

*QCD and Collider Physics*  
*Lecture III: Parton branching, DGLAP and shower*  
*Monte Carlo*

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# *Parton branching*

- Solution to the Dirac Equation
- Branching Probabilities
- Derivation of DGLAP equation
- Solution to DGLAP equation using moments
- Formulation of DGLAP equation for shower Monte Carlo
- Angular ordering
- Chudakov effect
- Parton distributions for LHC
- Master formula for LHC cross sections

## Dirac eqn. Massless fermions

- We choose an explicit representation for the gamma matrices. The Bjorken and Drell representation is,

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}, \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix},$$

- The Weyl representation is more suitable at high energy (this is the definition used in Peskin and Schroder)

$$\gamma^0 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}, \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix},$$

- In the Weyl representation upper and lower components have different helicities.

$$\gamma_L = \frac{1}{2}(1 - \gamma_5) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \gamma_R = \frac{1}{2}(1 + \gamma_5) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix},$$

# Weyl representation

- Both representations satisfy the same commutation relations.

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

- in the Weyl representation  $\gamma^0 \gamma^i = \begin{pmatrix} -\sigma^i & \mathbf{0} \\ \mathbf{0} & \sigma^i \end{pmatrix}$ .  $\sigma$  are the Pauli matrices.

- The fermions involved in high energy processes can often be taken to be massless.

- To derive an explicit solution for the massless Dirac equation  $\not{p}u(p) = 0$ , write out an explicit expression for  $\not{p} = \gamma^0 p^0 - \gamma^1 p^1 - \gamma^2 p^2 - \gamma^3 p^3$  in the Weyl representation, NB  $p^\pm = p^0 \pm p^3$ , and for a massless particle

$$p^+ p^- = (p^1)^2 + (p^2)^2$$

$$\not{p} = \begin{pmatrix} 0 & 0 & p^- & -p^1 + ip^2 \\ 0 & 0 & -p^1 - ip^2 & p^+ \\ p^+ & p^1 - ip^2 & 0 & 0 \\ p^1 + ip^2 & p^- & 0 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 0 & p^- & -p^1 + ip^2 \\ 0 & 0 & -p^1 - ip^2 & p^+ \\ p^+ & p^1 - ip^2 & 0 & 0 \\ p^1 + ip^2 & p^- & 0 & 0 \end{pmatrix} \begin{bmatrix} -p^1 + ip^2 \\ p^+ \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

# Spinor solutions of Dirac equation

- The normalized massless spinor solns of Dirac eqn are

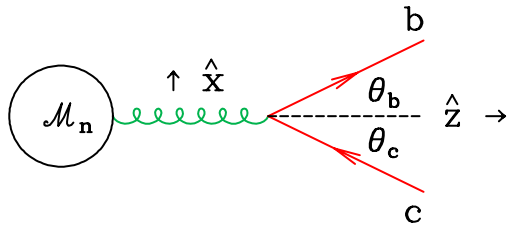
$$u_-(p) = \frac{1}{\sqrt{p^+}} \begin{bmatrix} -p^1 + ip^2 \\ p^+ \\ 0 \\ 0 \end{bmatrix}, \quad u_+(p) = \frac{1}{\sqrt{p^+}} \begin{bmatrix} 0 \\ 0 \\ p^+ \\ p^1 + ip^2 \end{bmatrix},$$

- For massless particles charge conjugation is almost trivial;  $u_{\pm}(p) = v_{\mp}(p)$
- In the Weyl representation the conjugate spinors are

$$u_-^\dagger(p) = \frac{1}{\sqrt{p^+}} [-p_1 - ip_2, p^+, 0, 0]$$

$$u_+^\dagger(p) = \frac{1}{\sqrt{p^+}} [0, 0, p^+, p^1 - ip^2]$$

# Parton branching - kinematics



$$p_a = \left( E_a + \frac{p_a^2}{4E_a}, 0, 0, E_a - \frac{p_a^2}{4E_a} \right)$$

$$p_b = (E_b, +E_b \sin \theta_b, 0, +E_b \cos \theta_b)$$

$$p_c = (E_c, -E_c \sin \theta_c, 0, +E_c \cos \theta_c)$$

- the kinematics and notation for the branching of parton  $a$  into  $b + c$ . We assume that

$$p_b^2, p_c^2 \ll p_a^2 \equiv t$$

- $a$  is an outgoing parton, which is called timelike branching since  $t > 0$ .
- The opening angle is  $\theta = \theta_b + \theta_c$ . Defining the energy fraction as

$$z = E_b/E_a = 1 - E_c/E_a ,$$

we have for small angles,  $t = 2E_b E_c (1 - \cos \theta) = z(1 - z)E_a^2 \theta^2$  so that  $\sqrt{t} = \sqrt{E_b E_c} \theta$ .

- using transverse momentum conservation  $E_b \theta_b = E_c \theta_c$

$$\theta = \frac{\theta_b}{1 - z} = \frac{\theta_c}{z}, \quad \theta_b - \theta_c = (1 - 2z)\theta$$

## Branching probabilities $g \rightarrow q\bar{q}$

■ Consider the case where

$$p_a = \left( E_a + \frac{p_a^2}{4E_a}, 0, 0, E_a - \frac{p_a^2}{4E_a} \right)$$

$$p_b \sim (E_b, +E_b\theta_b, 0, +E_b) \quad p_c \sim (E_c, -E_c\theta_c, 0, +E_c)$$

■ Thus for example

$$u_+^\dagger(p_b) = \sqrt{2E_b} \left[ 0, 0, 1, \frac{\theta_b}{2} \right], \quad u_+(p_c) \equiv v_-(p_c) = \sqrt{2E_c} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -\frac{\theta_c}{2} \end{bmatrix}$$

■ Hence for polarization vectors  $\varepsilon_{in} = (0, 1, 0, 0), \varepsilon_{out} = (0, 0, 1, 0)$

$$-gu_+^\dagger(p_b) \gamma^0 \gamma^1 v_-(p_c) = -g\sqrt{4E_b E_c} \begin{pmatrix} 1, \frac{\theta_b}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{\theta_c}{2} \end{pmatrix} = -g\sqrt{E_b E_c}(\theta_b - \theta_c)$$

$$-gu_+^\dagger(p_b) \gamma^0 \gamma^2 v_-(p_c) = -g\sqrt{4E_b E_c} \begin{pmatrix} 1, \frac{\theta_b}{2} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{\theta_c}{2} \end{pmatrix} = -ig\sqrt{E_b E_c}(\theta_b + \theta_c)$$

## Branching probabilities $g \rightarrow q\bar{q}$

$$-g\bar{u}_+(p_b)\not{\epsilon}^{\text{in}}v_-(p_c) = -g\sqrt{E_b E_c}(\theta_b - \theta_c) = -g(1 - 2z)\sqrt{t}$$

$$-g\bar{u}_+\not{\epsilon}^{\text{out}}v_-(p_c) = -ig\sqrt{E_b E_c}(\theta_b + \theta_c) = -ig\sqrt{z(1-z)}E_a\theta = -ig\sqrt{t}$$

and the matrix element relation for the branching is

$$|\mathcal{M}_{n+1}|^2 \sim \frac{g^2}{t} T_R F(z; \varepsilon_a, \lambda_b, \lambda_c) |\mathcal{M}_n|^2$$

where the colour factor is now  $\text{Tr}(t^A t^A)/8 = T_R = 1/2$ . The non-vanishing functions  $F(z; \varepsilon_a, \lambda_b, \lambda_c)$  for quark and antiquark helicities  $\lambda_b$  and  $\lambda_c$  are

$\varepsilon_a$	$\lambda_b$	$\lambda_c$	$F(z; \varepsilon_a, \lambda_b, \lambda_c)$
in	$\pm$	$\mp$	$(1 - 2z)^2$
out	$\pm$	$\mp$	1

Summing over the polarizations we get

$$2 \left[ (1 - 2z)^2 + 1 \right] = 4(z^2 + (1 - z)^2).$$

- Angular momentum argument for vanishing of amplitude in forward direction.



# Branching probabilities

The spin averaged splitting function integrated over azimuthal angle is,  $\int \frac{d\phi}{2\pi} F = \hat{P}_{ba}(z)$ .

$\hat{P}_{ba}(z)$  is the appropriate splitting function

$$d\sigma_{n+1} = d\sigma_n \frac{dt}{t} dz \frac{\alpha_S}{2\pi} \hat{P}_{ba}(z) + (\text{non - logarithmically enhanced terms})$$

- Including all the color factors we find the results for the unregulated branching probabilities.

$$\hat{P}_{qq}(z) = C_F \left[ \frac{1+z^2}{(1-z)} \right],$$

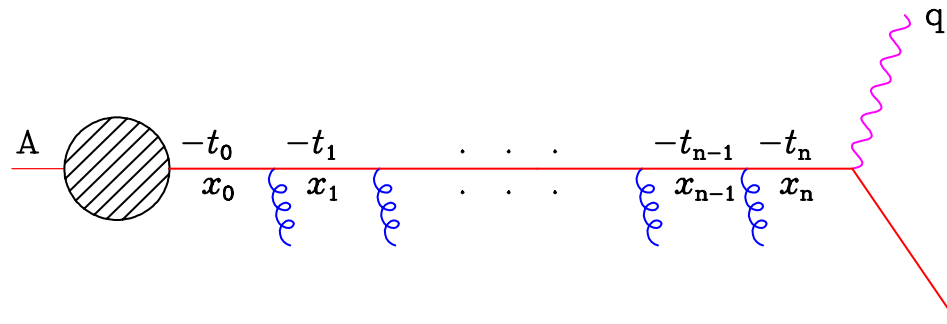
$$\hat{P}_{qg}(z) = T_R \left[ z^2 + (1-z)^2 \right], \quad T_R = \frac{1}{2},$$

$$\hat{P}_{gq}(z) = C_F \left[ \frac{1+(1-z)^2}{z} \right],$$

$$\hat{P}_{gg}(z) = C_A \left[ \frac{z}{(1-z)} + \frac{1-z}{z} + z(1-z) \right]$$

# DGLAP equation

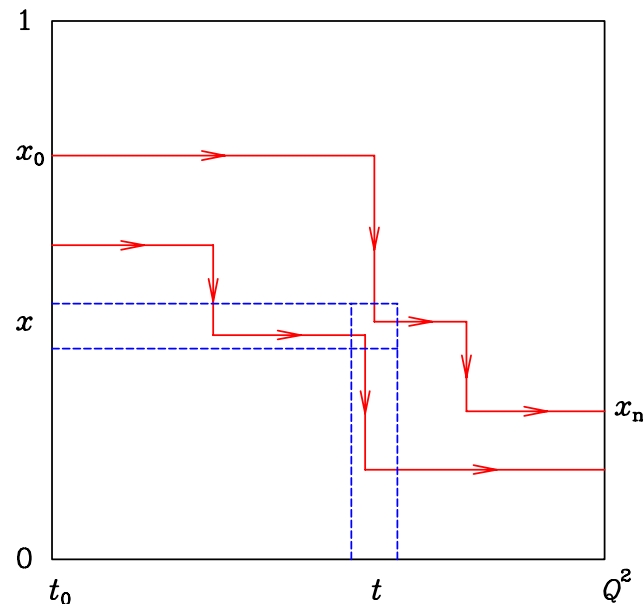
- Consider enhancement of higher-order contributions due to multiple small-angle parton emission, for example in deep inelastic scattering (DIS)



- Incoming quark from target hadron, initially with low virtual mass-squared  $-t_0$  and carrying a fraction  $x_0$  of hadron's momentum, moves to more virtual masses and lower momentum fractions by successive small-angle emissions, and is finally struck by photon of virtual mass-squared  $q^2 = -Q^2$ .
- Cross section will depend on  $Q^2$  and on momentum fraction distribution of partons seen by virtual photon at this scale,  $D(x, Q^2)$ .

# Branching paths

- To derive evolution equation for  $Q^2$ -dependence of  $D(x, Q^2)$ , first introduce pictorial representation of evolution.



- Represent sequence of branchings by path in  $(t, x)$ -space. Each branching is a step downwards in  $x$ , at a value of  $t$  equal to (minus) the virtual mass-squared after the branching.
- At  $t = t_0$ , paths have distribution of starting points  $D(x_0, t_0)$  characteristic of target hadron at that scale. Then distribution  $D(x, t)$  of partons at scale  $t$  is just the  $x$ -distribution of paths at that scale.

## Change in parton distribution

- Consider change in the parton distribution  $D(x, t)$  when  $t$  is increased to  $t + \delta t$ . This is number of paths arriving in element  $(\delta t, \delta x)$  minus number leaving that element, divided by  $\delta x$ .
- Number arriving is branching probability times parton density integrated over all higher momenta  $x' = x/z$ ,

$$\begin{aligned}\delta D_{\text{in}}(x, t) &= \frac{\delta t}{t} \int_x^1 dx' dz \frac{\alpha_S}{2\pi} \hat{P}(z) D(x', t) \delta(x - zx') \\ &= \frac{\delta t}{t} \int_0^1 \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) D(x/z, t)\end{aligned}$$

- For the number leaving element, must integrate over lower momenta  $x' = zx$ :

$$\begin{aligned}\delta D_{\text{out}}(x, t) &= \frac{\delta t}{t} D(x, t) \int_0^x dx' dz \frac{\alpha_S}{2\pi} \hat{P}(z) \delta(x' - zx) \\ &= \frac{\delta t}{t} D(x, t) \int_0^1 dz \frac{\alpha_S}{2\pi} \hat{P}(z)\end{aligned}$$

# Change in parton distribution – DGLAP equation

- Change in population of element is

$$\delta D(x, t) = \delta D_{\text{in}} - \delta D_{\text{out}} = \frac{\delta t}{t} \int_0^1 dz \frac{\alpha_S}{2\pi} \hat{P}(z) \left[ \frac{1}{z} D(x/z, t) - D(x, t) \right] .$$

- Introduce plus-prescription with definition

$$\int_0^1 dx f(x) [g(x)]_+ = \int_0^1 dx [f(x) - f(1)] g(x) .$$

Using this we can define regularized splitting function

$$P(z) = [\hat{P}(z)]_+ ,$$

- Plus-prescription, like the Dirac-delta function, is only defined under integral sign.
- Plus-prescription includes some of the effects of virtual diagrams.
- We obtain the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi ( DGLAP) evolution equation:

$$t \frac{\partial}{\partial t} D(x, t) = \int_x^1 \frac{dz}{z} \frac{\alpha_S}{2\pi} P(z) D(x/z, t) .$$

- Here  $D(x, t)$  represents parton momentum fraction distribution inside incoming hadron probed at scale  $t$ .
- In timelike branching, it represents instead hadron momentum fraction distribution produced by an outgoing parton. Boundary conditions and direction of evolution are different, but evolution equation remains the same.

# Quarks and gluons

- For several different types of partons, must take into account different processes by which parton of type  $i$  can enter or leave the element  $(\delta t, \delta x)$ . This leads to coupled DGLAP evolution equations of form

$$t \frac{\partial}{\partial t} D_i(x, t) = \sum_j \int_x^1 \frac{dz}{z} \frac{\alpha_S}{2\pi} P_{ij}(z) D_j(x/z, t) .$$

- Quark ( $i = q$ ) can enter element via either  $q \rightarrow qg$  or  $g \rightarrow q\bar{q}$ , but can only leave via  $q \rightarrow qg$ . Thus plus-prescription applies only to  $q \rightarrow qg$  part, giving

$$P_{qq}(z) = \hat{P}_{qq}(z)_+ = C_F \left( \frac{1+z^2}{1-z} \right)_+$$

$$P_{qg}(z) = \hat{P}_{qg}(z) = T_R [z^2 + (1-z)^2]$$

$$P_{gg}(z) = 2C_A \left[ \left( \frac{z}{1-z} + \frac{1}{2}z(1-z) \right)_+ + \frac{1-z}{z} + \frac{1}{2}z(1-z) \right] - \frac{2}{3}N_f T_R \delta(1-z) ,$$

$$P_{gq}(z) = P_{g\bar{q}}(z) = \hat{P}_{qg}(1-z) = C_F \frac{1+(1-z)^2}{z} .$$

- Using definition of the plus-prescription,  $P_{qq}$  and  $P_{gg}$  can be written in more common forms

$$P_{qq}(z) = C_F \left[ \frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right]$$

$$P_{gg}(z) = 2C_A \left[ \frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right] + \frac{1}{6} (11C_A - 4N_f T_R) \delta(1-z) .$$

# Solution by moments

- Given  $f_i(x, t)$  at some scale  $t = t_0$ , factorized structure of DGLAP equation means we can compute its form at any other scale.
- One strategy for doing this is to take moments (Mellin transforms) with respect to  $x$ :

$$\tilde{f}_i(N, t) = \int_0^1 dx x^{N-1} f_i(x, t) .$$

Inverse Mellin transform is  $f_i(x, t) = \frac{1}{2\pi i} \int_C dN x^{-N} \tilde{f}_i(N, t)$ . where contour  $C$  is parallel to imaginary axis to right of all singularities of integrand.

- After Mellin transformation, convolution in DGLAP equation becomes simply a product:

$$t \frac{\partial}{\partial t} \tilde{f}_i(N, t) = \sum_j \gamma_{ij}(N, \alpha_S) \tilde{f}_j(N, t)$$

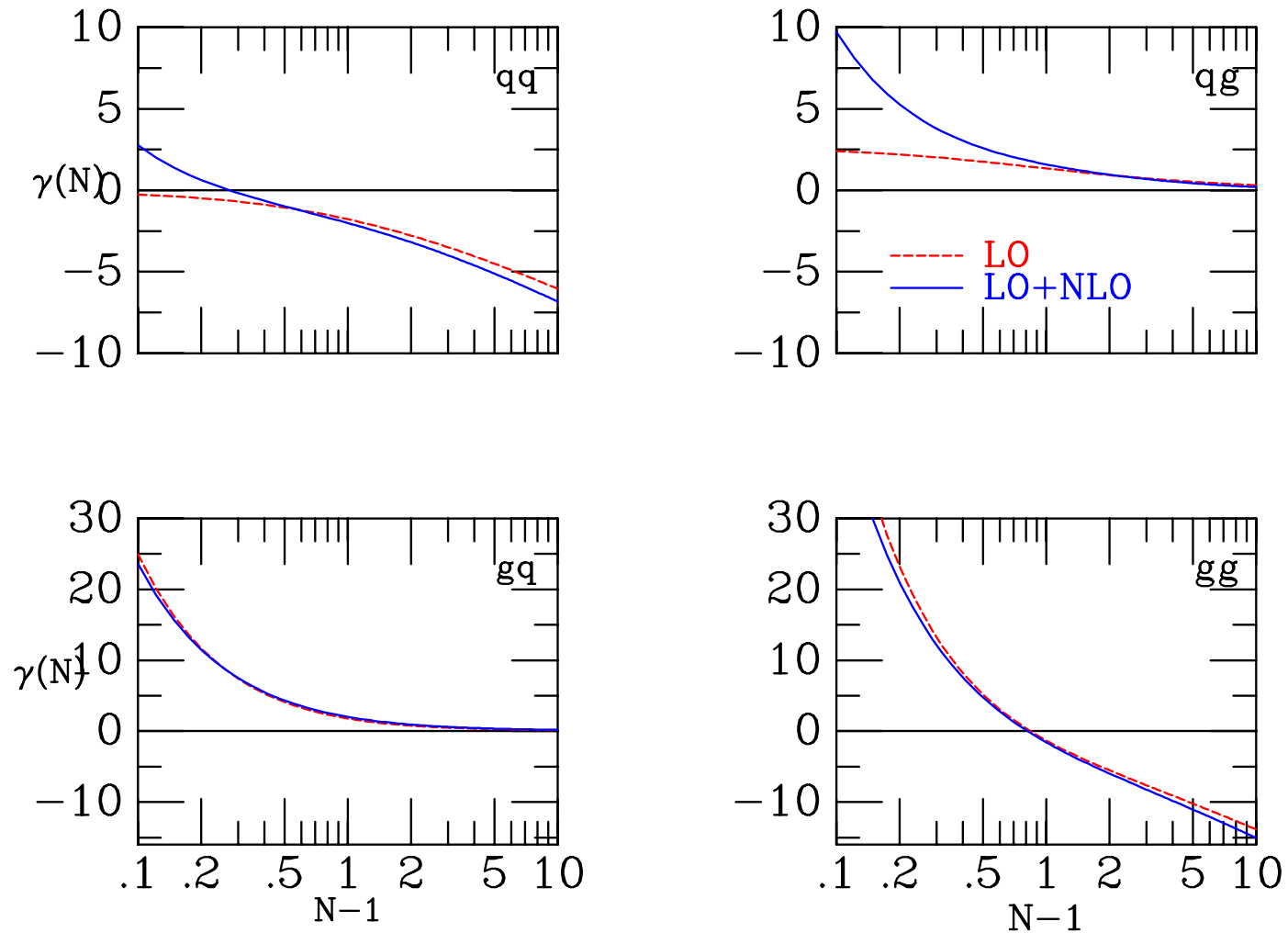
- Using the lowest order approximation for  $\gamma$  defined as

$$\gamma_{ij}^{(0)}(N) = \int_0^1 dx x^{N-1} P_{ij}(x)$$

$$t \frac{\partial}{\partial t} \tilde{f}_i(N, t) = \frac{\alpha_S}{2\pi} \sum_j \gamma_{ij}^{(0)}(N) \tilde{f}_j(N, t)$$



# Anomalous dimension matrix in moment space



- Behaviour of anomalous dimensions for real  $N$ ;
- At lowest order  $\gamma_{qq}(1) = 0$ ;
- behaviour of diagonal elements for large  $N$  is negative  $-\ln(N)$

# Scaling violation

- Consider combination of parton distributions which is flavour non-singlet, e.g.  $D_V = D_{q_i} - D_{\bar{q}_i}$  or  $D_{q_i} - D_{q_j}$ . Then mixing with the flavour-singlet gluons drops out and solution for fixed  $\alpha_S$  is

$$\tilde{D}_V(N, t) = \tilde{D}_V(N, t_0) \left( \frac{t}{t_0} \right)^{\gamma_{qq}(N, \alpha_S)},$$

- We see that dimensionless function  $D_V$ , instead of being scale-independent function of  $x$  as expected from dimensional analysis, has scaling violation: its moments vary like powers of scale  $t$  (hence the name anomalous dimensions).
- For running coupling  $\alpha_S(t)$ , scaling violation is power-behaved in  $\ln t$  rather than  $t$ . Using leading-order formula  $\alpha_S(t) = 1/b \ln(t/\Lambda^2)$ , we find

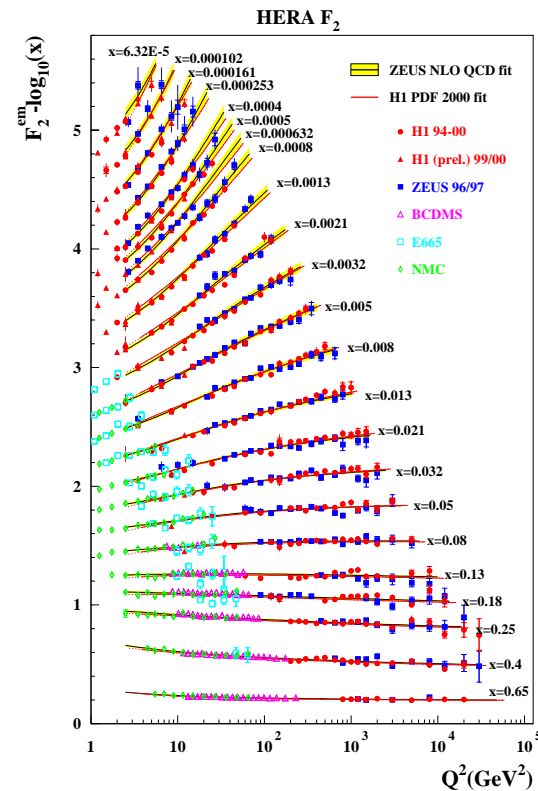
$$\tilde{D}_V(N, t) = \tilde{D}_V(N, t_0) \left( \frac{\alpha_S(t)}{\alpha_S(t_0)} \right)^{-d_{qq}(N)}$$

where  $d_{qq}(N) = \gamma_{qq}^{(0)}(N)/2\pi b$ .



$$\gamma_{qq}^0 = C_F \left[ -\frac{1}{2} + \frac{1}{(N(N+1))} - 2 \sum_{k=2}^N \frac{1}{k} \right]$$

# Combined data on $F_2$ proton



- Now  $d_{qq}(1) = 0$  and  $d_{qq}(N) < 0$  for all  $N > 1$ . Thus as  $t$  increases  $V$  decreases at large  $x$  and increases at small  $x$ . Physically, this is due to increase in the phase space for gluon emission by quarks as  $t$  increases, leading to loss of momentum. This is clearly visible in data:

# Sudakov form factor

- DGLAP equations are convenient for evolution of parton distributions. Expressed in terms of the unregulated branching probability we have,

$$t \frac{\partial}{\partial t} D(x, t) = \int_x^1 \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) D(x/z, t) - \int_0^1 dz \frac{\alpha_S}{2\pi} \hat{P}(z) D(x, t)$$

- To study structure of final states, slightly different form is useful. Consider again simplified treatment with only one type of branching. Introduce Sudakov form factor:

$$\Delta(t) \equiv \exp \left[ - \int_{t_0}^t \frac{dt'}{t'} \int dz \frac{\alpha_S}{2\pi} \hat{P}(z) \right] ,$$

$$t \frac{\partial}{\partial t} D(x, t) = \int \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) D(x/z, t) + \frac{D(x, t)}{\Delta(t)} t \frac{\partial}{\partial t} \Delta(t) ,$$

$$t \frac{\partial}{\partial t} \left( \frac{D}{\Delta} \right) = \frac{1}{\Delta} \int \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) D(x/z, t) .$$

# Sudakov form factor

- This is similar to DGLAP, except  $D$  replaced by  $D/\Delta$  and regularized splitting function  $P$  replaced by unregularized  $\hat{P}$ . Integrating,

$$D(x, t) = \Delta(t)D(x, t_0) + \int_{t_0}^t \frac{dt'}{t'} \frac{\Delta(t)}{\Delta(t')} \int \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) D(x/z, t') .$$

- This has simple interpretation. First term is contribution from paths that do not branch between scales  $t_0$  and  $t$ . Thus Sudakov form factor  $\Delta(t)$  is probability of evolving from  $t_0$  to  $t$  without branching. Second term is contribution from paths which have their last branching at scale  $t'$ . Factor of  $\Delta(t)/\Delta(t')$  is probability of evolving from  $t'$  to  $t$  without branching.
- Generalization to several species of partons straightforward. Species  $i$  has Sudakov form factor

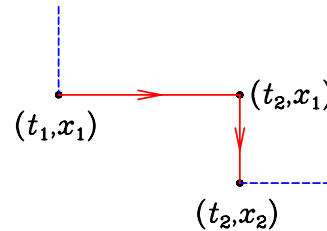
$$\Delta_i(t) \equiv \exp \left[ - \sum_j \int_{t_0}^t \frac{dt'}{t'} \int dz \frac{\alpha_S}{2\pi} \hat{P}_{ji}(z) \right] ,$$

which is probability of it evolving from  $t_0$  to  $t$  without branching. Then

$$t \frac{\partial}{\partial t} \left( \frac{D_i}{\Delta_i} \right) = \frac{1}{\Delta_i} \sum_j \int \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}_{ij}(z) D_j(x/z, t) .$$

# Monte Carlo method

- Monte Carlo branching algorithm operates as follows: given virtual mass scale and momentum fraction  $(t_1, x_1)$  after some step of the evolution, or as initial conditions, it generates values  $(t_2, x_2)$  after the next step.



- ★ Since probability of evolving from  $t_1$  to  $t_2$  without branching is  $\Delta(t_2)/\Delta(t_1)$ ,  $t_2$  can be generated with the correct distribution by solving  $\frac{\Delta(t_2)}{\Delta(t_1)} = \mathcal{R}$  where  $\mathcal{R}$  is random number (uniform on  $[0, 1]$ ).
- ★ If  $t_2$  is higher than hard process scale  $Q^2$ , this means branching has finished.
- ★ Otherwise, generate  $z = x_2/x_1$  with distribution proportional to  $(\alpha_S/2\pi)P(z)$ , where  $P(z)$  is appropriate splitting function, using a second random number  $\mathcal{R}'$

$$\int_{\epsilon}^{x_2/x_1} dz \frac{\alpha_S}{2\pi} P(z) = \mathcal{R}' \int_{\epsilon}^{1-\epsilon} dz \frac{\alpha_S}{2\pi} P(z)$$

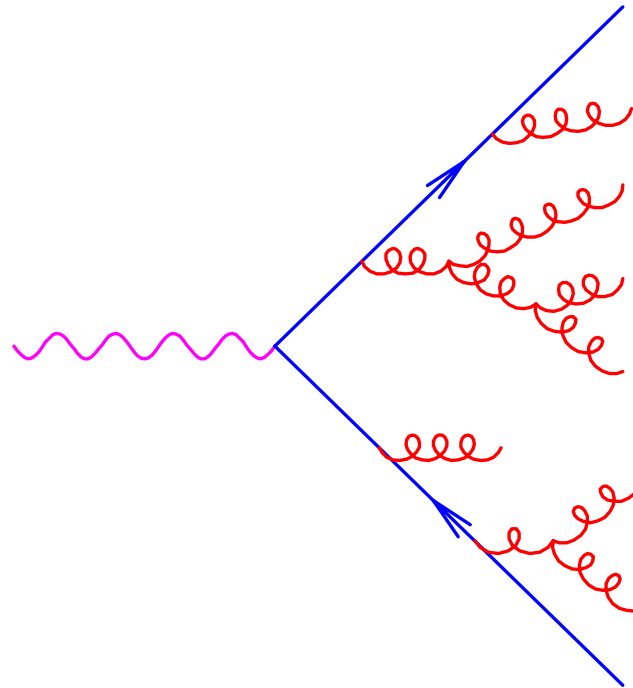
where  $\epsilon$  is cutoff for resolvable branching.

- In DIS,  $(t_i, x_i)$  values generated define virtual masses and momentum fractions of exchanged quark, from which momenta of emitted gluons can be computed. Azimuthal emission angles are then generated uniformly in the range  $[0, 2\pi]$ . More generally, e.g. when exchanged parton is a gluon, azimuths must be generated with polarization angular correlations.
- Each emitted (timelike) parton can itself branch. In that case  $t$  evolves downwards towards cutoff value  $t_0$ , rather than upwards towards hard process scale  $Q^2$ . Probability of evolving downwards without branching between  $t_1$  and  $t_2$  is now given by

$$\frac{\Delta(t_1)}{\Delta(t_2)} = \mathcal{R} .$$

Thus branching stops when  $\mathcal{R} < \Delta(t_1)$ .

# Parton Cascade



- Due to successive branching, parton cascade or shower develops. Each outgoing line is source of new cascade, until all outgoing lines have stopped branching. At this stage, which depends on cutoff scale  $t_0$ , outgoing partons have to be converted into hadrons via a hadronization model.



## Soft gluon emission

- Parton branching formalism discussed so far takes account of collinear enhancements to all orders in PT. There are also soft enhancements: When external line with momentum  $p$  and mass  $m$  (not necessarily small) emits a gluon with momentum  $q$ , the propagator factor is

$$\frac{1}{(p \pm q)^2 - m^2} = \frac{\pm 1}{2p \cdot q} = \frac{\pm 1}{2\omega E(1 - v \cos \theta)}$$

where  $\omega$  is emitted gluon energy,  $E$  and  $v$  are energy and velocity of parton emitting it, and  $\theta$  is angle of emission. This diverges as  $\omega \rightarrow 0$ , for any velocity and emission angle.

- Including numerator, soft gluon emission gives a colour factor times universal, spin-independent factor in amplitude

$$F_{\text{soft}} = \frac{p \cdot \epsilon}{p \cdot q}$$

where  $\epsilon$  is polarization of emitted gluon. For example, emission from quark gives numerator factor  $N \cdot \epsilon$ , where

## Soft gluon emission (continued)

$$\frac{N^\mu}{D} = \frac{(\not{p} + \not{q} + m)}{D} \gamma^\mu u(p) \rightarrow \frac{(2p^\mu - \gamma^\mu \not{p} + \gamma^\mu m)}{D} u(p) = p^\mu u(p) / p \cdot q .$$

(using Dirac equation for on-mass-shell spinor  $u(p)$  and  $D = (p + q)^2 - m^2 \rightarrow 2p \cdot q$ ).

- Universal factor  $F_{\text{soft}}$  coincides with classical eikonal formula for radiation from current  $p^\mu$ , valid in long-wavelength limit.
- No soft enhancement of radiation from off-mass-shell internal lines, since associated denominator factor  $(p + q)^2 - m^2 \rightarrow p^2 - m^2 \neq 0$  as  $\omega \rightarrow 0$ .
- Enhancement factor in amplitude for each external line implies cross section enhancement is sum over all pairs of external lines  $\{i, j\}$ :

$$d\sigma_{n+1} = d\sigma_n \frac{d\omega}{\omega} \frac{d\Omega}{2\pi} \frac{\alpha_S}{2\pi} \sum_{i,j} C_{ij} W_{ij}$$

where  $d\Omega$  is element of solid angle for emitted gluon,  $C_{ij}$  is a colour factor, and radiation function  $W_{ij}$  is given by

$$W_{ij} = \frac{\omega^2 p_i \cdot p_j}{p_i \cdot q p_j \cdot q} = \frac{1 - v_i v_j \cos \theta_{ij}}{(1 - v_i \cos \theta_{iq})(1 - v_j \cos \theta_{jq})} .$$

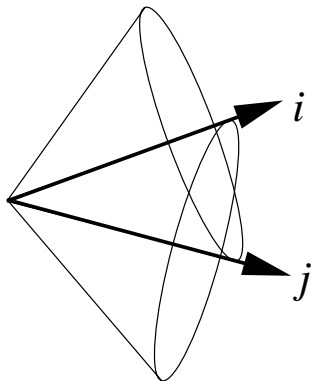
## Soft-Gluon emission (continued)

- Radiation function can be separated into two parts containing collinear singularities along lines  $i$  and  $j$ . Consider for simplicity massless particles,  $v_{i,j} = 1$ . Then  $W_{ij} = W_{ij}^i + W_{ij}^j$  where

$$W_{ij}^i = \frac{1}{2} \left( W_{ij} + \frac{1}{1 - \cos \theta_{iq}} - \frac{1}{1 - \cos \theta_{jq}} \right).$$

- This function has remarkable property of angular ordering. Write angular integration in polar coordinates w.r.t. direction of  $i$ ,  $d\Omega = d\cos \theta_{iq} d\phi_{iq}$ . Performing azimuthal integration, we find

$$\int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} W_{ij}^i = \frac{1}{1 - \cos \theta_{iq}} \quad \text{if } \theta_{iq} < \theta_{ij}, \text{ otherwise } 0.$$



Thus, after azimuthal averaging, contribution from  $W_{ij}^i$  is confined to cone, centred on direction of  $i$ , extending in angle to direction of  $j$ . Similarly,  $W_{ij}^j$ , averaged over  $\phi_{jq}$ , is confined to cone centred on line  $j$  extending to direction of  $i$ .

# Angular ordering

- To prove angular ordering property, write

$$1 - \cos \theta_{jq} = a - b \cos \phi_{iq}$$

where

$$a = 1 - \cos \theta_{ij} \cos \theta_{iq}, \quad b = \sin \theta_{ij} \sin \theta_{iq}.$$

Defining  $z = \exp(i\phi_{iq})$ , we have

$$I_{ij}^i \equiv \int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} \frac{1}{1 - \cos \theta_{jq}} = \frac{1}{i\pi b} \oint \frac{dz}{(z_+ - z)(z - z_-)}$$

where  $z$ -integration contour the unit circle and  $z_{\pm} = \frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1}$  Now only pole at  $z = z_-$  can lie inside unit circle, so

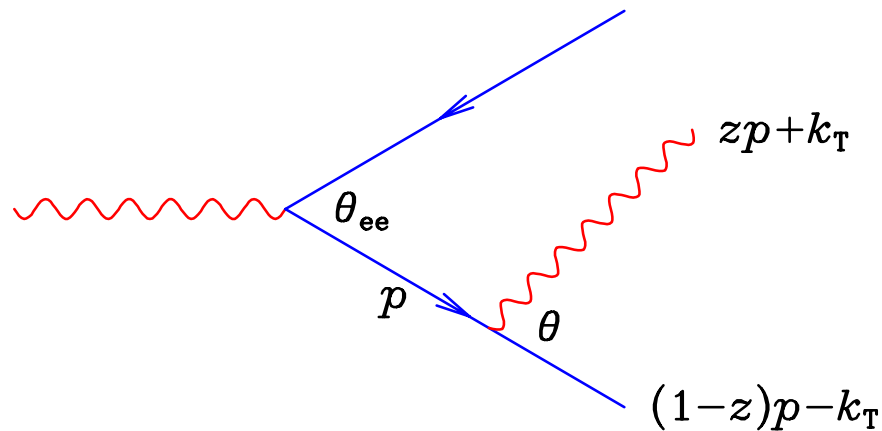
$$I_{ij}^i = \sqrt{\frac{1}{a^2 - b^2}} = \frac{1}{|\cos \theta_{iq} - \cos \theta_{ij}|}.$$

## Angular ordering (continued)

$$\begin{aligned}\int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} W_{ij}^i &= \frac{1}{2(1 - \cos \theta_{iq})} [1 + (\cos \theta_{iq} - \cos \theta_{ij}) I_{ij}^i] \\ &= \frac{1}{1 - \cos \theta_{iq}} \quad \text{if } \theta_{iq} < \theta_{ij}, \text{ otherwise } 0.\end{aligned}$$

- Angular ordering is coherence effect common to all gauge theories. In QED it causes Chudakov effect – suppression of soft bremsstrahlung from  $e^+e^-$  pairs, which has simple explanation in old-fashioned (time-ordered) perturbation theory.

# Chudakov effect



- Consider emission of soft photon at angle  $\theta$  from electron in pair with opening angle  $\theta_{ee} < \theta$ . For simplicity assume  $\theta_{ee}, \theta \ll 1$ .
- Transverse momentum of photon is  $k_T \sim zp\theta$  and energy imbalance at  $e \rightarrow e\gamma$  vertex is

$$\Delta E \sim k_T^2 / zp \sim zp\theta^2 .$$

- Time available for emission is  $\Delta t \sim 1/\Delta E$ . In this time transverse separation of pair will be  $\Delta b \sim \theta_{ee}\Delta t$ .
- For non-negligible probability of emission, photon must resolve this transverse separation of pair, so

$$\Delta b > \lambda/\theta \sim (zp\theta)^{-1}$$

where  $\lambda$  is photon wavelength.

## Chudakov effect (continued)

- This implies that  $\theta_{ee}(zp\theta^2)^{-1} > (zp\theta)^{-1}$ , and hence  $\theta_{ee} > \theta$ .  
Thus soft photon emission is suppressed at angles larger than opening angle of pair, which is angular ordering.
- Photons at larger angles cannot resolve electron and positron charges separately – they see only total charge of pair, which is zero, implying no emission.
- More generally, if  $i$  and  $j$  come from branching of parton  $k$ , with (colour) charge  $Q_k = Q_i + Q_j$ , then radiation outside angular-ordered cones is emitted coherently by  $i$  and  $j$  and can be treated as coming directly from (colour) charge of  $k$ .

# Coherent branching

- Angular ordering provides basis for coherent parton branching formalism, which includes leading soft gluon enhancements to all orders.
- In place of virtual mass-squared variable  $t$  in earlier treatment, use angular variable

$$\zeta = \frac{p_b \cdot p_c}{E_b E_c} \simeq 1 - \cos \theta$$

as evolution variable for branching  $a \rightarrow bc$ , and impose angular ordering  $\zeta' < \zeta$  for successive branchings. Iterative formula for  $n$ -parton emission becomes

$$d\sigma_{n+1} = d\sigma_n \frac{d\zeta}{\zeta} dz \frac{\alpha_S}{2\pi} \hat{P}_{ba}(z) .$$

Thus for the coherent branching process, the Sudakov form factor of the quark, for example, takes the form

$$\tilde{\Delta}_q(\tilde{t}) = \exp \left\{ - \int_{4t_0}^{\tilde{t}} \frac{dt'}{t'} \int_{\sqrt{t_0/t'}}^{1-\sqrt{t_0/t'}} \frac{dz}{2\pi} \alpha_S (z^2 (1-z)^2 t') \hat{P}_{qq}(z) \right\} .$$

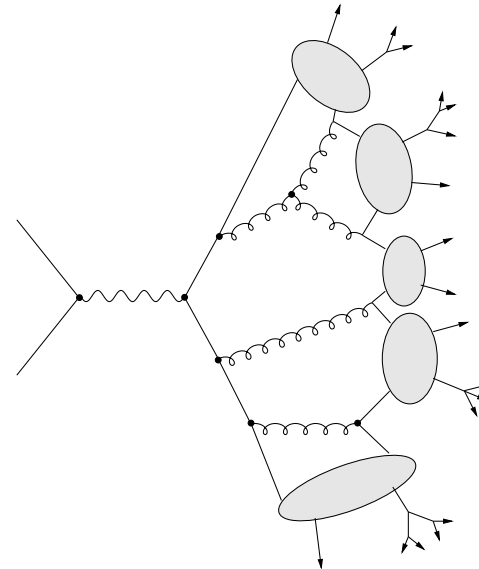
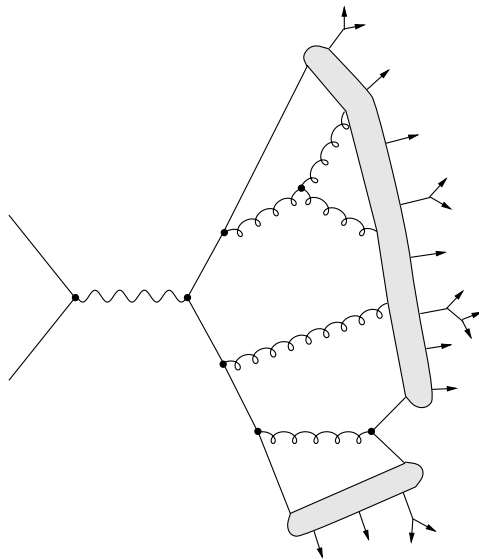
At large  $\tilde{t}$  this falls more slowly than the Sudakov factor without angular ordering, but still faster than any negative power of  $\tilde{t}$ .

- The slower fall implies less branching, which is due to the suppression of soft gluon emission by angular ordering.



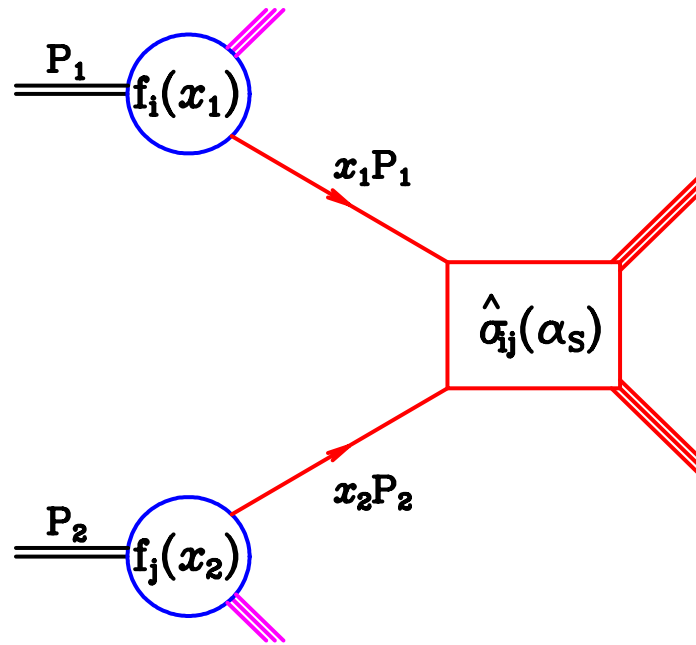
# Hadronization models/Underlying event

- Real events consist of partons, not hadrons;
- Since this is a non-perturbative process at low momentum scale, Monte Carlo generators use hadronization models;
- One widely-used model involves stretching a colour string across quarks and gluons, and breaking it up into hadrons. This string type of hadronization is implemented in the Pythia event generator;
- Another model breaks each gluon into a  $q\bar{q}$  pair and then groups quarks and anti-quarks into colourless 'clusters', which then give the hadrons. This cluster type hadronization is implemented in the Herwig event generator;



- A final detail needed in Monte Carlo is the underlying event. This is normally modelled through multiple extra  $2 \rightarrow 2$  scatterings.

# Ingredients for a parton calculation at LHC



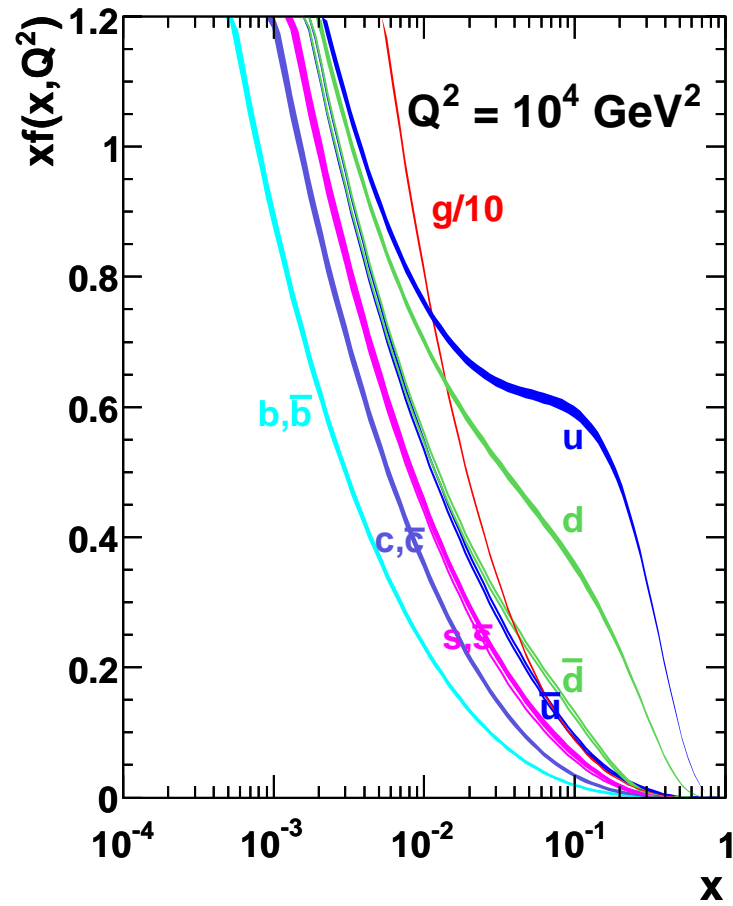
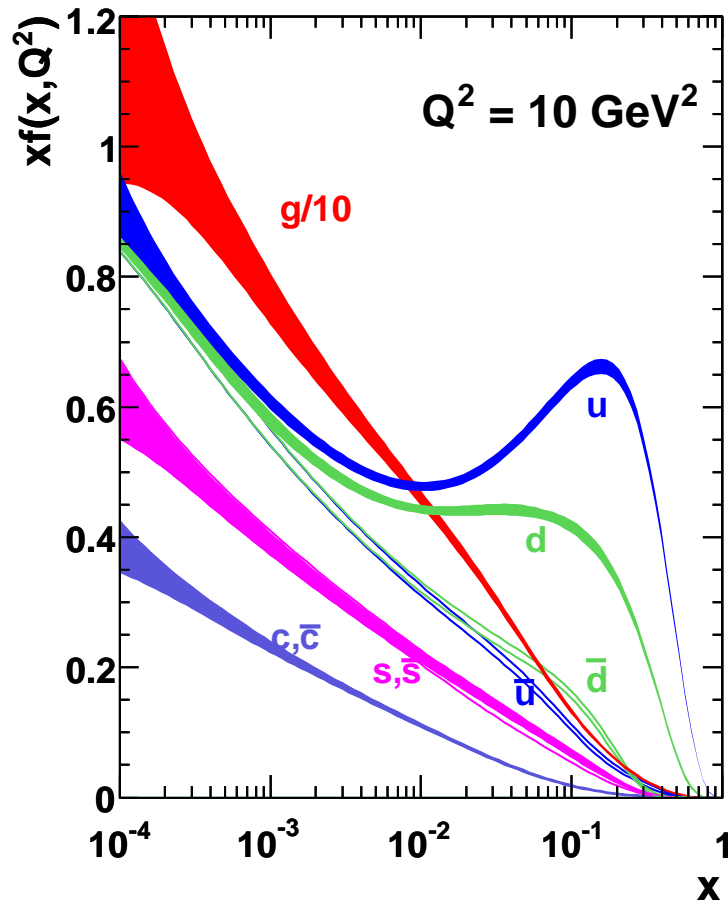
## ■ Factorization formula

$$\sigma(S) = \sum_{i,j} \int dx_1 dx_2 f_i(x_1, \mu^2) f_j(x_2, \mu^2) \hat{\sigma}_{ij}(\hat{s} = x_1 x_2 S, \alpha_S(\mu^2), Q^2/\mu^2)$$

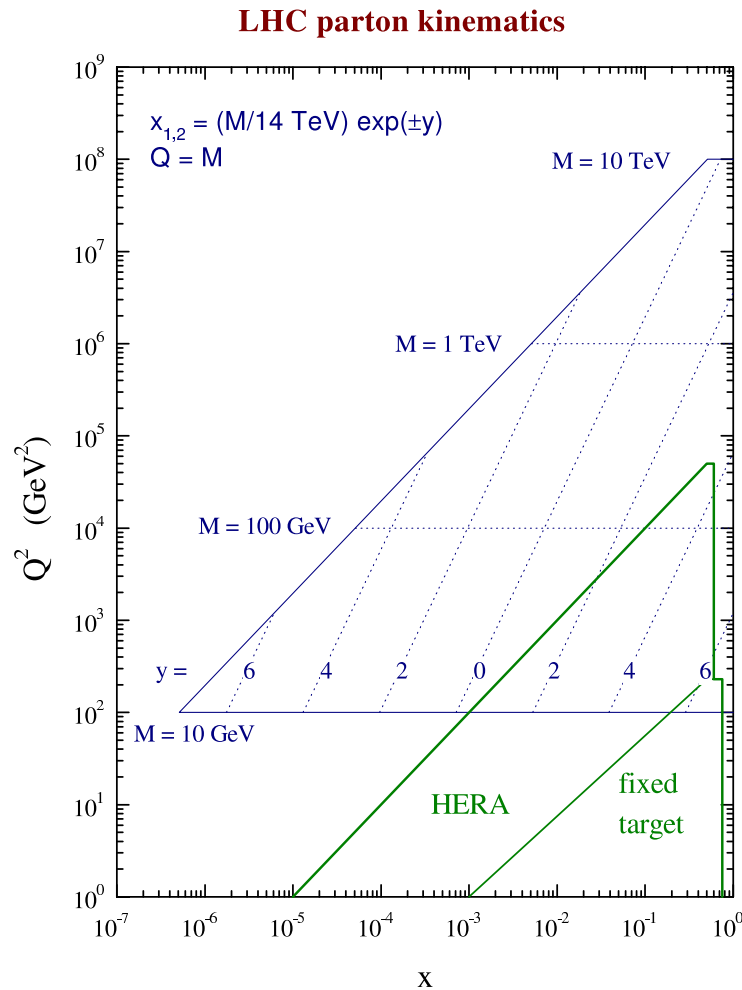
- Non-perturbative parton distributions  $f_i(x, \mu^2)$  with calculable scale dependence.
- Short distance cross section that depends on  $\alpha_s$  and factorization scale  $\mu$ .
- Value of the coupling  $\alpha_s$  with known scale dependence.

# Parton distributions

## MSTW 2008 NLO PDFs (68% C.L.)



# Extrapolation to LHC energies



- Determination of parton distribution functions requires fitting to data.
- Production of an object of Mass  $M$  and rapidity  $y$  probes parton distribution functions at  $x_{1,2} = M/\sqrt{s} \exp(\pm y)$ .

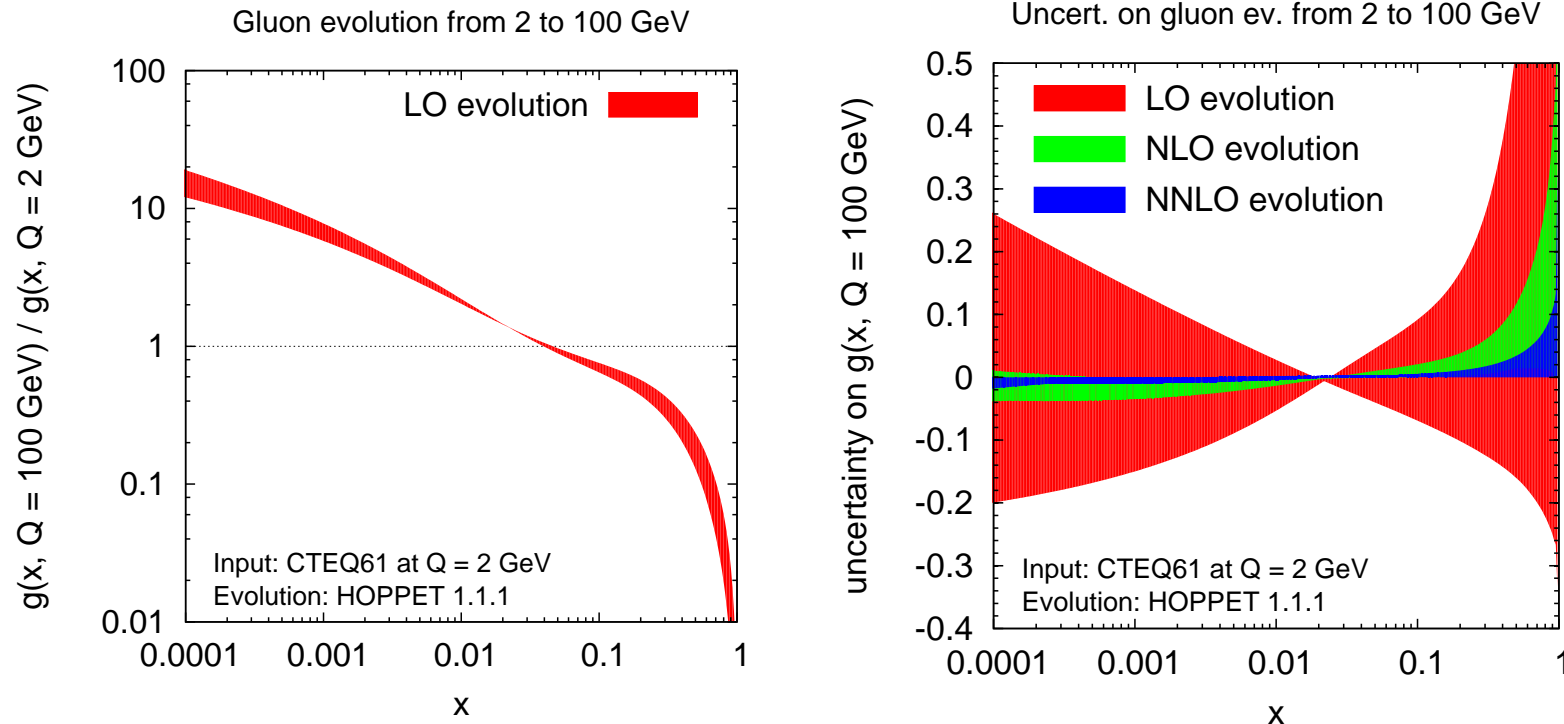
$$p_1 = \frac{\sqrt{s}}{2} (x_1, 0, 0, +x_1)$$

$$p_2 = \frac{\sqrt{s}}{2} (x_2, 0, 0, -x_2)$$

$$p_3 = (M \cosh y, 0, 0, M \sinh y)$$

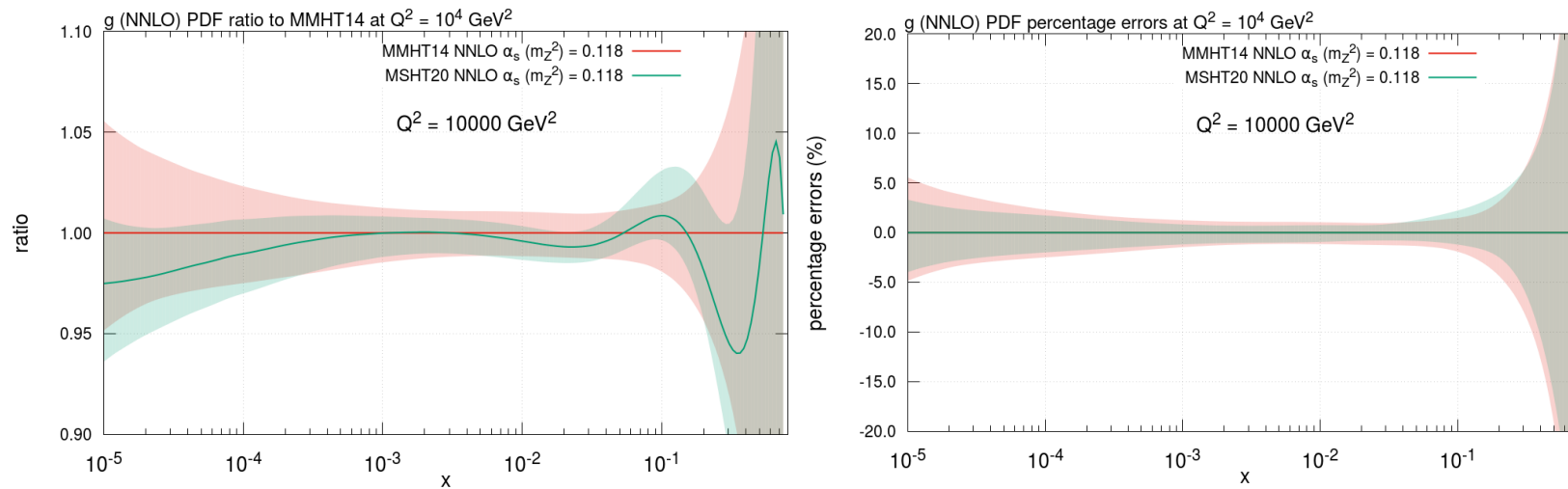
- Large extrapolation from HERA/fixed target to LHC, performed using DGLAP evolution.

# Gluon distribution at LHC energies



- The left-hand figure shows the change in the gluon distribution going from  $Q = 2 \text{ GeV}$  to  $Q = 100 \text{ GeV}$ .
- The right-hand figure shows the theoretical uncertainty band estimated using a scale variations of  $Q/2, Q, 2Q$  with LO, NLO and NNLO evolution.

# Gluon distribution and its errors



- Left figure shows the change from 2014 to 2020 (MHST) from 2012.04684.
- Data sets included are final HERA combined data on total and heavy flavour structure functions, final Tevatron data and a significant number of new LHC 7 and 8 TeV data sets on vector boson production, inclusive jets and top quark distributions.
- NNLO QCD corrections included for most data sets.
- NNLO fit is strongly favoured over the NLO.
- Modern parton distributions come with procedures to estimate errors;
- Through most of the range the errors on the gluon distribution at the level of a few percent.

# Recap

- Spinor solutions of the massless Dirac equation are classified by helicity and are simple.
- Parton evolution can be represented as a branching process from higher values of  $x$ ;
- Amplitudes for parton branching are proportional to  $\sqrt{t}$  leading to  $dt/t$  behavior for branching probability.
- DGLAP equation controls the change of parton distributions with scale. It predicts growth at small  $x$  and shrinkage at large  $x$  with increasing  $Q^2$ .
- The Sudakov factor  $\Delta(t, t_0)$  predicts the probability of no-branching between  $t_0$  and  $t$
- Angular ordering of soft radiation put a further dynamical constraint on the parton shower.
- Parton distributions for LHC involve extrapolations from HERA and fixed target energies, performed using the DGLAP equation and also LHC data. Errors below 5%.