Approximate CFTs, Holographic Ensembles & Tensor models

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Based on work with A. Belin, J. de Boer, D. Jafferis, J. Sonner

CERN

Huddle, ICTP

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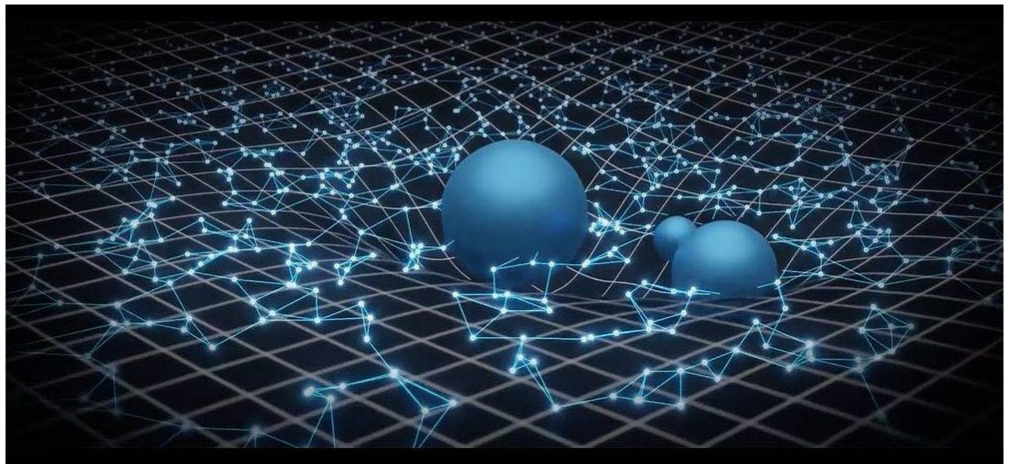

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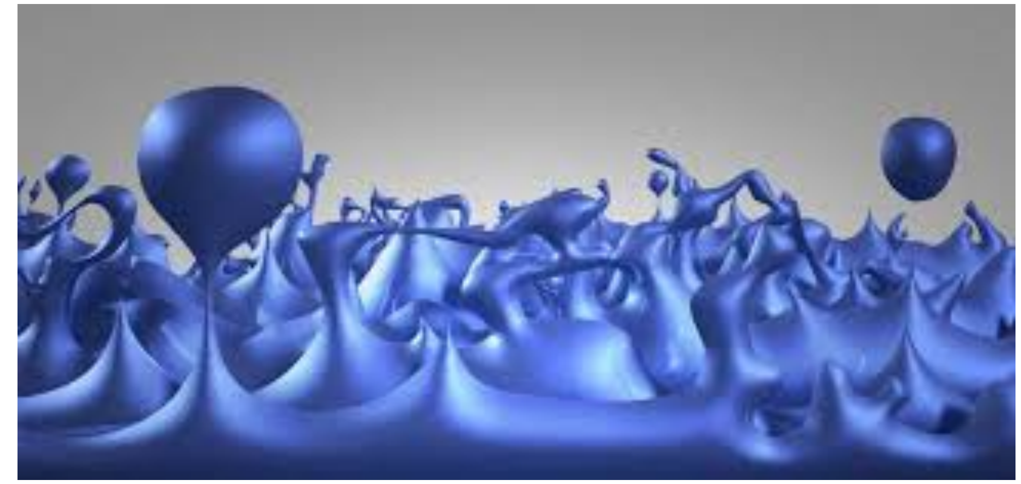
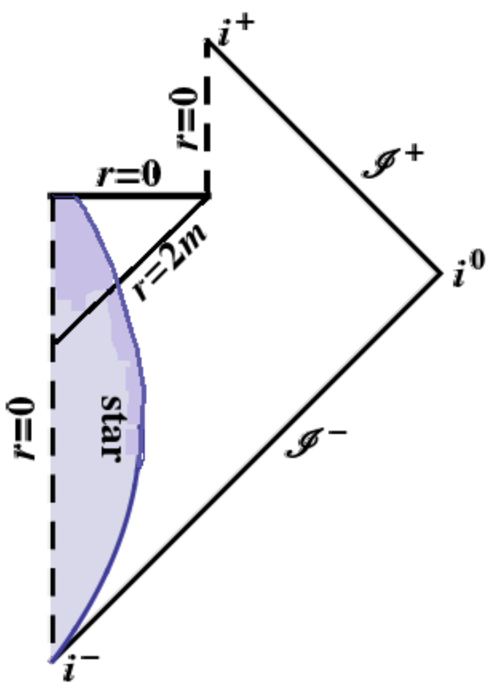
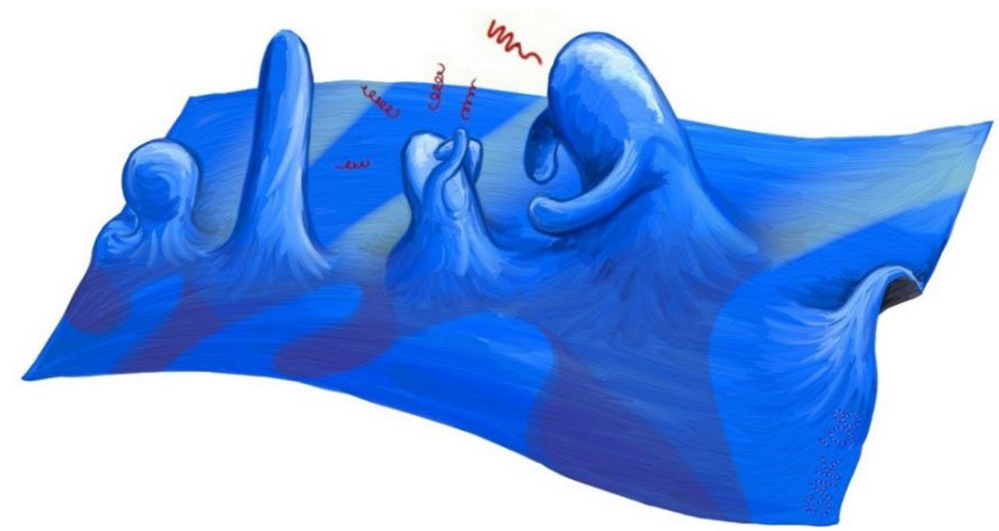
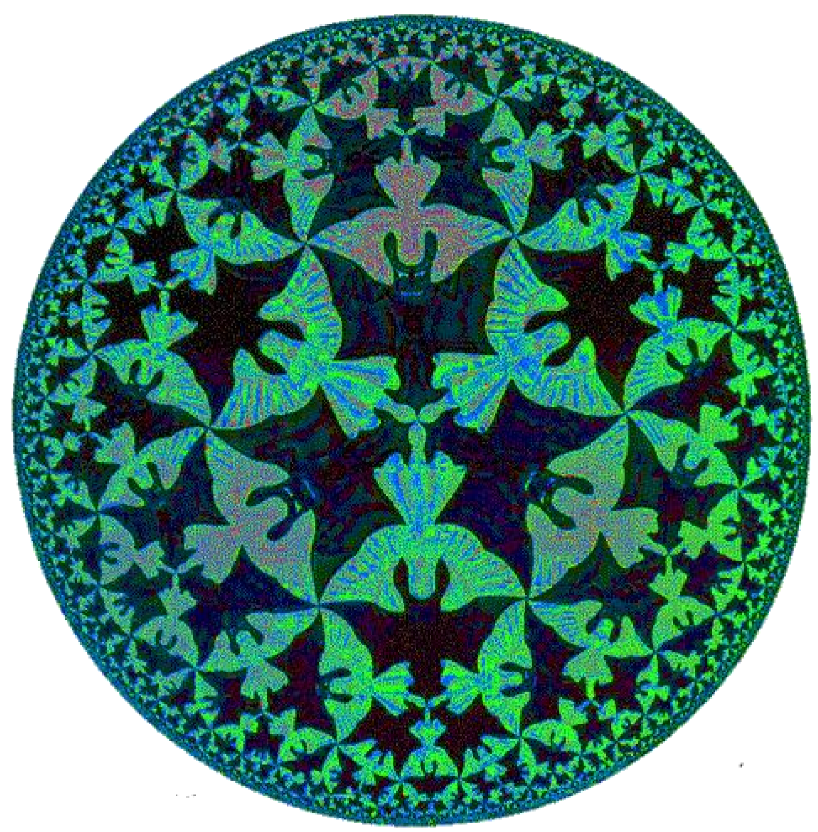




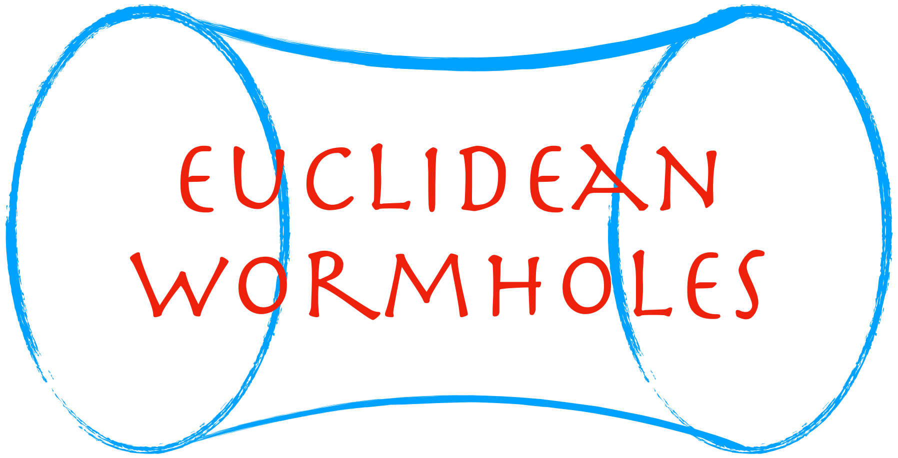
Information
Loss
Paradox





Strongly-coupled
QFT

A Partial Resolution?!



Saad, Shenker, Stanford '18 & '19;
Saad '19;
Penington, Shenker, Stanford, Yang '19

Altland, Bagrets '17;
Altland, Sonner '20;
Altland, Bagrets, *PN*, Sonner, Vielma *'21;*
more…

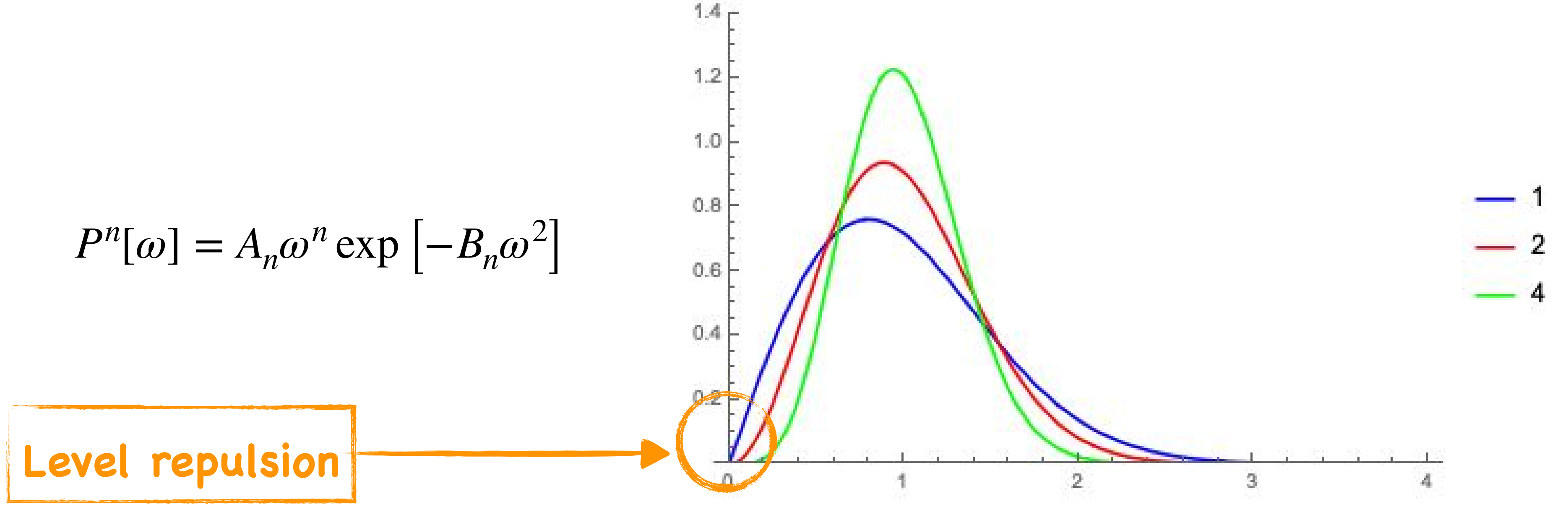
Quantum Ergodicity

Ergodicity in Quantum systems

[Berry-Taylor '77
Bohigas-Gioannoni-Schmidt '84]

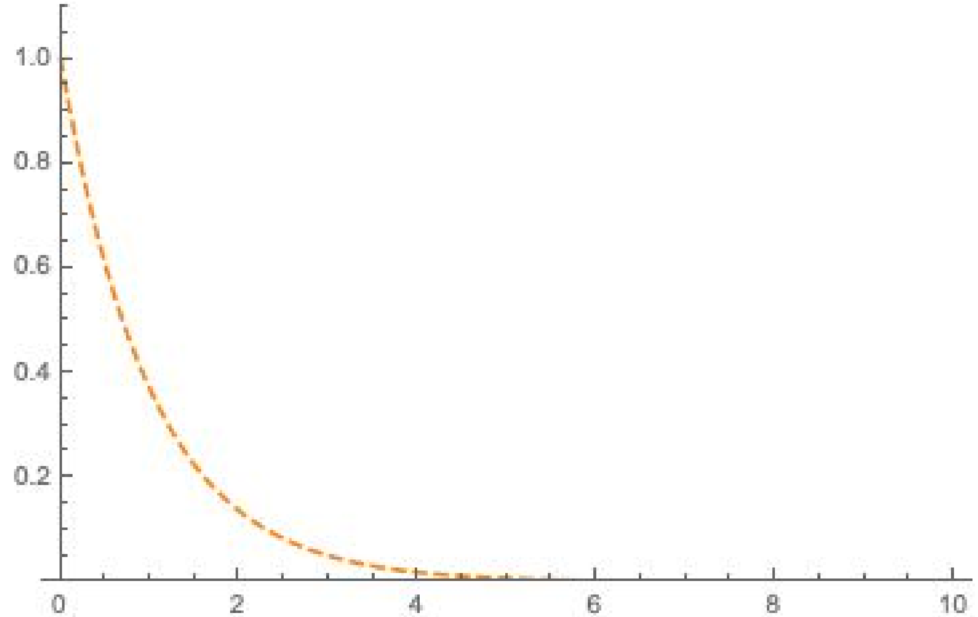
- In Chaotic Quantum systems, *spectral statistical properties* are 'similar' to those of a Random Matrix Theory (RMT)

$$P^n[\omega] = A_n \omega^n \exp\left[-B_n \omega^2\right]$$



- In Integrable Quantum systems, spectral statistical properties are described by Poisson Statistics

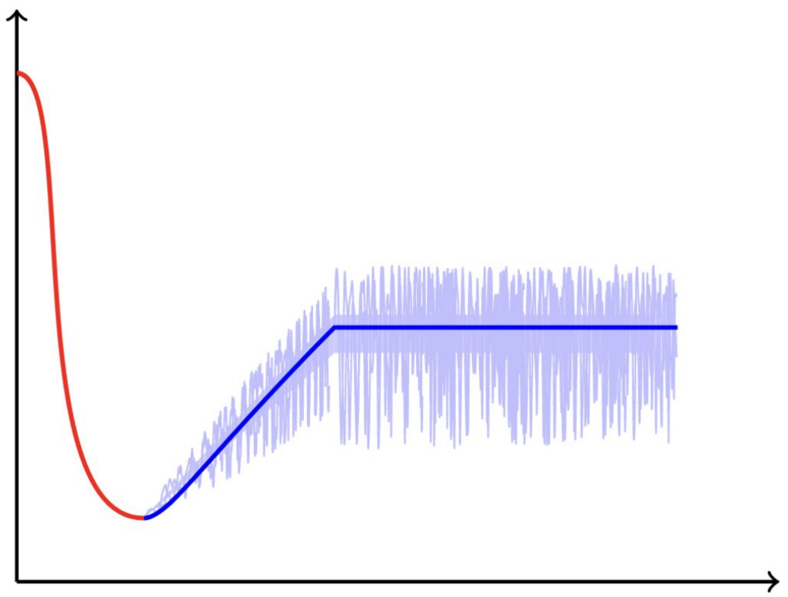
$$P^n[\omega] = \exp\left[-\omega\right]$$



* ω is the level spacing between eigenstates

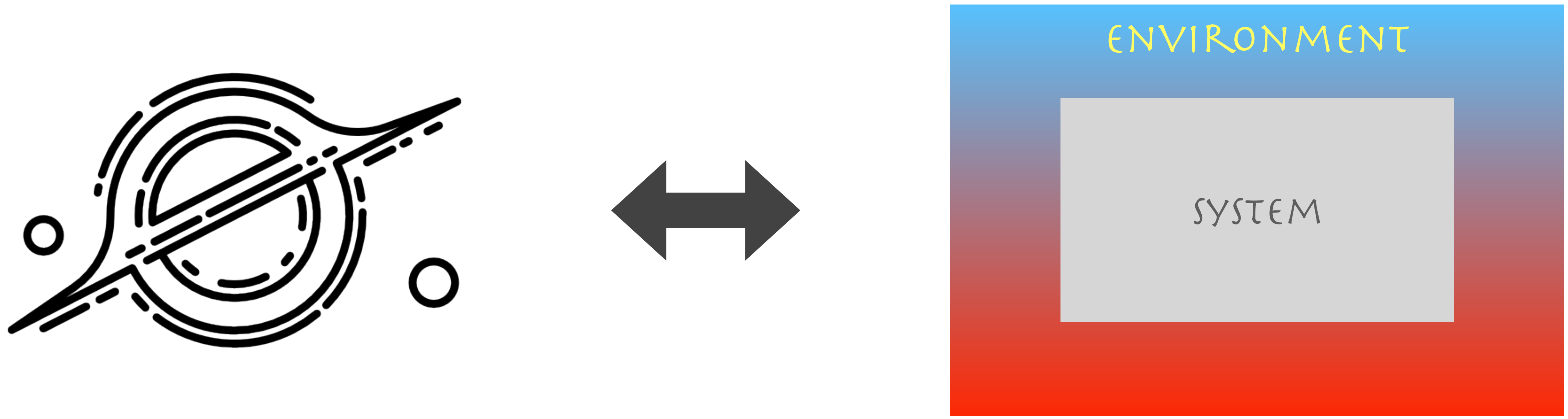
Quantum Ergodicity

- Level repulsion and Spectral Rigidity in energy eigenstates of RMT leads to a characteristic slope-ramp-plateau behavior of observables like Spectral Form factor and correlation functions in RMT.

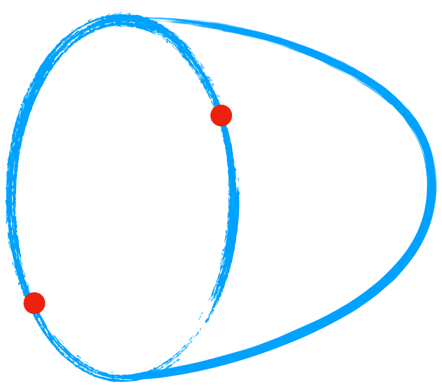


- Ergodic limit is defined as the energy domain in which RMT statistics persists for a many-body (chaotic) system. Corresponding time is called t_{ramp}.

Mysteries of Semiclassical duality



Computing observables like correlation functions even in this version of the duality is inconsistent:

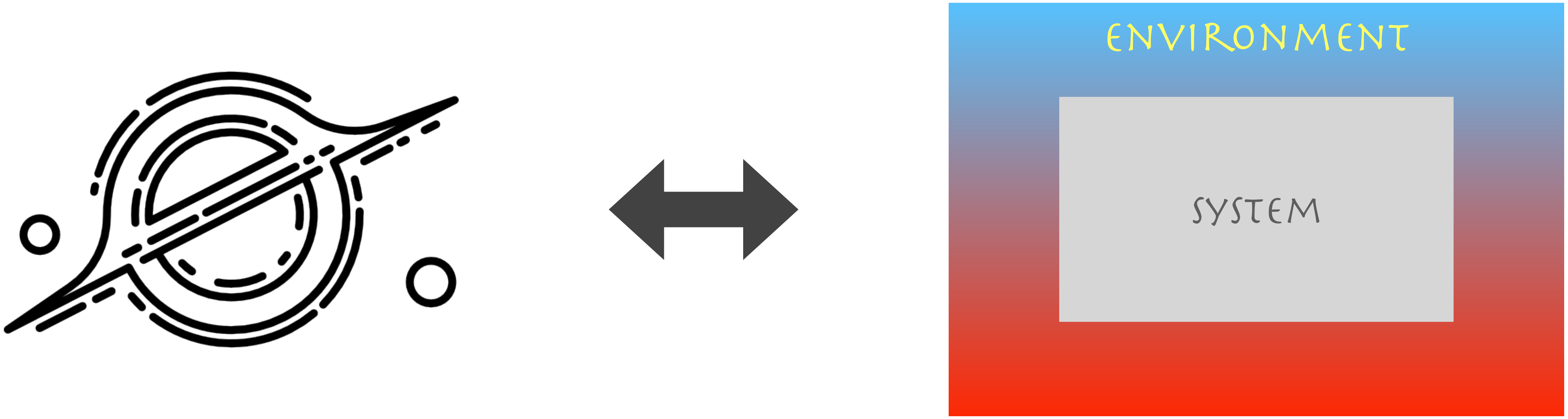


$$\mathrm{Tr}\left[e^{-\beta H}\,\mathcal{O}(t)\mathcal{O}^{\dagger}(0)\right]$$

Gives a decaying answer

Gives a non-zero answer at late times

Mysteries of Semiclassical duality



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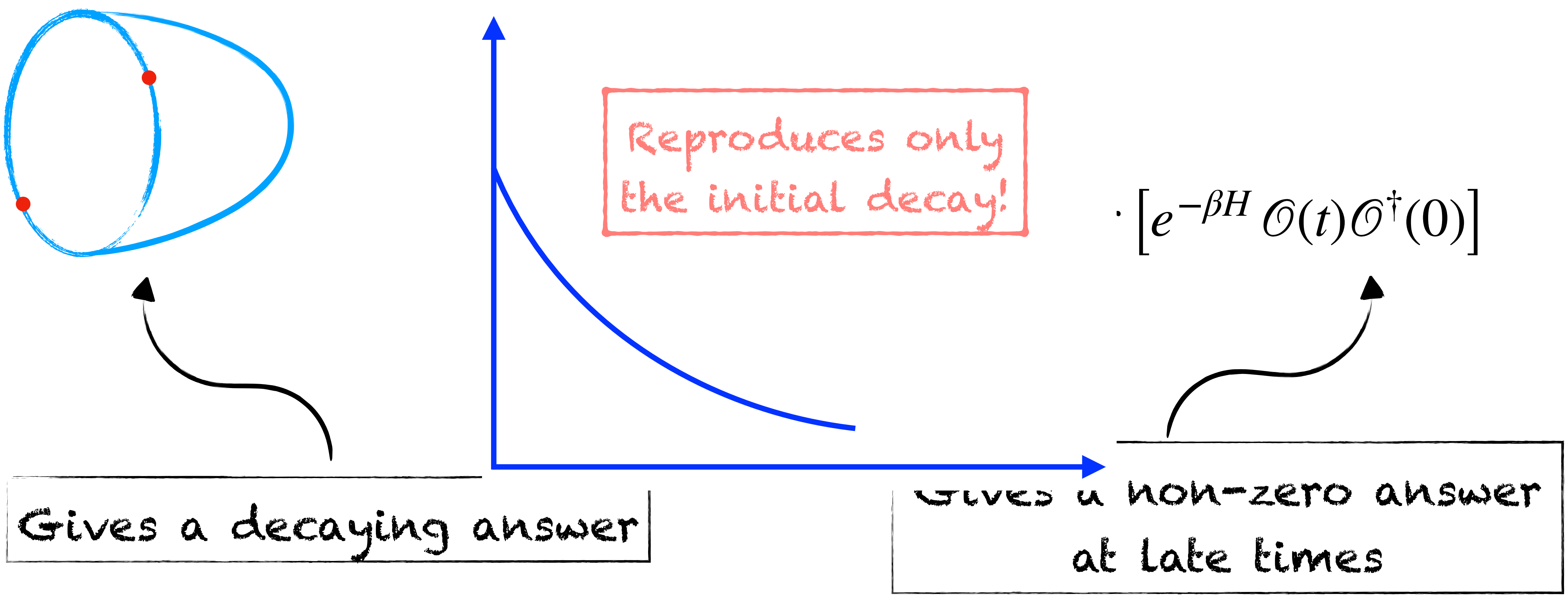
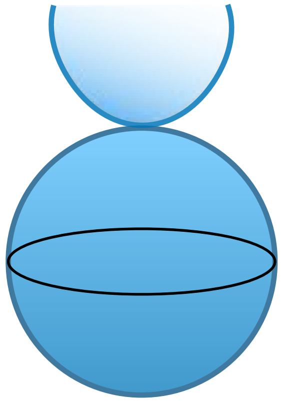


Overview

- Developing an analytic understanding of Quantum ergodicity in physical quantum systems:

  - EFT description of quantum ergodicity using the σ-model on $SU(2|2)/U(1|1)\times U(1|1)$

    [arXiv: 2105.12129]



  - 2D CFTs

    [arXiv: 2306.xxxxx]

    - Discussing the approximate CFTs and their importance
    - Relevance of the non-Gaussian corrections to the Gaussian RMT in context of holography
    - Tensor model of 3D gravity corresponding to simplicity gravity.

EFT of Quantum Ergodicity

Ergodic limit with σ-model

- Let us look at the resolvent of an operator which is defined as,

$$\tilde{R}(E,\omega) = \sum_{\beta} |\langle \alpha | \mathcal{O} | \beta \rangle|^2 \, \delta\left(E_\alpha - E_\beta - \omega\right)$$

$$= \int dE' \, \overline{\rho(E)\rho(E')\left|\langle E | \mathcal{O} | E' \rangle\right|^2} \, \delta\left(E - E' - \omega\right)$$

$$\left|\langle E | \mathcal{O} | E' \rangle\right|^2 = \mathcal{O}_i \, \mathcal{O}_j^* \, U_{iE'} \, U^\dagger_{Ei} \, U_{jE} \, U^\dagger_{E'j}$$

- $\mathcal{O} = e^{-(\beta + it)H}$ corresponds to the spectral form factor (SFF).

- Note that the averaging $\overline{\cdot}$ can be over of various kinds: disorder averaging, coarse graining, microcanonical averaging, etc.

The Resolvent

- The resolvent is related to the Fourier transform of the thermal 2-point function/SFF,

$$\mathrm{Tr}[\mathcal{O}(t)\,\mathcal{O}^\dagger(0)] = \int dE\,d\omega\,e^{-\beta E + i\omega t}\,\tilde{R}(E,\omega)$$

ω is conjugate to the time, t

Long time ⟷ small ω

- Recall the identity,

$$\rho(E) = \pm\,\mathrm{Im}\;\mathrm{Tr}\left[\frac{1}{E \mp i\varepsilon - H}\right]_{\lim \varepsilon\to 0}$$

$$\int dE'\;\overline{\rho(E)\rho(E')\left|\langle E|\mathcal{O}|E'\rangle\right|^2}\,\delta(E - E' - \omega)$$

Ergodic limit with σ-model

- The above observable can be rewritten as a path integral over some auxiliary graded-fields, $\Psi, \bar{\Psi}$,

$$\tilde{R}(E,\omega) = \Re \left[\partial_h^2 Z[h]\right]_{h=0}$$

$$Z[h] = \int \mathscr{D}\bar{\Psi}\mathscr{D}\Psi \exp\left[i\bar{\Psi}\cdot\left(z - H + h\right)\cdot\Psi\right]$$

$$h = \begin{pmatrix} 0 & h_+ P_b \otimes \mathcal{O} \\ -h_- P_b \otimes \mathcal{O}^\dagger & 0 \end{pmatrix}$$

$$\Psi = \begin{bmatrix} S^+ \\ \psi^+ \\ S^- \\ \psi^- \end{bmatrix} \qquad z = \begin{bmatrix} z_1 \\ z_2 \\ z_3^- \\ z_4^- \end{bmatrix}$$

- This action has U(2|2) causal symmetry, (weakly) explicitly broken by $z_i \neq z_j$

Rotations in the 4d graded space

$$U(2|2) \to U(1|1) \times U(1|1)$$

Symmetry broken b/w advanced & retarded section

Also spontaneously broken by the saddle point in the limit $\dim(H) \gg 1$

Ergodic limit with σ-model

$U(2|2) \to U(1|1) \times U(1|1)$

Symmetry broken b/w advanced & retarded section

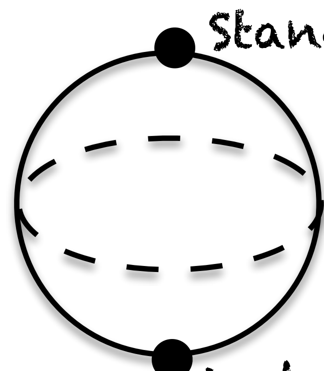
Also spontaneously broken by the saddle point in the limit $\dim(H) \gg 1$

- Effective description of Quantum ergodicity is captured by the (pseudo-)Goldstones of this symmetry breaking

$$\int dQ\, e^{-S[Q;\omega]} \quad \text{where} \quad Q \in \frac{U(2|2)}{U(1|1) \times U(1|1)} := \mathcal{M}(Q)$$

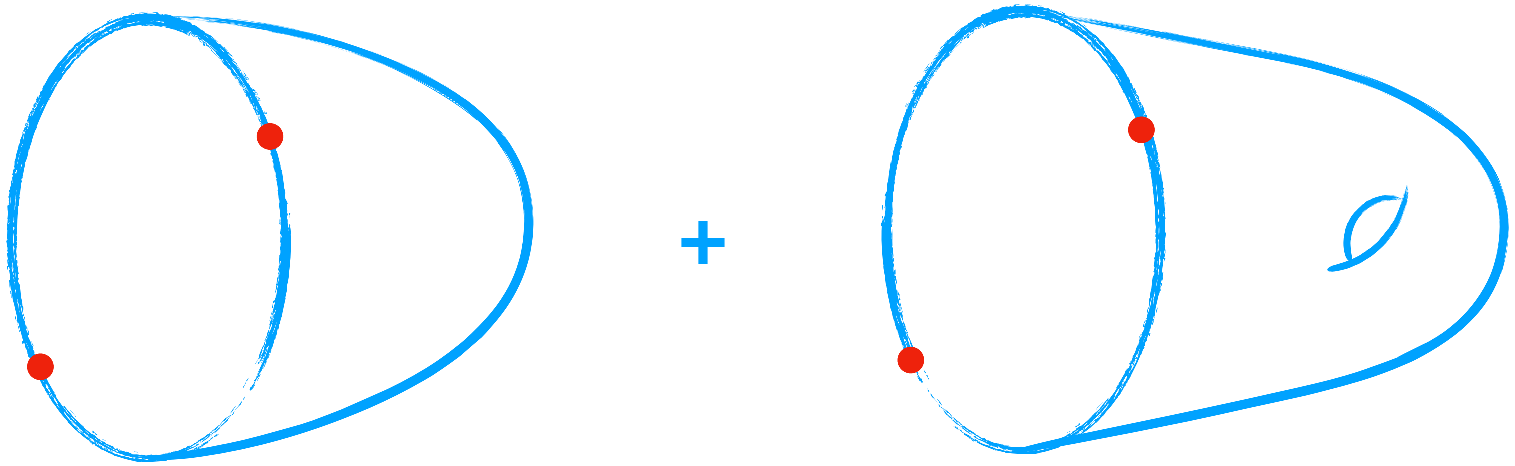
- There are two symmetry breaking saddle points in the limit, $\dim(H) \gg 1$:

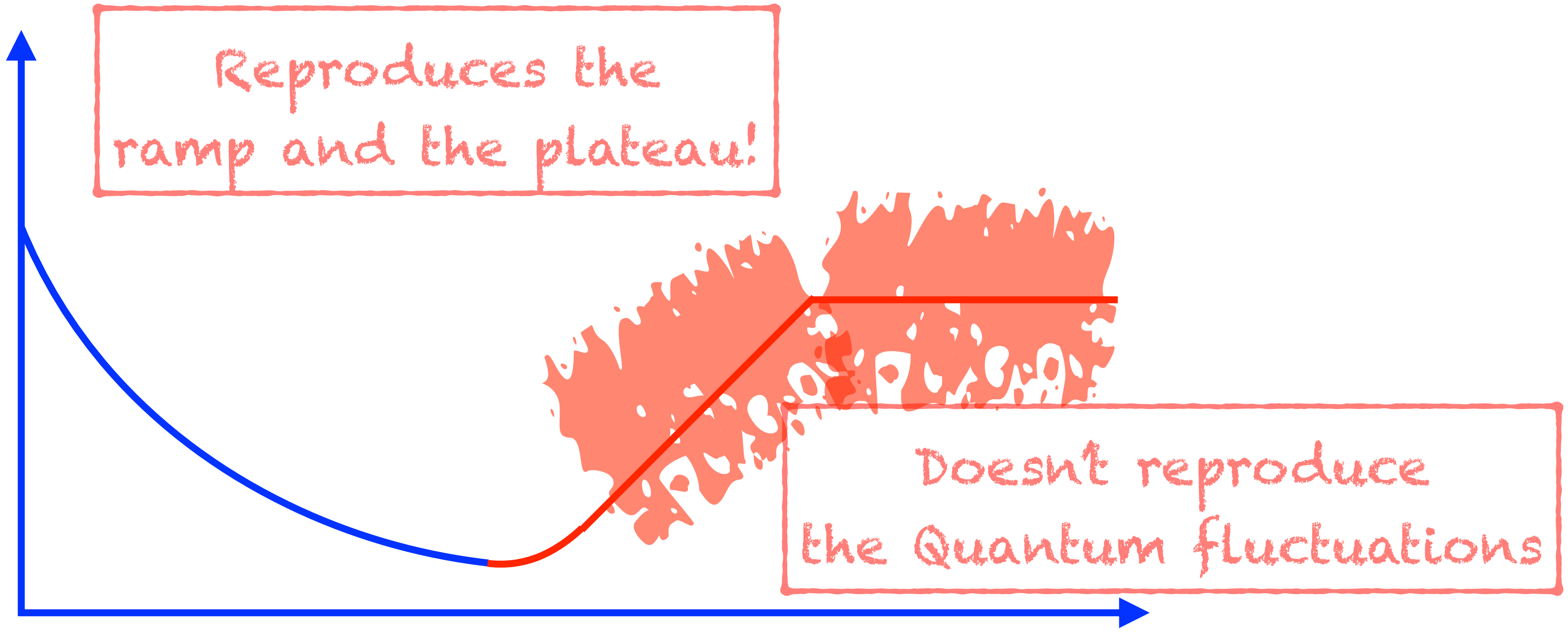
$$\mathcal{M} = H_2 \times S^2$$



Euclidean Wormholes

[Saad '19; Altland, Bagrets, *PN*, Sonner, Vielma '21]





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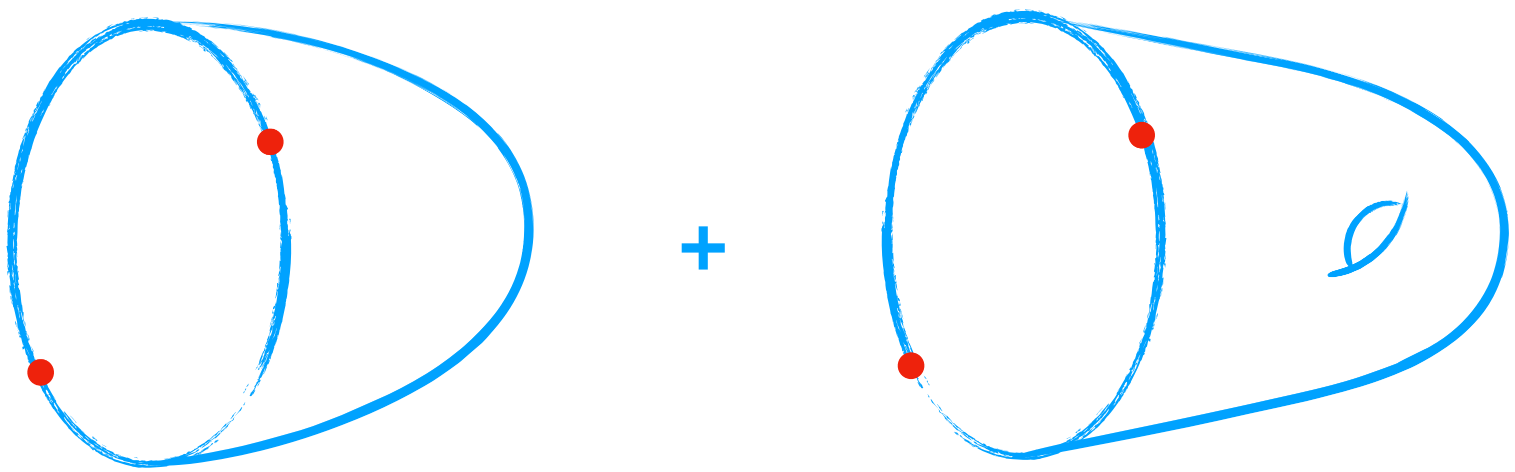
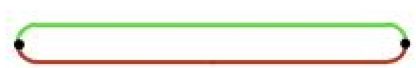
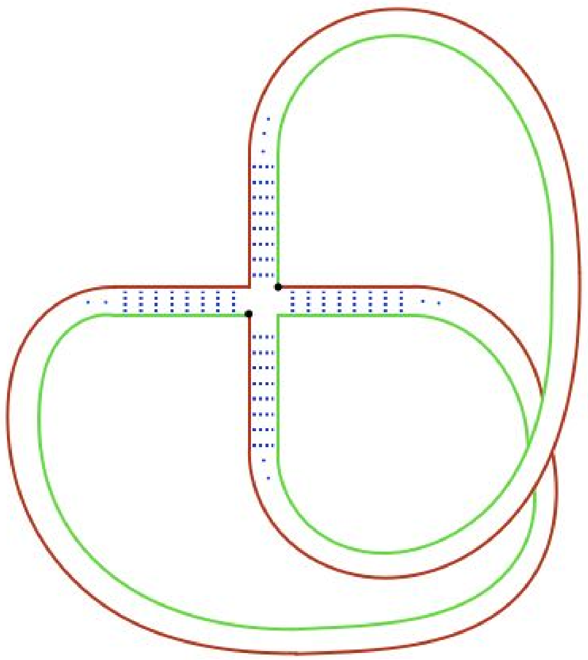
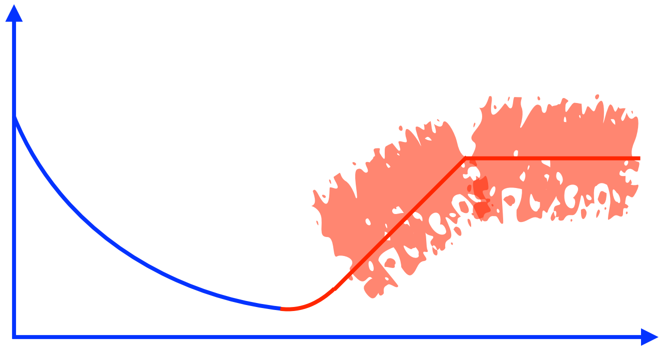
[Altland, Bagrets, *PN*, Sonner, Vielma '21]

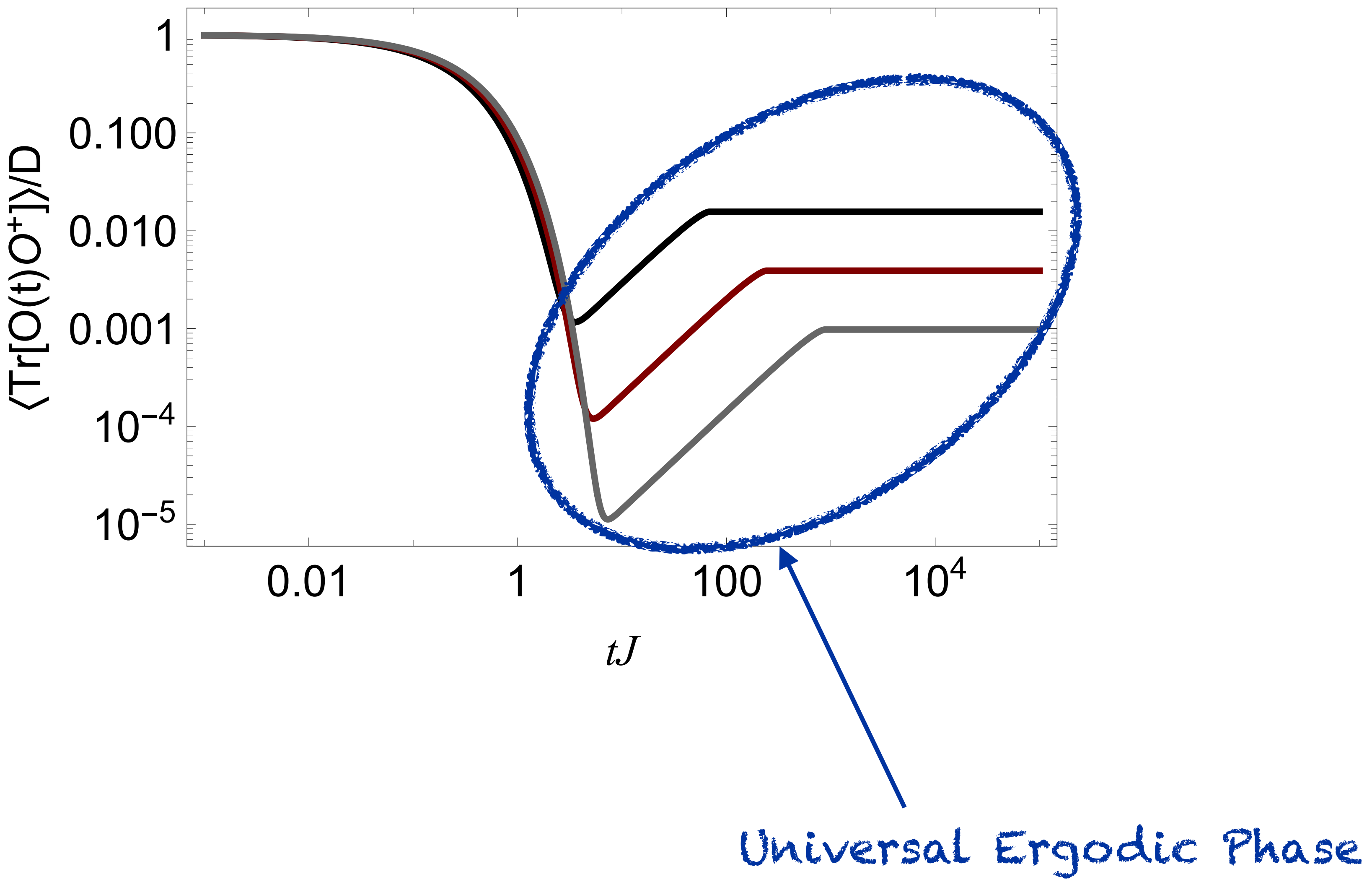




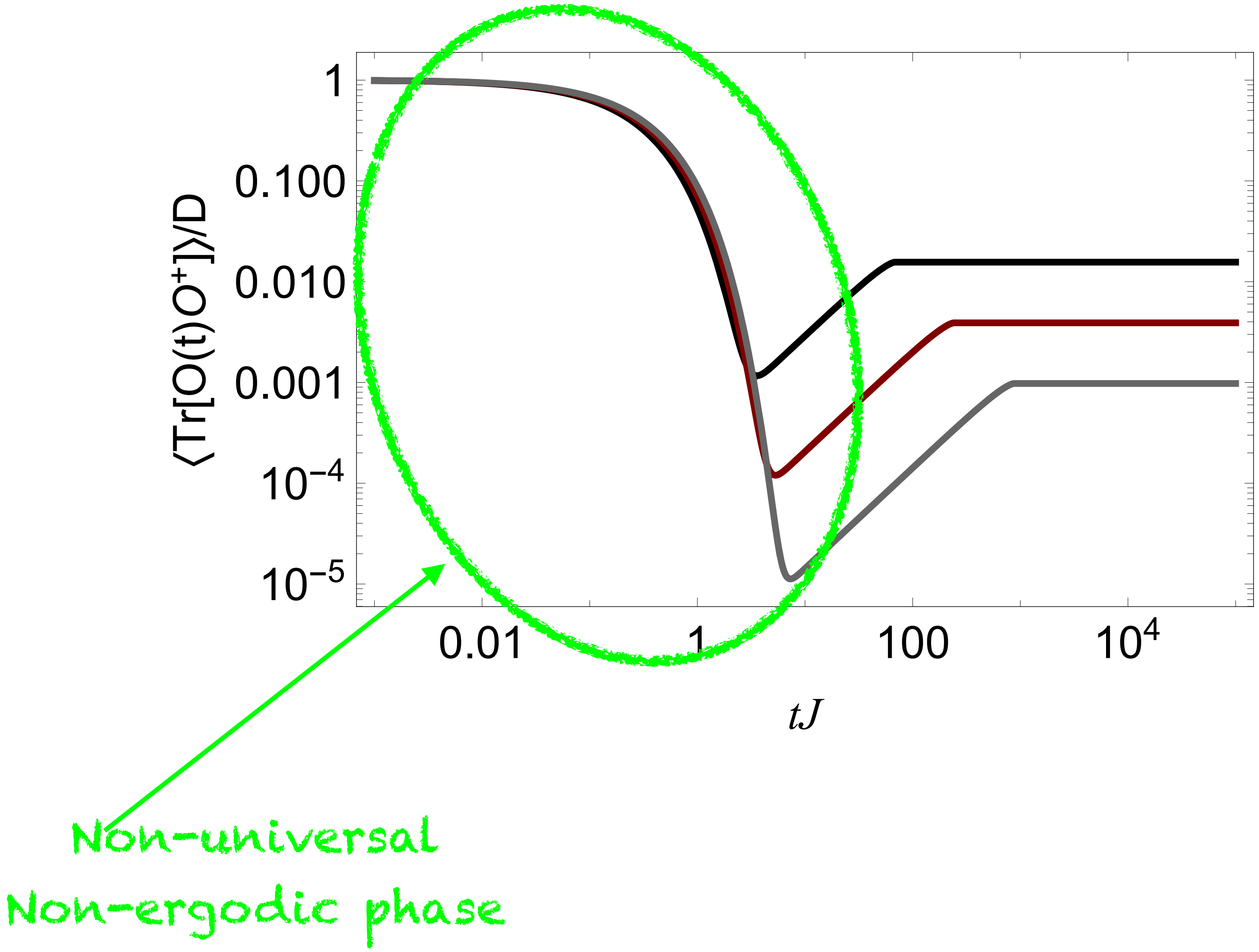




Non-ergodic Regime



Non-ergodic Regime



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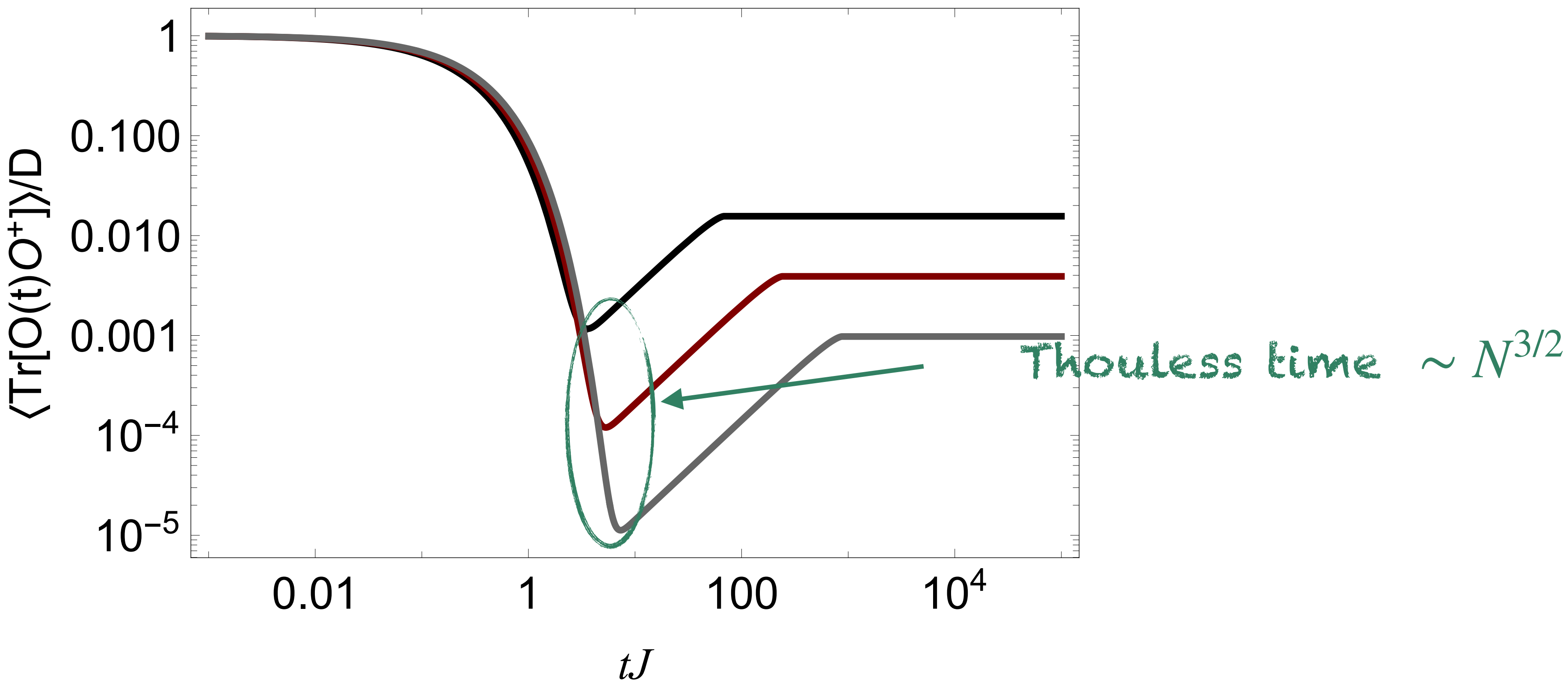


A study in CFT_2

CFTs

- CFTs are completely defined in terms of the *CFT data:*

$$\{\Delta_i, J_i, C_{ijk}\}$$

- CFT is constrained by consistency conditions imposed by conformal symmetry:
  - Crossing symmetry of the four-point function

$$\langle O_1(x_1) O_2(x_2) O_3(x_3) O_4(x_4) \rangle$$

  - Modular invariance of 1-point function on torus

$$\langle O(x_1) \rangle_{\mathbb{T}^2(\beta)} = \langle O(x_1) \rangle_{\mathbb{T}^2(2\pi/\beta)}$$

Crossing of all the higher point functions and modular invariance of higher genus surfaces are believed to follow from these requirements.

Asymptotic formulae in CFTs

- Most celebrated asymptotic formula in CFT_2 is the *Cardy Formula*

$$\rho_0(E) \approx e^{2\pi\sqrt{\frac{c}{3}E}}$$

  - It has been shown to be valid for all states above blackhole threshold in CFT_2. [Hartman, Keller, Stoica '14]

- Similarly, asymptotic formulae can be derived for OPE coefficients involving heavy states: [Collier, Maloney, Maxfield, Tsiares '18]

$$c_{ijk} \approx \frac{c_{DOZZ}(P_i, P_j, P_k)}{\sqrt{\prod_l \rho_0(P_l)}}$$

$\{i, j, k\} \in \text{heavy}$

How do we interpret these asymptotic formulae?

Asymptotic formulae in CFTs

How do we interpret these asymptotic formulae?

- Asymptotic formulae like above can be understood in two ways
  - Coarse-grained description
  - Statistical description

It is not so simple in a CFT though!

- What are the issues?
  - Generic CFTs do not have marginal deformations
  - It might not be possible to develop a coarse-grained/ statistical description of consistent CFTs

Defining Approximate CFT₂s

[Belin, de Boer, Jafferis, *PN*, Sonner
... to appear soon]

- We define an *approximate CFT* as a list of data

$$\{\Delta_i, J_i, C_{ijk}\}$$

  such that they are *approximately consistent with CFT constraints*!

- What are the approximations?
  - $\Delta_{\max}$ is the cutoff in the conformal dimensions below which we require the operators to satisfy the CFT constraints
  - $g_{\max}$ is the maximum genera of the Riemann surface on which we study CFT constraints
  - $n_{\max}$ is the maximum number of external operator insertions that satisfy CFT constraints
  - $z^L_{\min}$ is the minimum Lorentzian distance that two insertions can approach each other.
  - $\mathbb{T}$ is a tolerance parameter governing the validity of the CFT constraints under consideration (possibly governed by some non-perturbative physics).

$$\text{crossing, modular covariance} < \mathbb{T}$$

Defining Approximate CFT$_2$s

[Belin, de Boer, Jafferis, *PN*, Sonner ... to appear soon]

- Is the data of heavy operators ($\Delta_i > \Delta_{\text{max}}$) entirely free?
  - Heavy operators still contribute to the "light" observables that we are interested in constraining.

    However this always involves sum over the entire heavy spectrum (or some dense subset of it)
  - Therefore, by imposing the approximate constraints on the light spectrum, we also ensure that the violation of "heavy" observables is small on average up to the tolerance parameter $\mathbb{T}$.
- Thus an approximate CFT is labeled by the

$$\tilde{C} = \{\mathbb{O}_{\text{rest.}}, \mathbb{T}\}$$

- Reduced set of constraints defines an *island* ($\mathscr{A}$) in the theory space of approximate CFTs.

Holographic Approximate CFT$_2$s

- In holographic theories,

$$N \sim \frac{l^{d-1}}{G_N}$$

- We can reasonably conclude that the different parameters defining the approximate CFTs scale as follows,

$$n_{\max} \ll N\,, \qquad \Delta_{\max} \ll N\,, \qquad z^L_{\min} \ll \log N\,, \qquad g_{\max} \ll N$$

- EFT nature of the bulk semi-classical physics mean that the tolerance parameter,

$$\mathbb{T} \sim e^{-N}$$

Averaging over Approximate CFT$_2$s

- We define a joint probability distribution over the space $\mathscr{A}$

$$P\left(\{\Delta_i, J_i, C_{ijk}\}\right)$$

- An observable computed within this ensemble on the space $\mathscr{A}$ is an average over the same observable computed in individual elements of $\mathscr{A}$

$$\overline{\langle\,\cdot\,\rangle} \equiv \int P\left(\{\Delta_i, J_i, C_{ijk}\}\right)\ \langle\,\cdot\,\rangle_{\{\Delta_i, J_i, C_{ijk}\}}$$

- In particular, one can compute CFT constraints, $\mathscr{C}R(u,v)$, in the ensemble

$$\left|\overline{\mathscr{C}\mathscr{R}(u,v)}\right| < \mathbb{T}$$

- However, within an ensemble, we must also ensure that the moments of the CFT constraints are also obeyed,

$$\sigma_{(n)}\left[\mathscr{C}\mathscr{R}(u,v)\right] < \mathbb{T}^n$$

Imposing these constraints on the higher moments imposes new constraints on the ensemble over $\mathscr{A}$

It introduces non-Gaussianities in the statistics of the OPE data.

Ensemble for Gravitational theory

- In theory of pure gravity in AdS_3, identity is the only operator below the blackhole threshold

- In our analysis we additionally introduce "conical defect operators" that have scaling dimensions below the black hole threshold
  - We do not average over the scaling dimensions
  - We do average over their OPE coefficients.

- A preliminary ensemble is defined by, [Chandra, Collier, Hartman, Maloney '22]

$$\overline{\rho(h,\bar{h})} \approx e^{2\pi\sqrt{h-\frac{c}{24}}}\ e^{2\pi\sqrt{\bar{h}-\frac{c}{24}}}$$

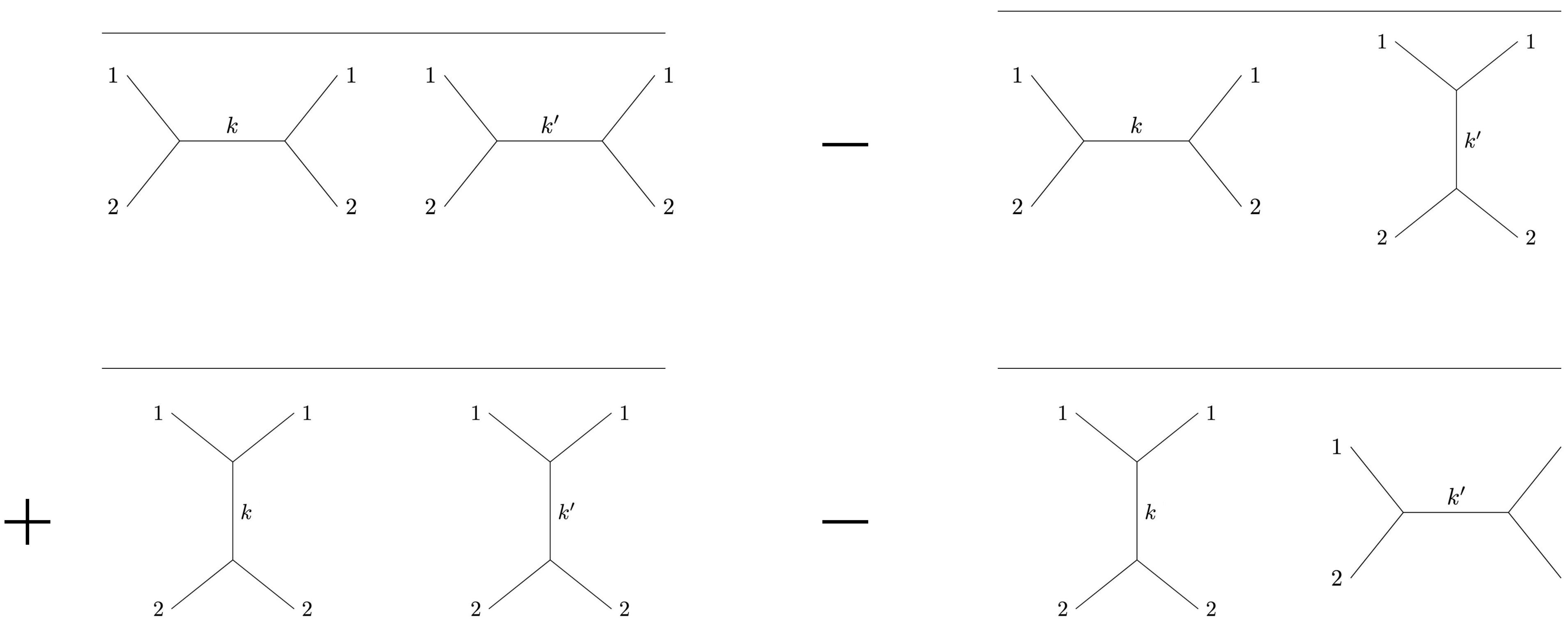
$$\overline{C_{ijk}C_{lmn}} = C_0(h_i,h_j,h_k)C_0(\bar{h}_i,\bar{h}_j,\bar{h}_k)\ \left(\delta_{i,l}\delta_{j,m}\delta_{k,n} + \text{perm}\,.\right)$$

- We see that the above ensemble is not sufficient to ensure that certain CFT constraints are approximately true:
  - Modular invariance of the genus-3 partition function [Belin, de Boer, Liska '21]
  - Variance of the crossing equation of the defect operators.

Variance of the crossing equation

- Various of the crossing equation is inherently a 2-sided computation from the gravitational point of view

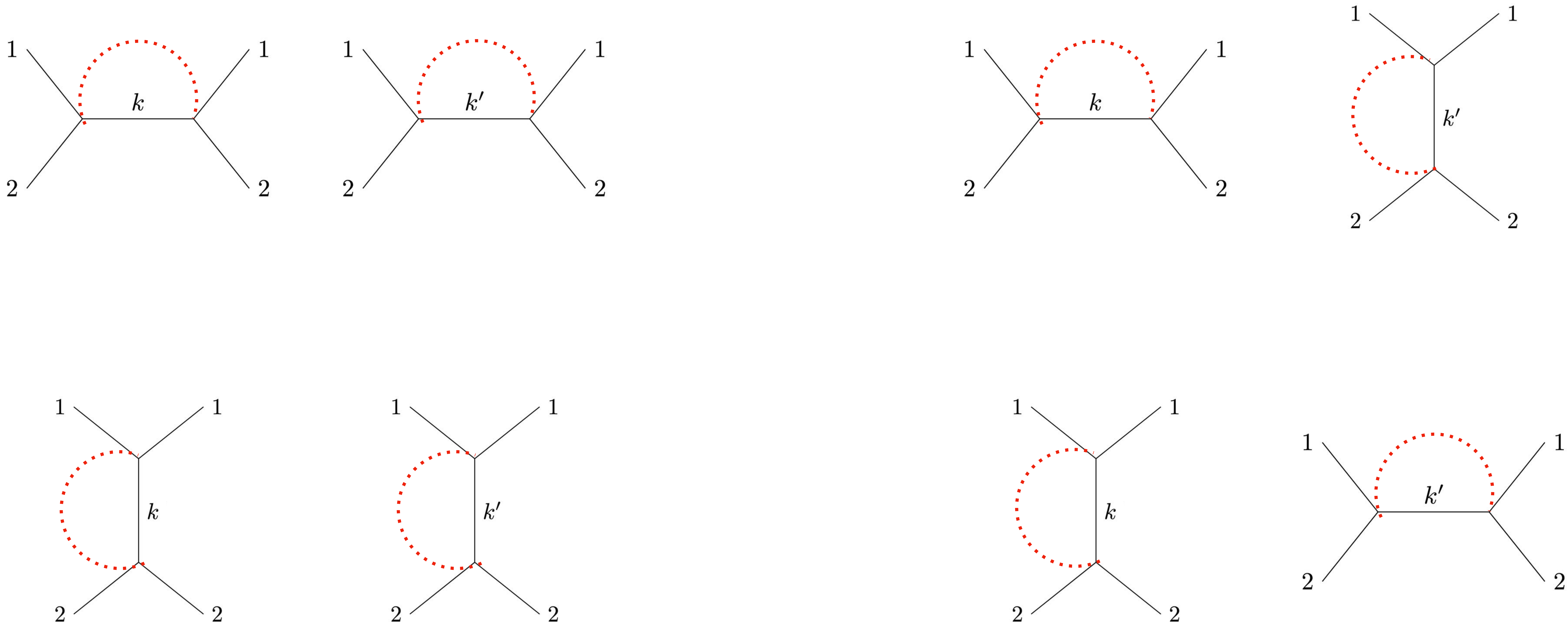
- Various terms arising in the expression of the variance of the crossing equation



- All the above expressions are quartic in OPE coefficients

Variance of the crossing equation

Disconnected contributions through quadratic correlations of the OPE coefficients



denotes the quadratic correlations of the OPE coefficients in our ensemble

Variance of the crossing equation

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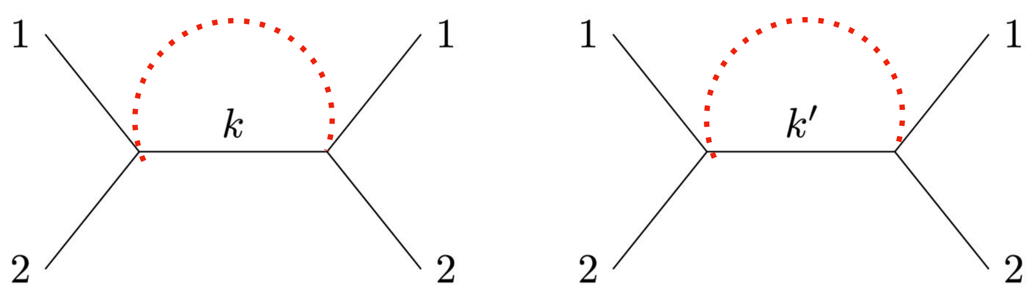
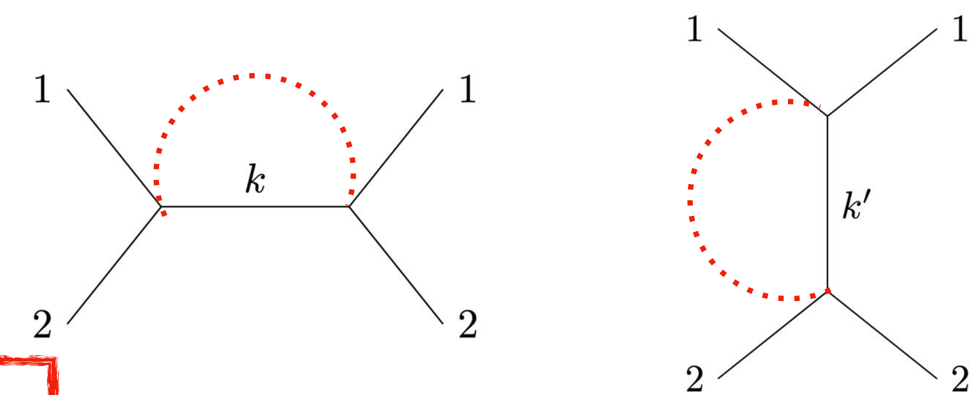
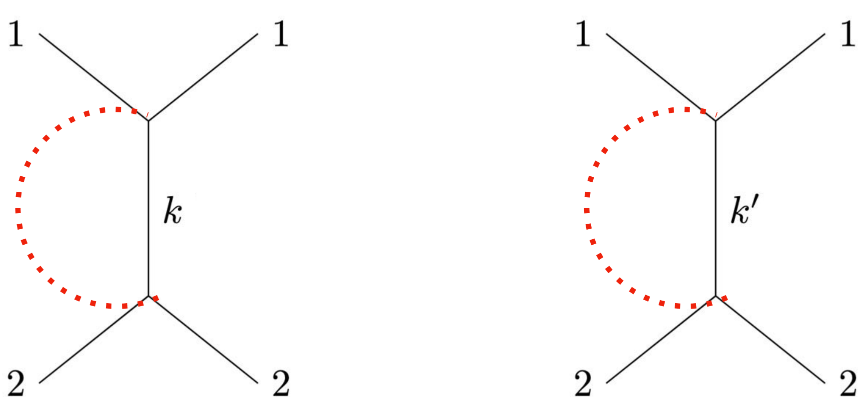
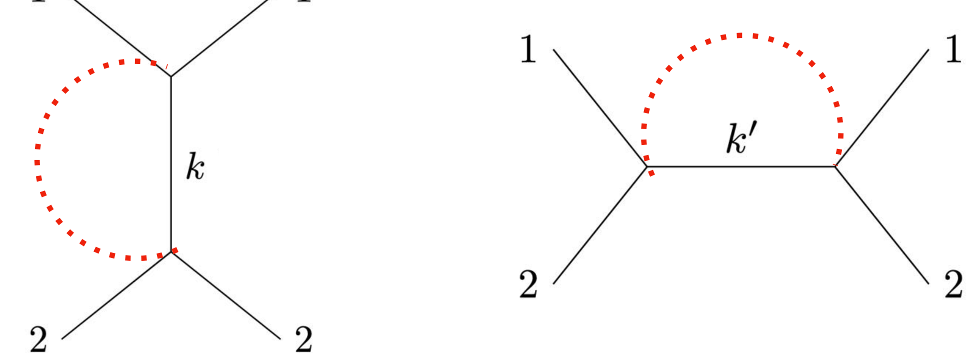




These contributions cancel in the variance computations!





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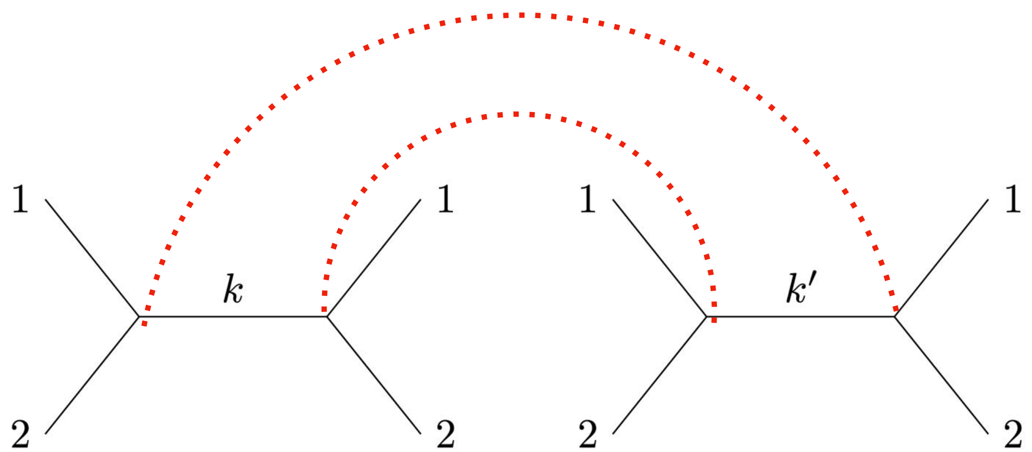

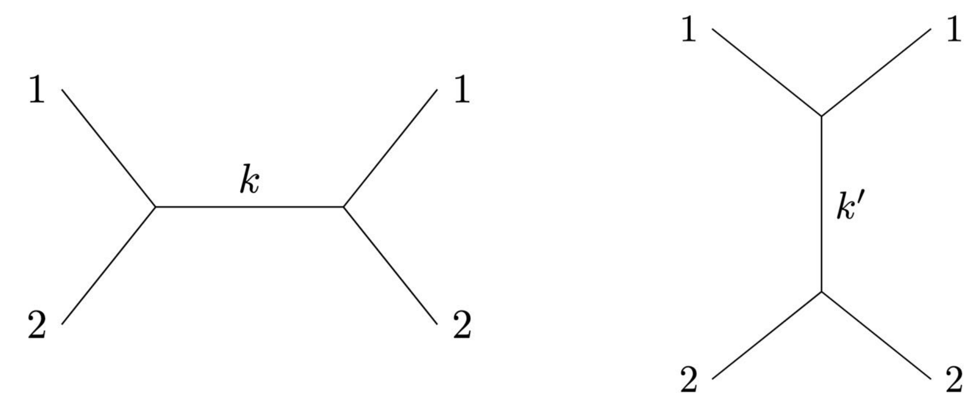

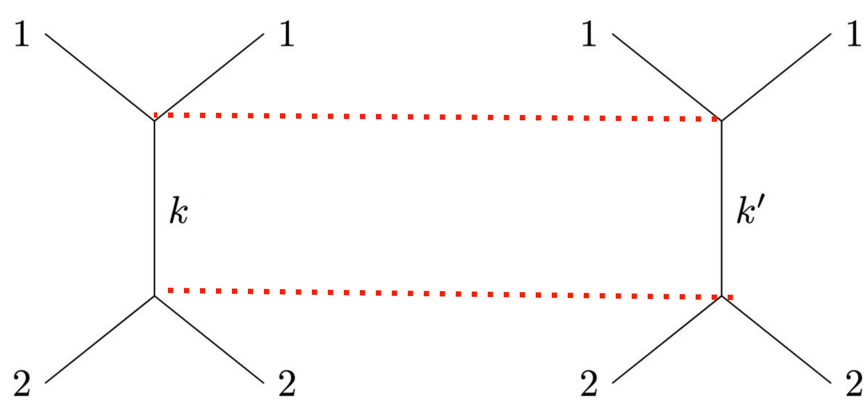

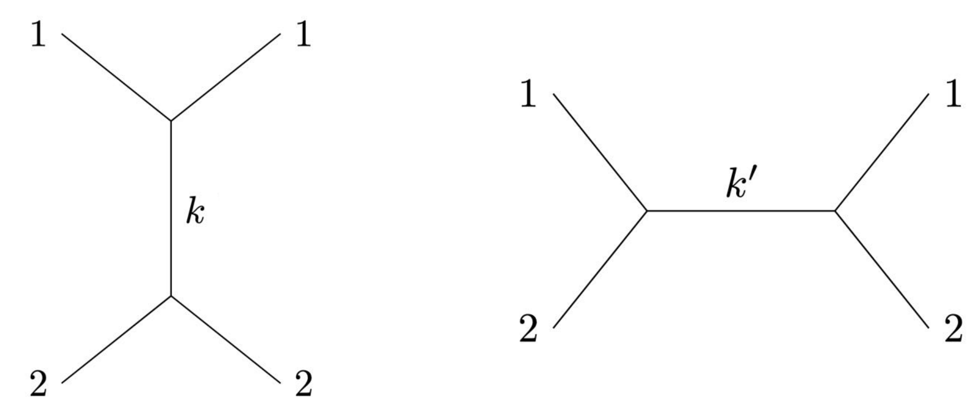

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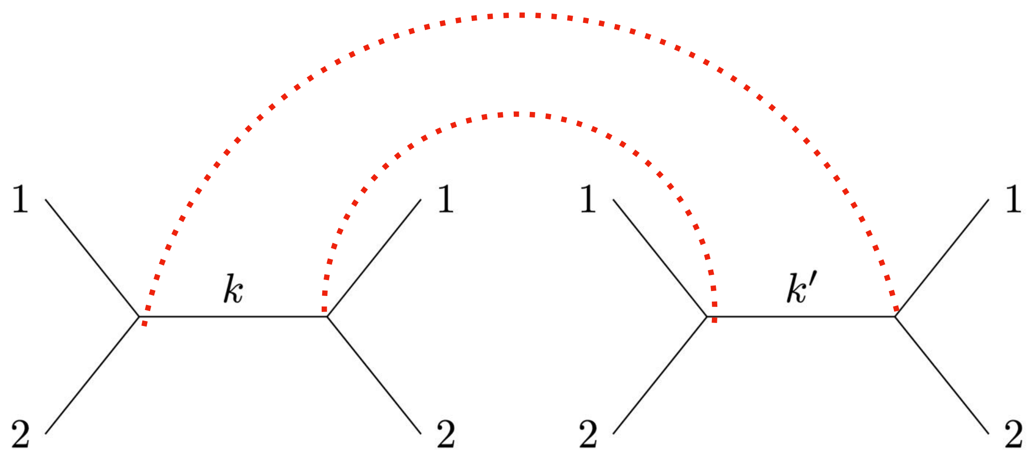
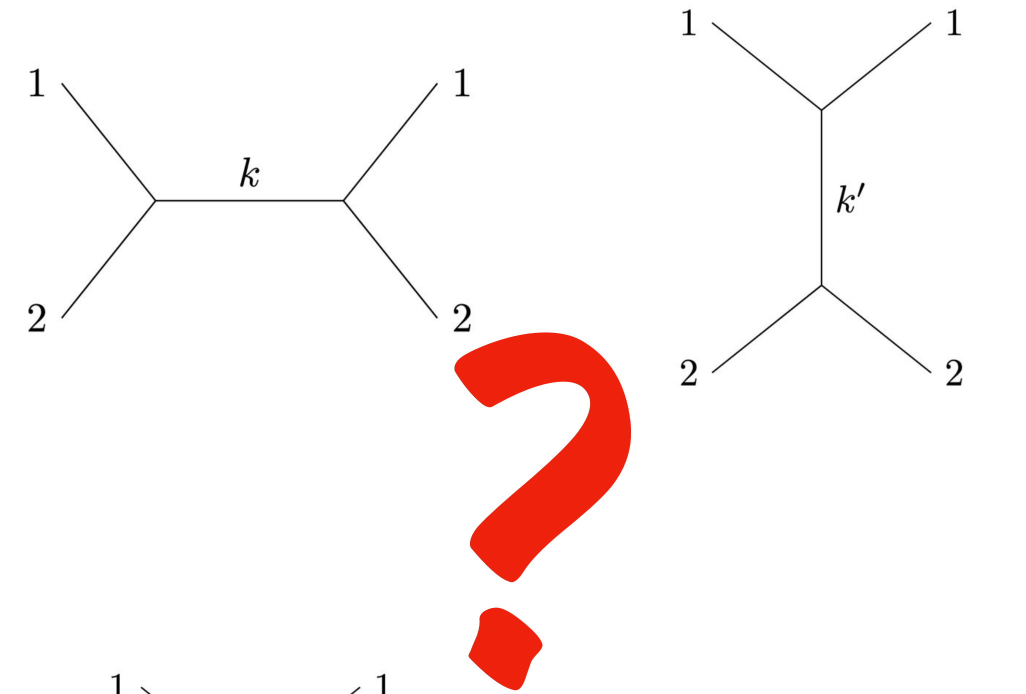
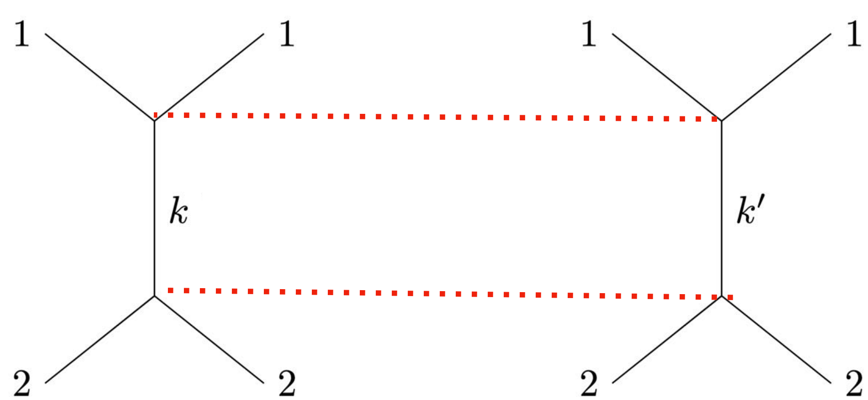
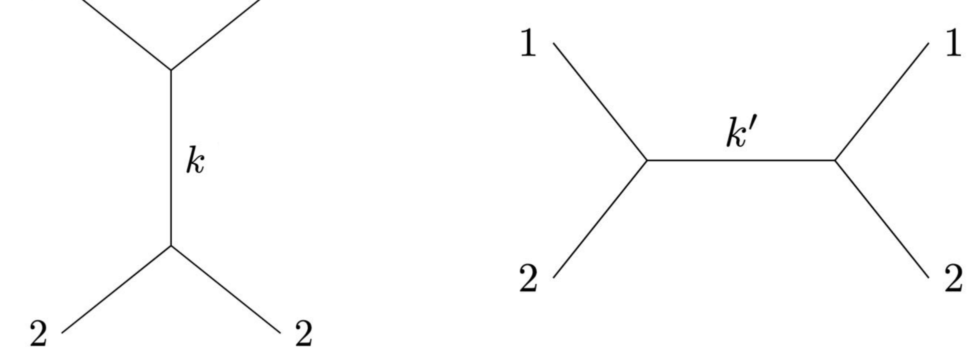
Variance of the crossing equation

Connected contributions through quadratic correlations of the OPE coefficients





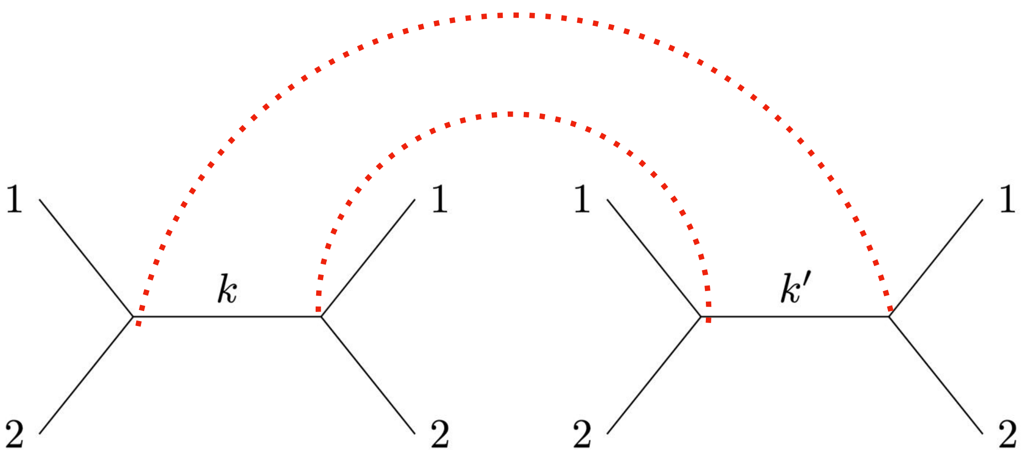


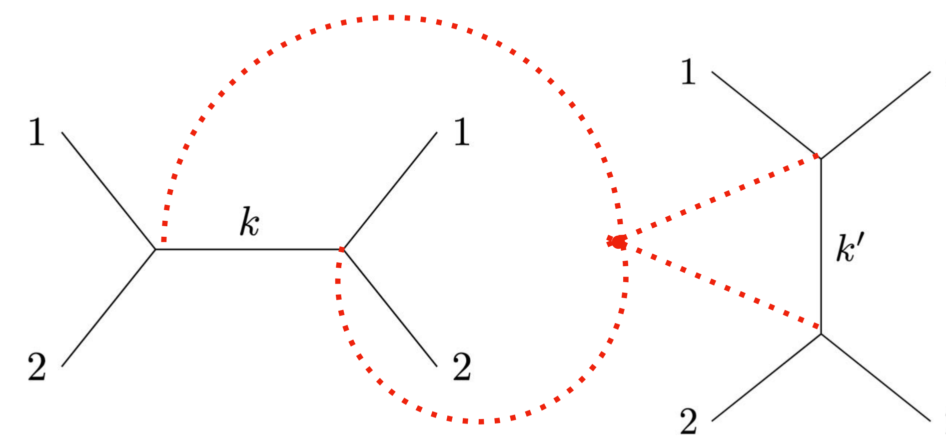


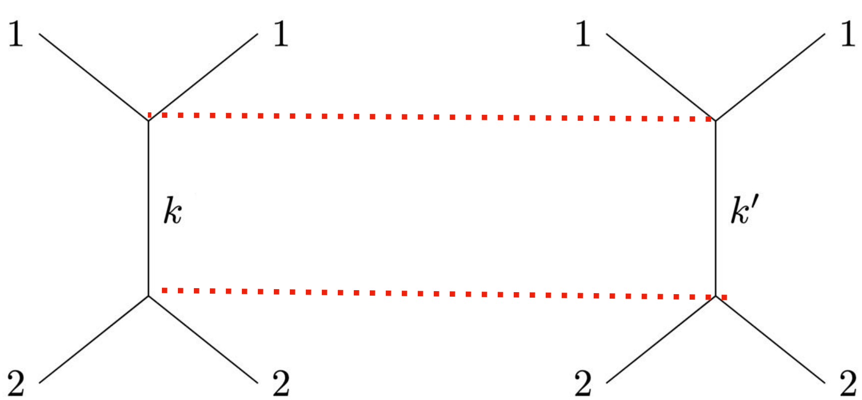
 denotes the quadratic correlations of the OPE coefficients in our ensemble

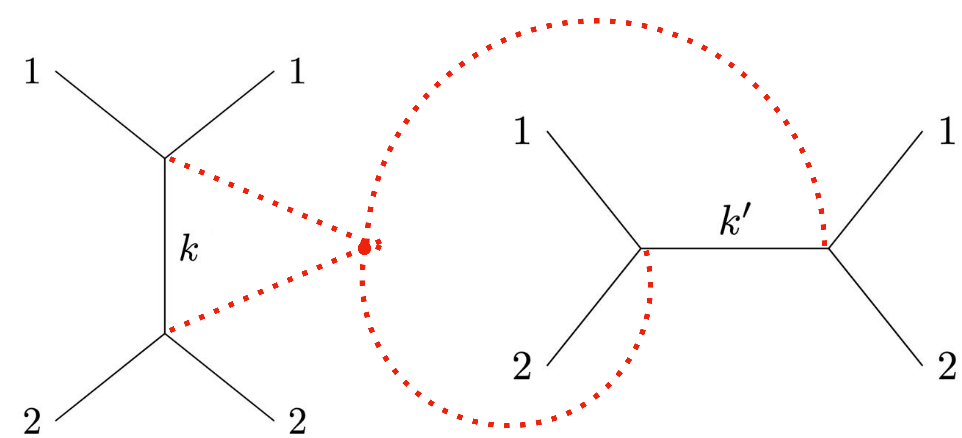
Non-Gaussian corrections to the ensemble

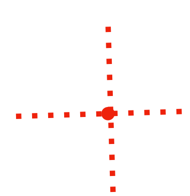
We need to introduce the quartic correlations between the OPE coefficients







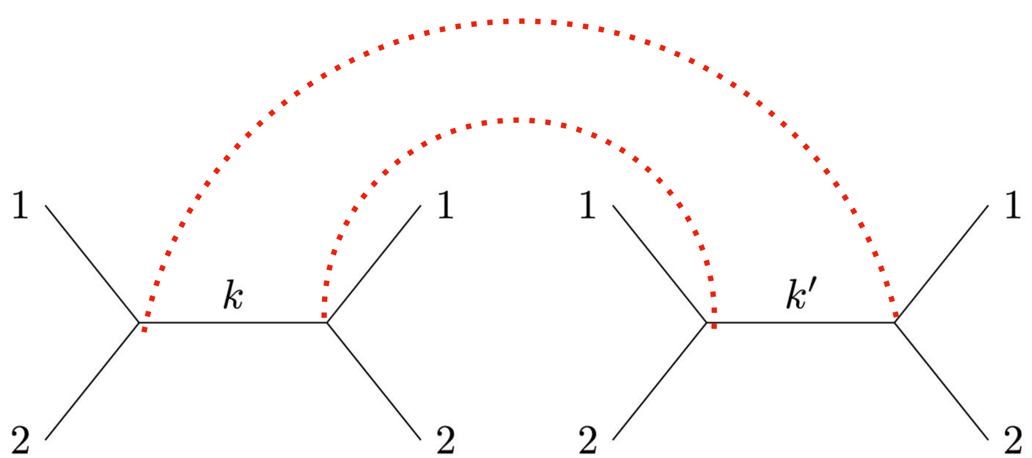


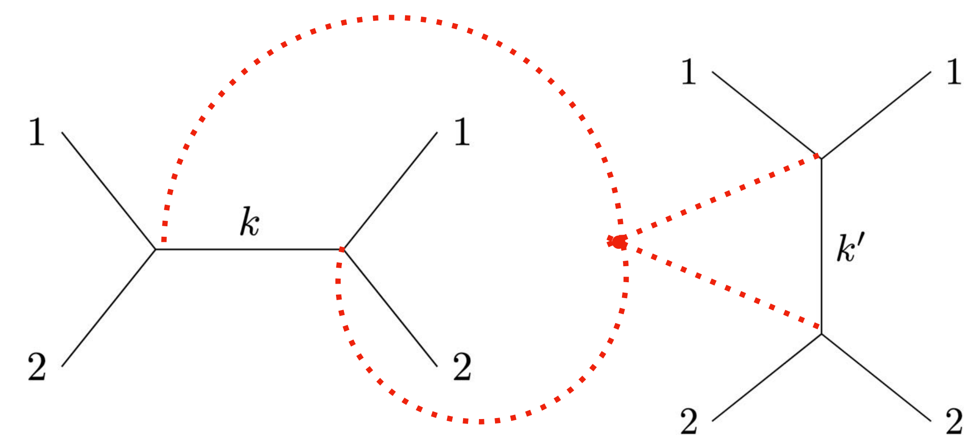


denotes the quartic correlations of the OPE coefficients in our ensemble

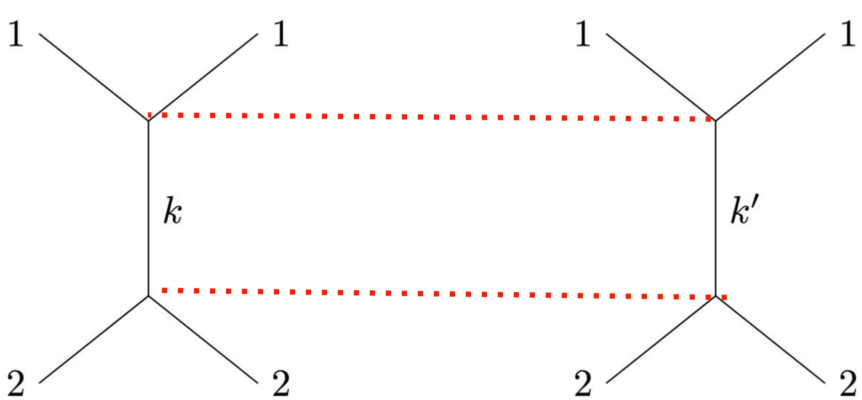
Non-Gaussian corrections to the ensemble

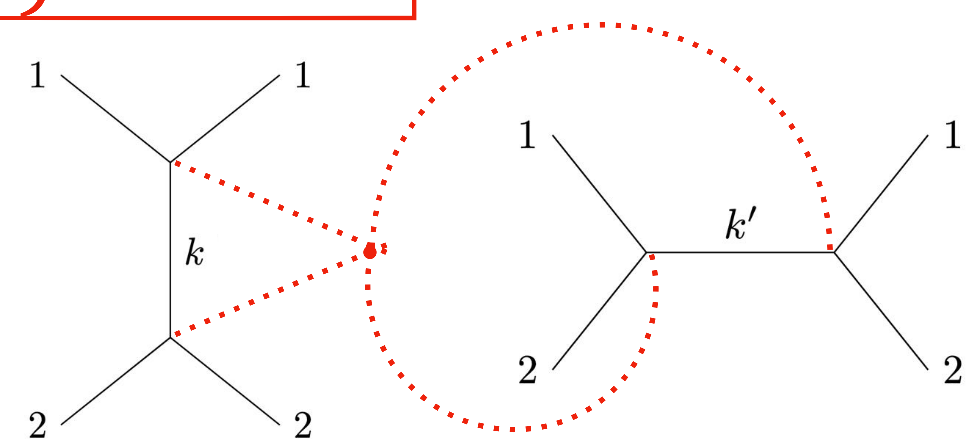
We need to introduce the quartic correlations between the OPE coefficients

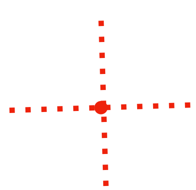




$$\overline{C_{126'}C_{346'}C_{14\tilde{6}}C_{23\tilde{6}}} \approx \begin{Bmatrix} O_1 & O_2 & O_{\tilde{6}} \\ O_3 & O_4 & O_{6'} \end{Bmatrix} \overline{C^2_{126'}}\;\overline{C^2_{346'}}$$







denotes the quartic correlations of the OPE coefficients in our ensemble

Tensor model for 3D gravity

- The ensemble over the CFT data can be defined explicitly as a probability measure that weighs the data by a factor that decays away from the solutions of CFT constraints,

$$P\left(\{\Delta_i, J_i, C_{ijk}\}\right) \propto \exp\left(-a \sum_I |F_I|^2\right)$$

In the $a \to \infty$ limit, the constraints are imposed exactly.

- Since all the constraints are imposed in terms of the CFT data, the above probability distribution over the theories can be viewed as a random matrix-tensor theory over matrices $\Delta_{i,j}, J_{i,j}, C_{i,j,k}$ (or equivalently, $h_i, \bar{h}_i, C_{i,j,k}$ for 2D CFTs)

$$\mathcal{Z} = \int D[h, \bar{h}]\, D[C]\; e^{-a\, S[h,\bar{h},C]}$$

- For 2D CFTs with non-universal light spectrum, the CFT data over light operators is not integrated over above, while the constraints are still imposed.

Tensor model for 3D gravity

- Imposing the 4-point crossing,

$$\sum_p C_{i_1 i_2 p} C_{i_3 i_4 p} \mathscr{F}(p \,|\, z) \overline{\mathscr{F}}(p \,|\, \bar{z}) = \sum_q C_{i_1 i_4 q} C_{i_2 i_3 q} \mathscr{F}(q \,|\, 1 - z) \overline{\mathscr{F}}(q \,|\, 1 - \bar{z})$$

- The Virasoro fusion kernel can be used to 'invert' the conformal blocks [Ponsot, Teschner '99]

$$\mathscr{F}_t(P_t \,|\, 1 - z) = \int_{\mathscr{C}} \frac{dP_s}{2} \mathbb{F}_{P_t P_s} \mathscr{F}_s(P_s \,|\, z)$$

- The crossing constraint can thereby be imposed in terms of the CFT data,

$$M^{i_1 i_2}_{i_3 i_4}(P_s) = \sum_q \left(C_{i_1 i_2 q} C^*_{i_3 i_4 q} \delta^{(2)} \left(P_s - P_q \right) - C_{i_1 i_4 q} C^*_{i_2 i_3 q} \left| \mathbb{F}_{P_s P_q} \right|^2 \right) = 0$$

- The fusion kernel is related to the Virasoro 6-j symbols,

$$|\mathbb{F}_{P_q P_s}|^2 := \left| \mathbb{F}_{P_q P_s} \begin{bmatrix} P_2 P_1 \\ P_3 P_4 \end{bmatrix} \right|^2 = \begin{Bmatrix} \mathcal{O}_q \mathcal{O}_2 \mathcal{O}_1 \\ \mathcal{O}_s \mathcal{O}_4 \mathcal{O}_3 \end{Bmatrix}$$

Tensor model for 3D gravity

- This gives rise to a particular term for our potential of the matrix-tensor theory,

$$V_4 = \sum_{i_1\cdots i_4}' \int_{\mathscr{C}} \left| M^{i_1 i_2}_{i_3 i_4} \right|^2 dP_s$$

$$= \sum_{i_1\cdots i_4}' \sum_{p,q} \left(C_{i_1 i_2 p} C^*_{i_3 i_4 p} C^*_{i_1 i_2 q} C_{i_3 i_4 q} \delta^{(2)}\left(P_p - P_q\right) - C_{i_1 i_2 p} C^*_{i_3 i_4 p} C^*_{i_1 i_4 q} C_{i_2 i_3 q} \begin{Bmatrix} \mathscr{O}_q & \mathscr{O}_2 & \mathscr{O}_1 \\ \mathscr{O}_p & \mathscr{O}_4 & \mathscr{O}_3 \end{Bmatrix} \right)$$

Tensor model for 3D gravity

- Imposing the 1-point modular crossing on torus,

$$M_j(P) = \sum_i \tilde{C}_{iij}\delta^{(2)}\left(P - P_j\right) - C_{iij}\left|\mathbb{S}\left[P_j\right]_{P_iP}\right|^2 = 0$$

$\mathbb{S}\left[P_j\right]_{P_iP}$ corresponds to modular S kernel

- Corresponding potential for the ensemble is given by,

$$V_{\mathrm{S}} = \sum_j \int_C \frac{d^2P}{4}\left|M_j(P)\right|^2$$

- The index j corresponds to external operator insertion, and can be identity operator $\mathbb{I}$. We separate the contributions corresponding to the identity,

$$V_{\mathrm{S}} = V_{\mathrm{S},\mathbb{I}} + V_{\mathrm{S},\not{\mathbb{I}}}$$

- The contribution of $\mathbb{I}$ imposes the asymptotic density of states corresponding to Cardy formula. The non-identity piece provides the quadratic potential for the OPE coefficients,

$$V_{\mathrm{S},\not{\mathbb{I}}} = {\sum_{i,j,k}}' C_{iij}C^*_{kkj}\left(\delta^{(2)}\left(P_i - P_k\right) - \left|\mathbb{S}\left[\mathcal{O}_j\right]_{P_iP_k}\right|^2\right)$$

Tensor model for 3D gravity

- The full potential for the matrix-tensor model can thereby be written as,

$$S = a\,\left(V_S + g_4\,V_4\right)$$

- The potential for the OPE coefficients of the heavy operators is given by,

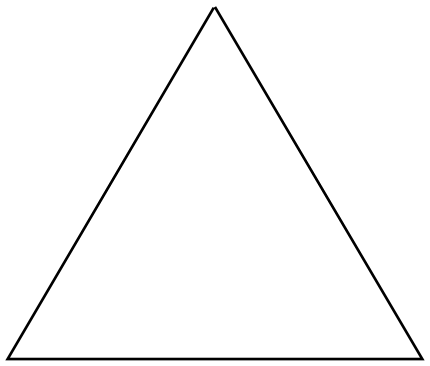
$$V_C = {\sum_{i,j,k}}' C_{iij} C^*_{kkj} \left(\delta^{(2)}\left(P_i - P_k\right) - \left| \mathbb{S}\left[\mathcal{O}_j\right]_{P_i P_k} \right|^2 \right) - 2 {\sum_{i,j,k}}' C_{ijk} C^*_{ijk} \begin{Bmatrix} \mathcal{O}_k & \mathcal{O}_i & \mathcal{O}_i \\ \mathbb{1} & \mathcal{O}_j & \mathcal{O}_j \end{Bmatrix}$$

$$+ {\sum_{i_1 \cdots i_4, p, q}}' \left(C_{i_1 i_2 p} C^*_{i_3 i_4 p} C^*_{i_1 i_2 q} C_{i_3 i_4 q} \delta^{(2)}\left(P_p - P_q\right) - C_{i_1 i_2 p} C^*_{i_3 i_4 p} C^*_{i_1 i_4 q} C_{i_2 i_3 q} \begin{Bmatrix} \mathcal{O}_q \mathcal{O}_2 \mathcal{O}_1 \\ \mathcal{O}_p \mathcal{O}_4 \mathcal{O}_3 \end{Bmatrix} \right)$$

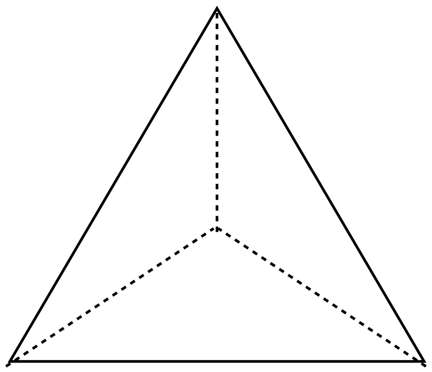
Simplicial gravity

[Ambjørn ’91; Godfrey, Gross ’91; Boulatov ’92]

- Fundamental field of the tensor model:



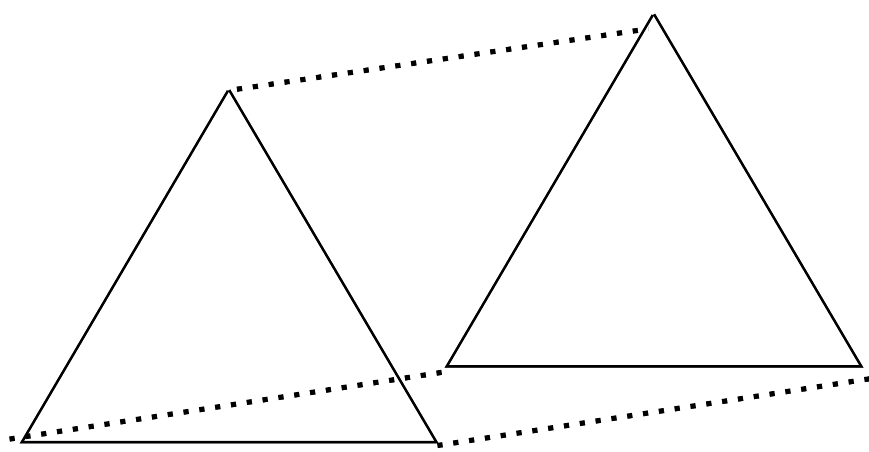
- Vertices of the tensor model



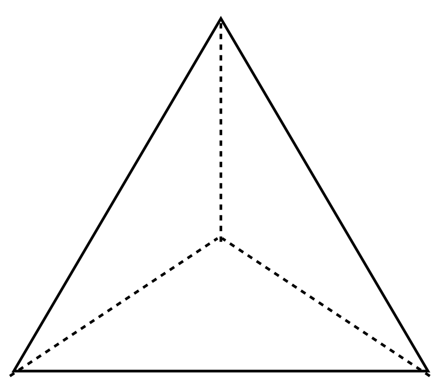
Simplicial gravity

$$V_C = \sum_{i,j,k}{}' C_{iij} C^*_{kkj} \left(\delta^{(2)}\left(P_i - P_k\right) - \left| \mathbb{S}\left[\mathcal{O}_j\right]_{P_i P_k} \right|^2 \right) - 2 \sum_{i,j,k}{}' C_{ijk} C^*_{ijk} \begin{Bmatrix} \mathcal{O}_k & \mathcal{O}_i & \mathcal{O}_i \\ \mathbb{1} & \mathcal{O}_j & \mathcal{O}_j \end{Bmatrix}$$

$$+ \sum_{i_1 \cdots i_4, p, q}{}' \left(C_{i_1 i_2 p} C^*_{i_3 i_4 p} C^*_{i_1 i_2 q} C_{i_3 i_4 q} \delta^{(2)}\left(P_p - P_q\right) - C_{i_1 i_2 p} C^*_{i_3 i_4 p} C^*_{i_1 i_4 q} C_{i_2 i_3 q} \begin{Bmatrix} \mathcal{O}_q \mathcal{O}_2 \mathcal{O}_1 \\ \mathcal{O}_p \mathcal{O}_4 \mathcal{O}_3 \end{Bmatrix} \right)$$



$$\sim \begin{Bmatrix} \mathcal{O}_k & \mathcal{O}_i & \mathcal{O}_i \\ \mathbb{1} & \mathcal{O}_j & \mathcal{O}_j \end{Bmatrix}$$

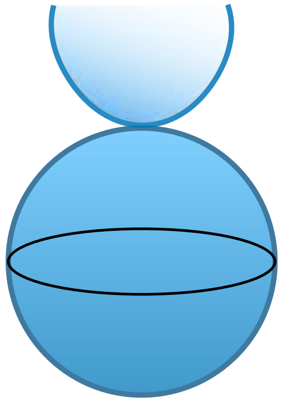


$$\sim \begin{Bmatrix} \mathcal{O}_q & \mathcal{O}_2 & \mathcal{O}_1 \\ \mathcal{O}_p & \mathcal{O}_4 & \mathcal{O}_3 \end{Bmatrix}$$

Summary

- I described how to analytically understand the emergence of Quantum ergodicity in physical quantum systems

  - EFT description of quantum ergodicity using the σ-model on $SU(2|2)/U(1|1) \times U(1|1)$ [arXiv: 2105.12129]



  - 2D CFTs [arXiv: 2306.xxxxx]

    - Discussing the approximate CFTs and their importance

    - Relevance of the non-Gaussian corrections to the Gaussian RMT in context of holography

    - Tensor model of 3D gravity corresponding to simplicity gravity.

THANK YOU!

C0 formula

$$C_0(P_i,P_j,P_k)=\frac{1}{\sqrt{2}}\frac{\Gamma_b(2Q)}{\Gamma_b(Q)^3}\frac{\prod\Gamma_b\left(\frac{Q}{2}\pm iP_i\pm iP_j\pm iP_k\right)}{\prod_{a\in\{i,j,k\}}\Gamma_b\left(Q+2iP_a\right)\Gamma_b\left(Q+2iP_a\right)}$$

$$C_0(P_i,P_j,P_k)\propto\frac{C_{\mathrm{DOZZ}}(P_i,P_j,P_k)}{\sqrt{\prod_l S_0(P_l)\rho_0(P_l)}}$$

Reflection coefficient in Liouville theory

Potential for spectrum

$$
\begin{aligned}
V_{\mathrm{S},\mathbb{I}} =& 256 \sum_i{}' \int \frac{d^2 P}{4} \left| \sinh(2\pi b P) \sinh\left(\frac{2\pi}{b} P\right) \right|^2 |\cosh(2\pi Q P_i)|^2 - 8 \sum_i{}' |\cosh(2\pi Q P_i)|^2 \\
& - 32 \sum_i{}' \left| \sinh(2\pi b P_i) \sinh\left(\frac{2\pi}{b} P_i\right) \right|^2 \\
+& \frac{1}{2} \sum_{i,k}{}' \left(\delta^{(2)}(P_i - P_k) - 16\, |\cos(4\pi P_i P_k)|^2 \right) + 32 \int \frac{d^2 P}{4} \sum_{i,k}{}' |\cos(4\pi P P_i) \cos(4\pi P P_k)|^2 \\
& + \cdots
\end{aligned}
$$