - \vec{R} \cdot \vec{E}(t)$ $\quad \vec{R} = \sum_n q_n \vec{r}_n$

Electrical conductivity

$$|\sigma_{ij}(t)| \leq \frac{2}{\hbar}\theta(t)\Delta_0 J_i \Delta_0 R_j, \quad |\sigma_{ij}(t)| \leq \theta(t)\Delta_0 J_i \sqrt{\frac{\sum_n q_n^2}{m k_B T}}$$

- Paramagnetic spin system: $H(t) = -\vec{M} \cdot (\vec{B} + \vec{h}(t))$ $\quad M_i = \gamma \sum_n \sigma_i^{(n)}$

Magnetic susceptibility

$$|\chi_{M,ij}(t)| \leq \frac{2}{\hbar}\theta(t)\Delta_0 M_i \Delta_0 M_j, \quad |\chi_{M,ij}(t)| \leq \frac{2\gamma\theta(t)}{\hbar}\sqrt{\frac{\chi_0 \langle M_i^2 \rangle_0 (|\vec{B}|^2 - B_j^2)}{k_B T}}$$

$$|\chi_{M,ii}(t)| \leq \frac{2}{\hbar^2}\theta(t)\langle M_i \{H_0 - E_0, M_i\}\rangle_0 t$$

Quantum Fisher Information (I)

- For a given O, $F_Q = 2\sum_{n,m} \frac{(p_n - p_m)^2}{p_n + p_m} |\langle n|O|m\rangle|^2$ quantifies (mixed) state distinguishability along the flow $\rho \to e^{-i\theta O} \rho \, e^{i\theta O}$

  $\rho = \sum_n p_n |n\rangle\langle n|$

- Quantum metrology: Cramer-Rao bound $(\Delta\theta)^2 \geq \frac{1}{M F_Q}$
- Related to linear response at thermal equilibrium [Hauke et al.,2016]

$$F_Q(T) = -\frac{4}{\pi}\int_0^\infty d\omega \tanh\left(\frac{\hbar\omega}{2k_B T}\right) \operatorname{Im} \tilde{\chi}_{OO}(\omega, T)$$

Quantum Fisher Information (II)

- Performing the integral over ω

$$F_Q(T) = -\frac{16 k_B T}{\hbar} \int_0^\infty dt \, \overbrace{\operatorname{csch}\left(\pi k_B T \frac{t}{\hbar}\right)}^{\sim t^{-1} \text{ as } t \to 0} \operatorname{Im} C_O$$

- Heisenberg and Bogoliubov bounds diverge

- QSL approach removes the divergence:

$$|F_Q(T)| \leq \frac{4}{k_B T} \langle O\{H - E_0, O\}\rangle_0 \qquad (\Delta\theta)^2 \geq \frac{k_B T}{4M} \frac{1}{\langle O\{H - E_0, O\}\rangle_0}$$

Outline

- QSLs on operator flows
  - Correlation functions
  - Dynamical susceptibilities
  - Fisher Information

Quantum speed limits on operator flows and correlation functions

Nicoletta Carabba[1], Niklas Hörnedal[1,2], and Adolfo del Campo[1,3]

[1]Department of Physics and Materials Science, University of Luxembourg, L-1511 Luxembourg, G. D. Luxembourg
[2]Fysikum, Stockholms Universitet, 106 91 Stockholm, Sweden
[3]Donostia International Physics Center, E-20018 San Sebastián, Spain

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- Dispersion bound on Krylov complexity

- Geometric QSL
  - Saturation
  - Hamiltonian flows (Wegner, Toda)
  - Krylov complexity

Operator growth in Krylov space (I)

- Evolution in the Heisenberg picture $O_t = e^{iHt} O_0 e^{-iHt} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \mathbb{L}^n O$

$$\mathcal{K} = \mathrm{Span}\,\{\mathbb{L}^n O_0\}_{n=0}^{\infty} = \mathrm{Span}\,\{O_0, [H, O_0], [H, [H, O_0]], \ldots\}$$

- **Krylov basis** through Lanczos algorithm:

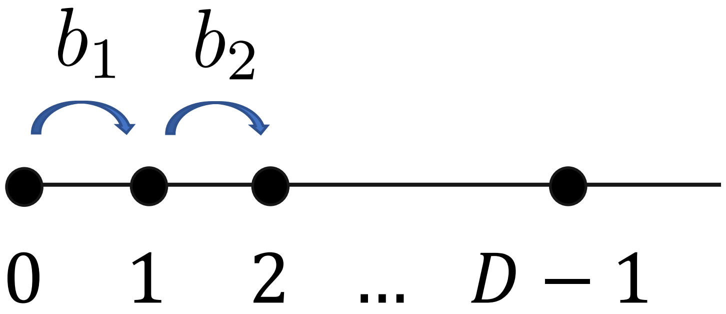
$$|A_{n+1}) = \mathbb{L}|O_n) - b_n|O_{n-1})$$

$$b_n = \|A_n\|,\ O_n = A_n/b_n$$

$$\{\mathbb{L}^n O\}_{n=0}^{\infty} \to \{O_n\}_{n=0}^{D-1}$$

Operator growth in Krylov space (II)

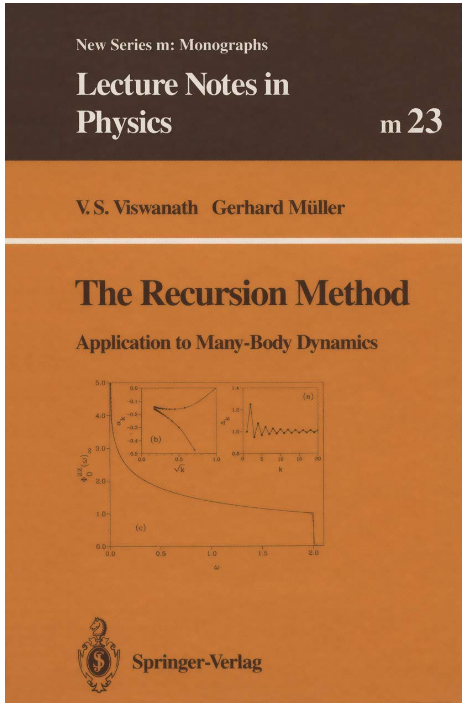
- Operator growth as a hopping problem



$$\mathbb{L} = \begin{pmatrix} 0 & b_1 & 0 & \cdots & & & 0 \\ b_1 & 0 & b_2 & & & & \\ 0 & b_2 & 0 & b_3 & & & \\ 0 & 0 & b_3 & 0 & \ddots & & \\ \vdots & & & \ddots & & b_{D-2} & \vdots \\ & & & & b_{D-2} & 0 & b_{D-1} \\ 0 & & & \cdots & 0 & b_{D-1} & 0 \end{pmatrix} = \mathbb{L}_+ + \mathbb{L}_-$$

- **Krylov complexity** = mean position: $\mathbb{K} = \sum_{n=0}^{D-1} n|O_n)(O_n|, \quad K = (O_t|\mathbb{K}|O_t)$

Operator growth in Krylov space (III)



PHYSICAL REVIEW X **9**, 041017 (2019)

A Universal Operator Growth Hypothesis

Daniel E. Parker,[1,*] Xiangyu Cao,[1,†] Alexander Avdoshkin,[1,‡] Thomas Scaffidi,[1,2,§] and Ehud Altman[1,¶]

Conjecture: maximally chaotic systems have $b_n \propto n$ and exponential growth of complexity

Geometry of Krylov Complexity

Pawel Caputa,[1] Javier M. Magan,[2] and Dimitrios Patramanis[1]

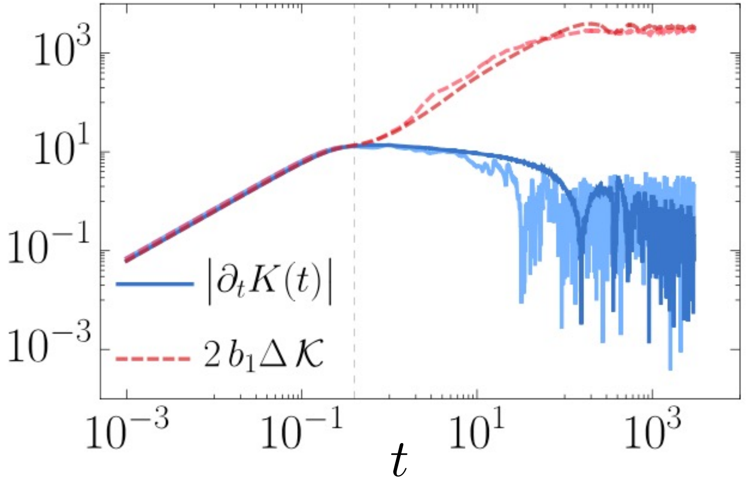
$$\begin{cases} |O_t) = e^{i\mathbb{L}t}|O_0) = D(\xi = it)|O_0) \\ D(\xi) = e^{\xi\mathbb{L}_+ - \xi^*\mathbb{L}_-} \end{cases}$$

Dispersion bound (I)

Heisenberg-Robertson uncertainty relation ⇒ universal upper bound on complexity rate:

$$|\partial_t K(t)| \leq 2b_1 \Delta \mathbb{K}$$

- Explicit models with symmetries **saturate the bound**
- Numerical example from RMT shows deviation

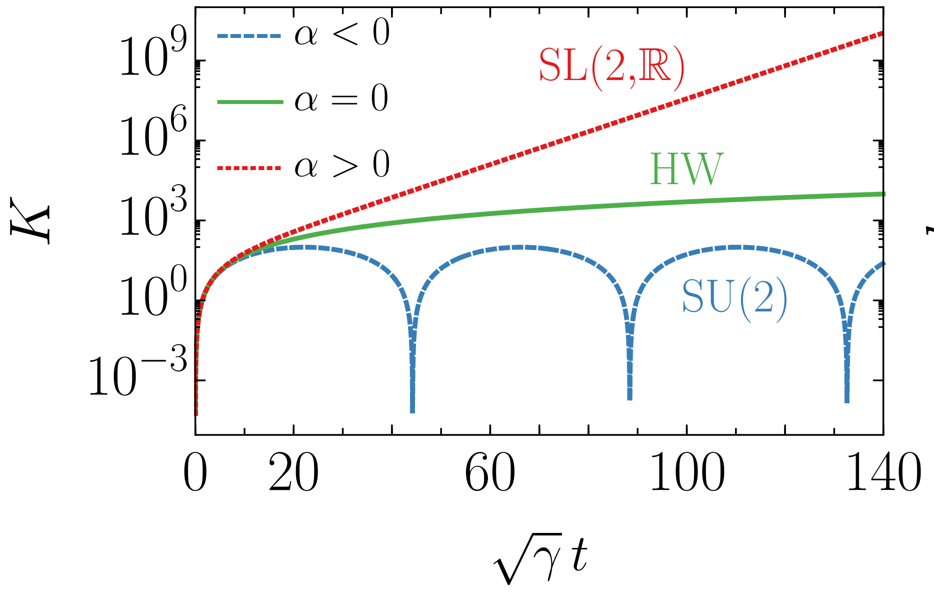


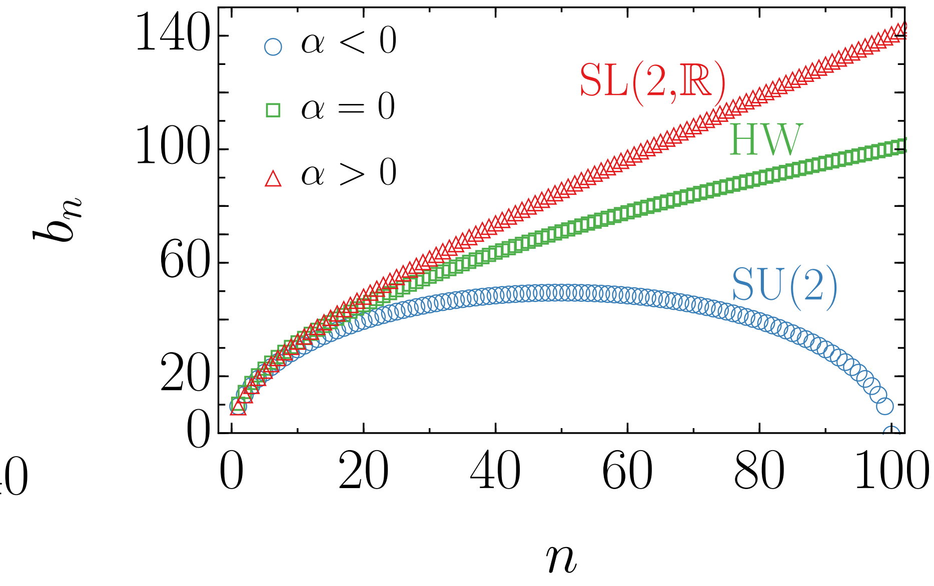
Dispersion bound (II)

- **Saturation** ⇔ closure of the complexity algebra

$$\begin{cases} \mathbb{L} = \mathbb{L}_+ + \mathbb{L}_-, \quad \mathbb{B} = \mathbb{L}_+ - \mathbb{L}_-, \quad \tilde{\mathbb{K}} = [\mathbb{L}, \mathbb{B}] \\ [\mathbb{L}, \mathbb{B}] = \tilde{\mathbb{K}}, \quad [\tilde{\mathbb{K}}, \mathbb{L}] = \alpha \mathbb{B}, \quad [\tilde{\mathbb{K}}, \mathbb{B}] = \alpha \mathbb{L} \end{cases}$$

$$\begin{cases} \tilde{\mathbb{K}} = \alpha \mathbb{K} + \gamma \mathbb{I} \\ b_n = \sqrt{\frac{1}{4}\alpha n(n-1) + \frac{1}{2}\gamma n} \end{cases}$$





- SU(2) realized by a **qubit**: chaos is not necessary for maximal complexity growth
- SL(2,R) realized by maximally chaotic systems (**SYK**)

Outline

- ❑ Algebraic QSLs on operator flows

- ❑ Dispersion bound on Krylov complexity →

Ultimate speed limits to the growth of operator complexity

Niklas Hörnedal, Nicoletta Carabba, Apollonas S. Matsoukas-Roubeas & Adolfo del Campo

Communications Physics **5**, Article number: 207 (2022) | Cite this article

- ❑ Geometric QSL
  - Saturation
  - Hamiltonian flows (Wegner, Toda)
  - Krylov complexity

Geometric OQSL (I)

- Geometric formulation of a new MT-type of Operator QSL (OQSL):
  - ✓ arbitrary driven generator (→ Wegner and Toda flows)
  - ✓ proof of saturation (e.g. Toda flow, Krylov complexity)

arXiv > quant-ph > arXiv:2301.04372

Quantum Physics

[Submitted on 11 Jan 2023]

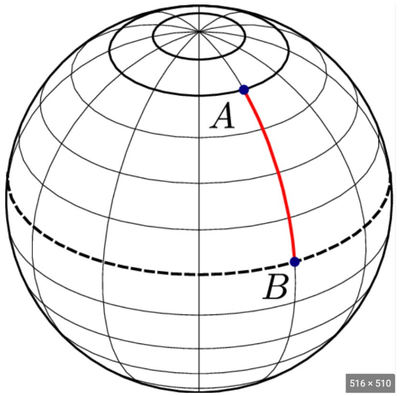
Geometric Operator Quantum Speed Limit, Wegner Hamiltonian Flow and Operator Growth

Niklas Hörnedal, Nicoletta Carabba, Kazutaka Takahashi, Adolfo del Campo

Geometric OQSL (II)

- Unitary flow $\Rightarrow O_t$ lies on a $(2d^2-1)$-sphere of radius $||O|| = 1$
- The minimum length of its trajectory is given by the geodesic

$$\tau \geq \frac{\text{geodesic}(O_0, O_\tau)}{\frac{1}{\tau}\text{length}(\{O_\tau\})} = \frac{\arccos \mathrm{Re}(O_0|O_\tau)}{\frac{1}{\tau}\int_0^\tau \|\dot{O}_t\|dt}$$



- Velocity of the flow: $\mathcal{V}_\tau \equiv \frac{1}{\tau}\int_0^\tau \|\dot{O}_t\|dt = \frac{1}{\tau}\int_0^\tau \Delta\mathbb{L}_t \, dt$

→ MT-type QSL

$d < \infty$

Saturation (I)

$$O_0 = \frac{1}{\sqrt{2}}(|1\rangle\langle 2| + |2\rangle\langle 1|) \qquad H|i\rangle = E_i|i\rangle$$

H is time-independent and O_0 is a traceless 2-level operator

$$\omega = E_2 - E_1, \quad (O_0|O_t) = \cos\omega t, \quad \mathcal{V} = \omega$$

$\Rightarrow$ The OQSL is identically saturated $\forall\, t$

- Dynamics confined to 2 energy eigenspaces $\Rightarrow$ geodesic evolution

Saturation (II)

- In general, P_0 must subtracted = projection of O over $\ker(\mathbb{L})$:

$$P_0 = \sum_i |i\rangle\langle i|O|i\rangle\langle i| \qquad \mathbb{L}P_0 = 0$$

- O has support only in 2 energy eigenspaces ⇒ **3 Liouville eigenspaces**

$$O = P_0 + P_\omega + P_{-\omega}, \quad P_\omega = |2\rangle\langle 2|O|1\rangle\langle 1|, \quad P_{-\omega} = (1 \leftrightarrow 2)$$

⇒ $O_t - P_0$ saturates the OQSL

- **Refined OQSL**: $$\tau \geq \frac{\sqrt{\|O\|^2 - \|P_0\|^2}}{\mathcal{V}_\tau} \arccos\left(\mathrm{Re}\, \frac{(O_0|O_t) - \|P_0\|^2}{\|O\|^2 - \|P_0\|^2} \right)$$

Wegner Hamiltonian flow

- (Block)-diagonalization through the unitary flow [Wegner, 1994]:

$$\frac{dH(l)}{dl} = [\eta(l), H(l)], \quad H_T(l) = \sum_n H_{nn}(l)|n\rangle\langle n|, \quad \lim_{l\to l_f} H(l) = H_T(l_f)$$

- Wegner choice for the generator: $\eta(l) = [H_T, H]$

- Dephasing-like monotonic decay of the off-diagonal components

$$\frac{d\,\mathrm{Tr}\,H^2_{\text{off-diag}}}{dl} = -2\sum_{i,j}(H_{ii} - H_{jj})^2|(H_{\text{off-diag}})_{ij}|^2$$

Toda flow

- Different choice $\eta_{nm}(l) = H_{nm}(l)\mathrm{sgn}\,(m-n)$
- XY model: $H(l) = \frac{1}{2}\sum_{n=1}^{N-1} v_n(l)\left(X_n X_{n+1} + Y_n Y_{n+1}\right) + \frac{1}{2}\sum_{n=1}^{N} h_n(l) Z_n$

⇒ Toda equations [Toda, 1967]:

$$\begin{cases} \dfrac{dh_n(l)}{dl} = 2(v_n^2(l) - v_{n-1}^2(l)), \\ \dfrac{dv_n(l)}{dl} = v_n(l)(h_{n+1}(l) - h_n(l)). \end{cases}$$

Ansatz:

$$v_n(l) = v_n \cos\theta(l)$$

$$\theta(l) = \arccos\left(\frac{\mathrm{Tr}\, H(l)H(0)}{\|H\|^2}\right)$$

saturation →

$$\|H\| \arccos \frac{(H(0)|H(l))}{\|H\|^2} = \int_0^l \|[\eta(s), H(s)]\| ds$$

OQSL on Krylov complexity (I)

$$K = (O_t|\mathbb{K}|O_t) = (O_0|\mathbb{K}_t|O_0)$$

- **Super-Heisenberg picture**: unitary flow generated by $\mathbb{S} = [\mathbb{L}, \cdot]$

$$\dot{\mathbb{K}} = i[\mathbb{L}, \mathbb{K}] \qquad \mathbb{K}_t = e^{-i\mathbb{L}t}\mathbb{K}_0 e^{i\mathbb{L}t} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!}\mathbb{S}^n(\mathbb{K}_0)\, t^n$$

- OQSL on Krylov complexity:

$$t \geq ||\mathbb{K}_0|| \frac{\arccos\left(\frac{(\mathbb{K}_0|\mathbb{K}_t)}{||\mathbb{K}_0||^2}\right)}{||[\mathbb{L}, \mathbb{K}_0]||}$$

- Complexity algebra ⇒ $\mathbb{K}_t \in \mathrm{Span}\{\mathbb{I}, \mathbb{K}_0, \mathbb{B}\}$ ⇒ Evolution contained in a 3-dimensional space

⇒ The **refined OQSL** must be **saturated**

$$\mathbb{P}_0 = ?$$

OQSL on Krylov complexity (II)

- **Velocity** of the flow: maximized by [Parker, 2019] conjecture

$$\mathcal{V}_{\mathbb{K}}^2 = \frac{\|\mathbb{B}\|^2}{\|\mathbb{K}_0\|^2} = \frac{2}{\|\mathbb{K}_0\|^2}\sum_{n=1}^{D-1} b_n^2 \quad \underline{\text{complexity algebras}} \rightarrow \quad \frac{\alpha(D-2)+3\gamma}{2D-1}$$

- Geodesic argument requires finite dimension: $SU(2) \Rightarrow \mathcal{V}_{\mathbb{K}}^2 = \frac{|\alpha|(D+1)}{2(2D-1)}$

- Commutation relations of the algebra → Complexity **autocorrelation:**

$$\begin{aligned}\mathbb{S}^{2n}(\mathbb{K}_0) &= (-1)^n\alpha^{n-1}(\alpha\mathbb{K}_0+\gamma\mathbb{I}) \\ \mathbb{S}^{2n+1}(\mathbb{K}_0) &= (-1)^{n+1}\alpha^n\mathbb{B}\end{aligned} \quad \rightarrow \quad (\mathbb{K}_0|\mathbb{K}_t) = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}(\mathbb{K}_0|\mathbb{S}^n(\mathbb{K}_0))\, t^n$$

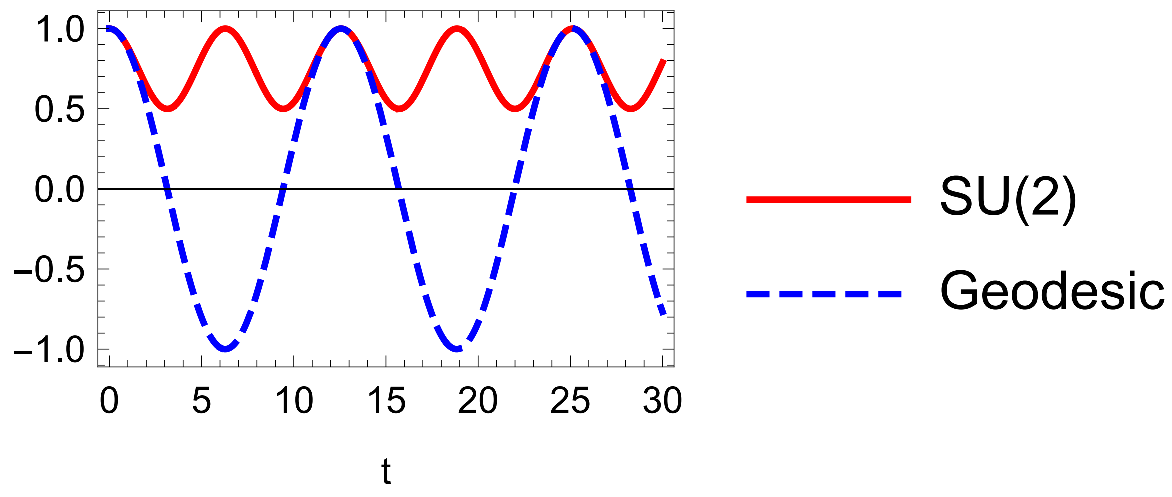
OQSL on Krylov complexity (III)

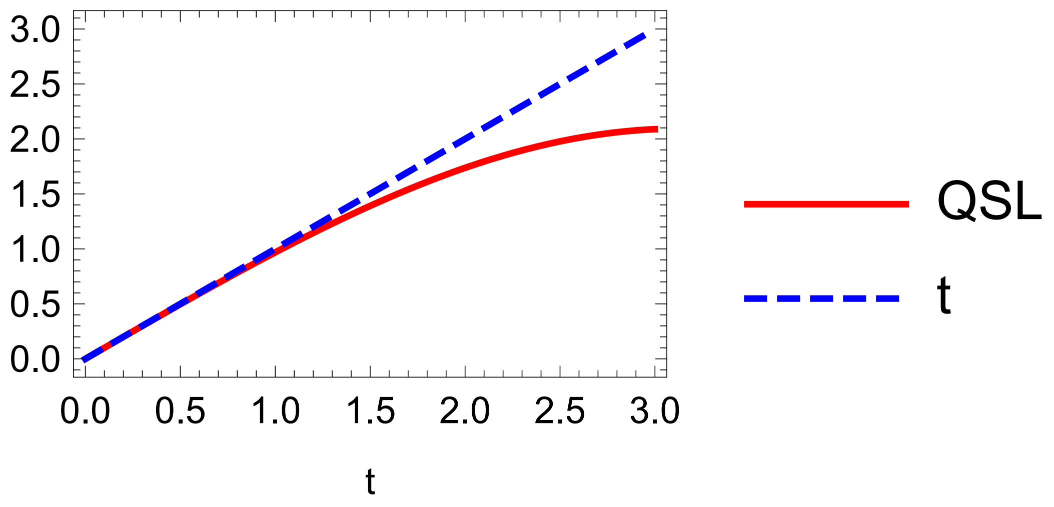
- For $SU(2)$: $(\mathbb{K}_0|\mathbb{K}_t) = (\|\mathbb{K}\|^2 + \frac{\gamma}{\alpha}\mathrm{Tr}\,\mathbb{K})\cos\sqrt{|\alpha|}t - \frac{\gamma}{\alpha}\mathrm{Tr}\,\mathbb{K}$

  to be compared with a geodesic trajectory $\|\mathbb{K}\|^2\cos\mathcal{V}_{\mathbb{K}}t$

- OQSL: $t \geq \frac{1}{\mathcal{V}_{\mathbb{K}}}\arccos\left[\left(1 - \frac{3(D-1)}{2(2D-1)}\right)\cos\sqrt{|\alpha|}t + \frac{3(D-1)}{2(2D-1)}\right]$

Deviation due to the stationary component $\mathbb{P}_0$

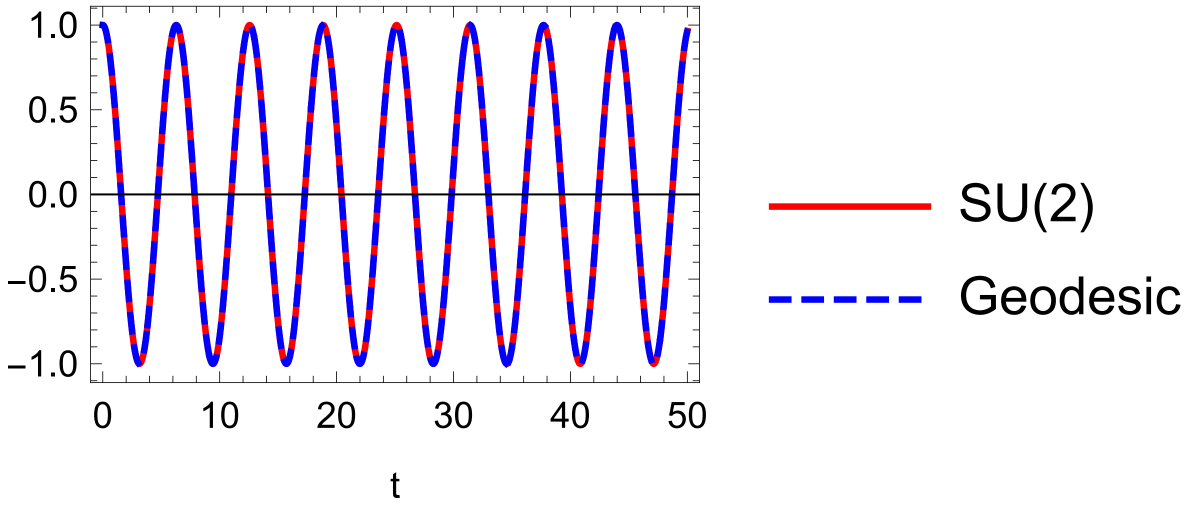


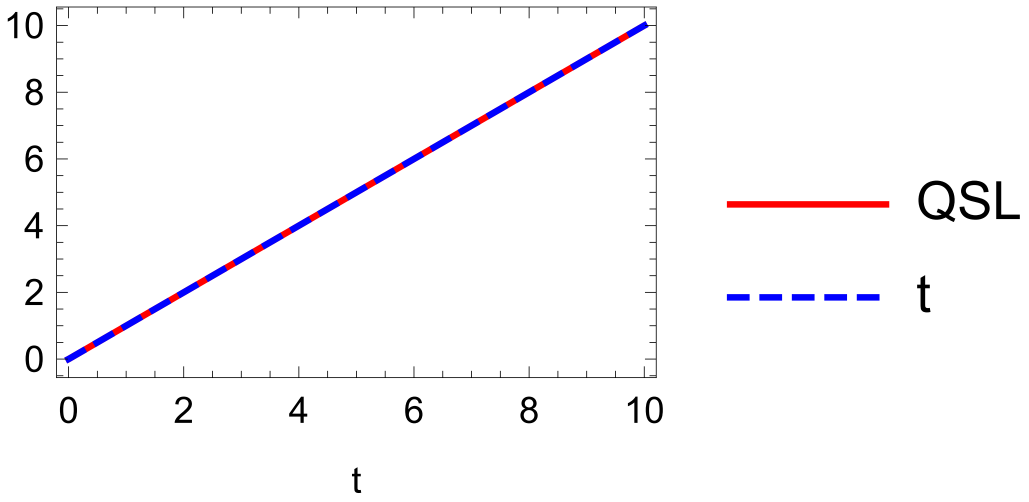


Saturation of the refined OQSL

- $\mathrm{Tr}\,\mathbb{K}_t \neq 0 \;\;\Rightarrow\;\; \mathbb{P}_0 \neq 0$ but it could have more components
- We try $\overline{\mathbb{K}}_t = \mathbb{K}_t - \dfrac{\mathrm{Tr}\,\mathbb{K}}{D}\,\mathbb{I} \;\;\Rightarrow \mathcal{V}_{\overline{\mathbb{K}}} = \sqrt{|\alpha|}$ for $SU(2)$
- $\mathrm{Tr}\,\overline{\mathbb{K}} = 0 \Rightarrow$ $\boxed{(\overline{\mathbb{K}}_0|\overline{\mathbb{K}}_t) = \|\overline{\mathbb{K}}\|^2 \cos\sqrt{|\alpha|}t}$

$$\mathbb{P}_0 = \frac{\mathrm{Tr}\,\mathbb{K}}{D}\,\mathbb{I}$$





Conclusions

- MT & ML operator QSL: Quantum 6, 884 (2022).
  - ✓MT-to-ML crossover on the decay of symmetric autocorrelation functions
  - ✓ML-type bounds on dynamical susceptibilities & Fisher information
- Operator growth at the maximal rate when the complexity algebra is closed

  Communications Physics **5**, Article number: 207 (2022)
- Geometric OQSL (MT):
  - ✓Driven evolution: Wegner flow
  - ✓Saturation: Toda flow & Krylov complexity with closed algebra
  - ✓Equivalence of the saturation of OQSL and the dispersion bound

arXiv > quant-ph > arXiv:2301.04372

Thank you for your attention!