QCD and Collider Physics
Lecture IV: Applications of the QCD parton model and modern techniques for tree graphs

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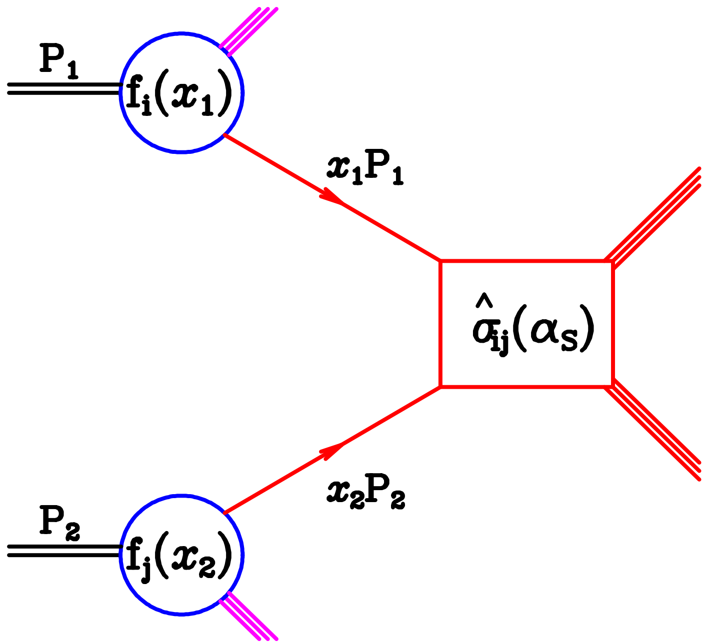
Plan

- Factorization formula at Hadron colliders
- Parton luminosity
- Lepton pair production and QCD radiative corrections
- Top properties and LO production
- Top at NLO and NNLO
- Higgs production at the LHC
- Spinor methods for tree graphs.

Ingredients for a parton calculation



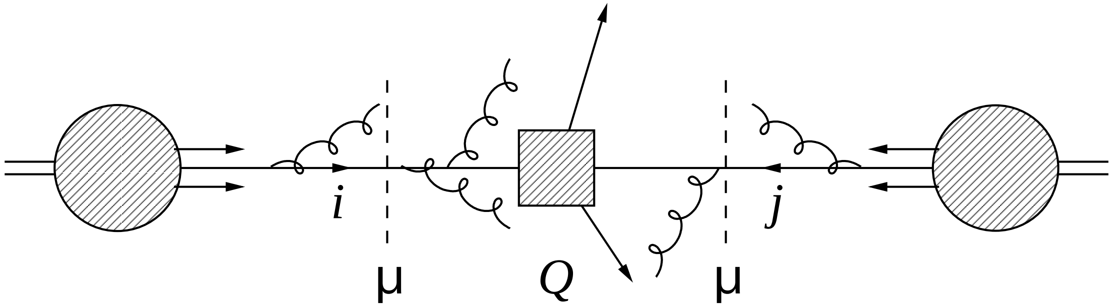
- Factorization formula

$$\sigma(S) = \sum_{i,j} \int dx_1 dx_2 f_i(x_1, \mu^2) f_j(x_2, \mu^2) \hat{\sigma}_{ij}(\hat{s} = x_1 x_2 S, \alpha_s(\mu^2), Q^2/\mu^2)$$

- Non-perturbative parton distributions $f_i(x, \mu^2)$ with calculable scale dependence.
- Short distance cross section that depends on α_s and factorization scale μ.
- Value of the coupling α_s with known scale dependence.

Hadron-hadron processes

- In hard hadron-hadron scattering, constituent partons from each incoming hadron interact at short distance (large momentum transfer Q^2).



- For hadron momenta P_1, P_2 ($S = 2P_1 \cdot P_2$), form of cross section is

$$\sigma(S) \quad = \quad \sum_{i,j} \int dx_1 dx_2 D_i(x_1, \mu^2) D_j(x_2, \mu^2) \hat{\sigma}_{ij}(\hat{s} = x_1 x_2 S, \alpha_s(\mu^2), Q^2/\mu^2)$$

where μ^2 is factorization scale and $\hat{\sigma}_{ij}$ is subprocess cross section for parton types i, j.

  - ⋆ Notice that factorization scale is in principle arbitrary: affects only what we call part of subprocess or part of initial-state evolution (parton shower).
  - ⋆ Unlike e^+e^- or ep, we may have interaction between spectator partons, leading to soft underlying event and/or multiple hard scattering.

Factorization of the cross section

- Why does the factorization property hold and when it should fail?
- For a heuristic argument Consider the simplest hard process involving two hadrons

$$H_1(P_1) + H_2(P_2) \to V + X.$$

- Do the partons in hadron H_1, through the influence of their colour fields, change the distribution of partons in hadron H_2 before the vector boson is produced? Soft gluons which are emitted long before the collision are potentially troublesome.
- A simple model from classical electrodynamics. The vector potential due to an electromagnetic current density J is given by

$$A^\mu(t,\vec{x}) = \int dt' d\vec{x}' \ \frac{J^\mu(t',\vec{x}')}{|\vec{x}-\vec{x}'|} \ \delta(t' + |\vec{x}-\vec{x}'| - t) \ ,$$

where the delta function provides the retarded behaviour required by causality.

- Consider a particle with charge e travelling in the positive z direction with constant velocity β. The non-zero components of the current density are

$$
\begin{aligned}
J^t(t',\vec{x}') &= e\delta(\vec{x}' - \vec{r}(t'))\ , \\
J^z(t',\vec{x}') &= e\beta\delta(\vec{x}' - \vec{r}(t')),\ \ \vec{r}(t') = \beta t'\hat{z},
\end{aligned}
$$

$\hat{z}$ is a unit vector in the z direction. At an observation point (the supposed position of hadron H_2) described by coordinates x, y and z, the vector potential (either performing the integrations using the current density given above, or by Lorentz transformation of the scalar potential in the rest frame of the particle) is

$$
\begin{aligned}
A^t(t,\vec{x}) &= \frac{e\gamma}{\sqrt{[x^2+y^2+\gamma^2(\beta t - z)^2]}} \\
A^x(t,\vec{x}) &= 0 \\
A^y(t,\vec{x}) &= 0 \\
A^z(t,\vec{x}) &= \frac{e\gamma\beta}{\sqrt{[x^2+y^2+\gamma^2(\beta t - z)^2]}},
\end{aligned}
$$

where $\gamma^2 = 1/(1-\beta^2)$. Target hadron H_2 is at rest near the origin, so that $\gamma \approx s/m^2$.

- Note that for large γ and fixed non-zero $(\beta t - z)$ some components of the potential tend to a constant independent of γ, suggesting that there will be non-zero fields which are not in coincidence with the arrival of the particle, even at high energy.
- However at large γ the potential is a pure gauge piece, $A^\mu = \partial^\mu \chi$ where χ is a scalar function
- Covariant formulation using the vector potential A has large fields which have no effect.
- For example, the electric field along the z direction is

$$E^z(t,\vec{x}) = F^{tz} \equiv \frac{\partial A^z}{\partial t} + \frac{\partial A^t}{\partial z} = \frac{e\gamma(\beta t - z)}{[x^2 + y^2 + \gamma^2(\beta t - z)^2]^{\frac{3}{2}}}.$$

The leading terms in γ cancel and the field strengths are of order $1/\gamma^2$ and hence of order m^4/s^2. The model suggests the force experienced by a charge in the hadron H_2, at any fixed time before the arrival of the quark, decreases as m^4/s^2.

Parton luminosity

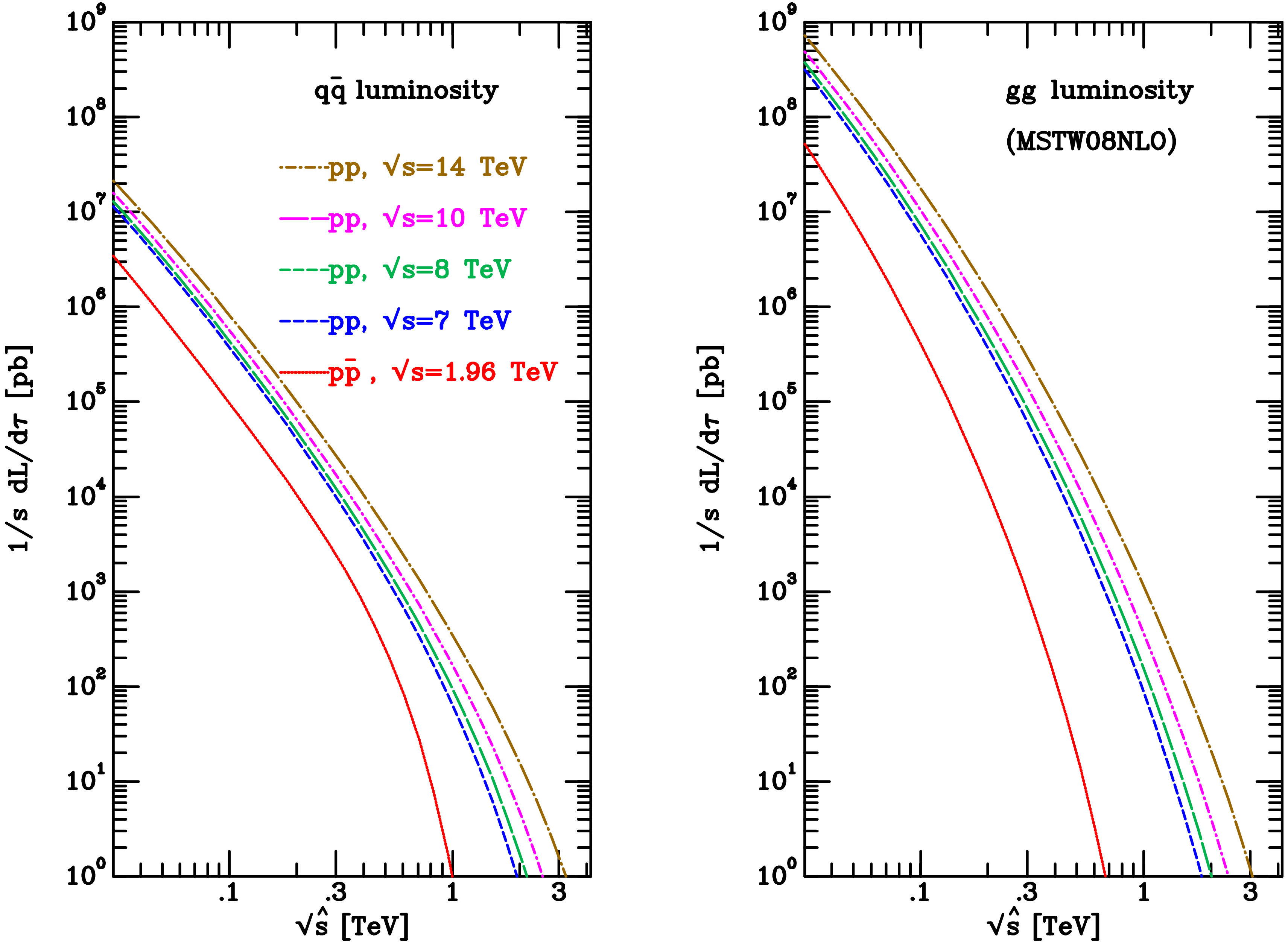
- Parton luminosity is determined by the parton distribution functions, $f_i(x_1, \mu^2)$ and $f_j(x_2, \mu^2)$.
- $f_j(x_i, \mu^2)$ need to be determined by data.
- the available centre-of-mass energy-squared of the parton-parton collision, $\hat{s}$, is less than the overall hadron-hadron collision energy, s, by a factor of $x_1 x_2 \equiv \tau$.
- Define differential parton luminosities

$$\tau \frac{dL_{ij}}{d\tau} = \frac{1}{1+\delta_{ij}} \int_0^1 dx_1 dx_2$$
$$\times \Big[\big(x_1 f_i(x_1, \mu^2)\; x_2 f_j(x_2, \mu^2) \big) + (1 \leftrightarrow 2) \Big] \delta(\tau - x_1 x_2).$$

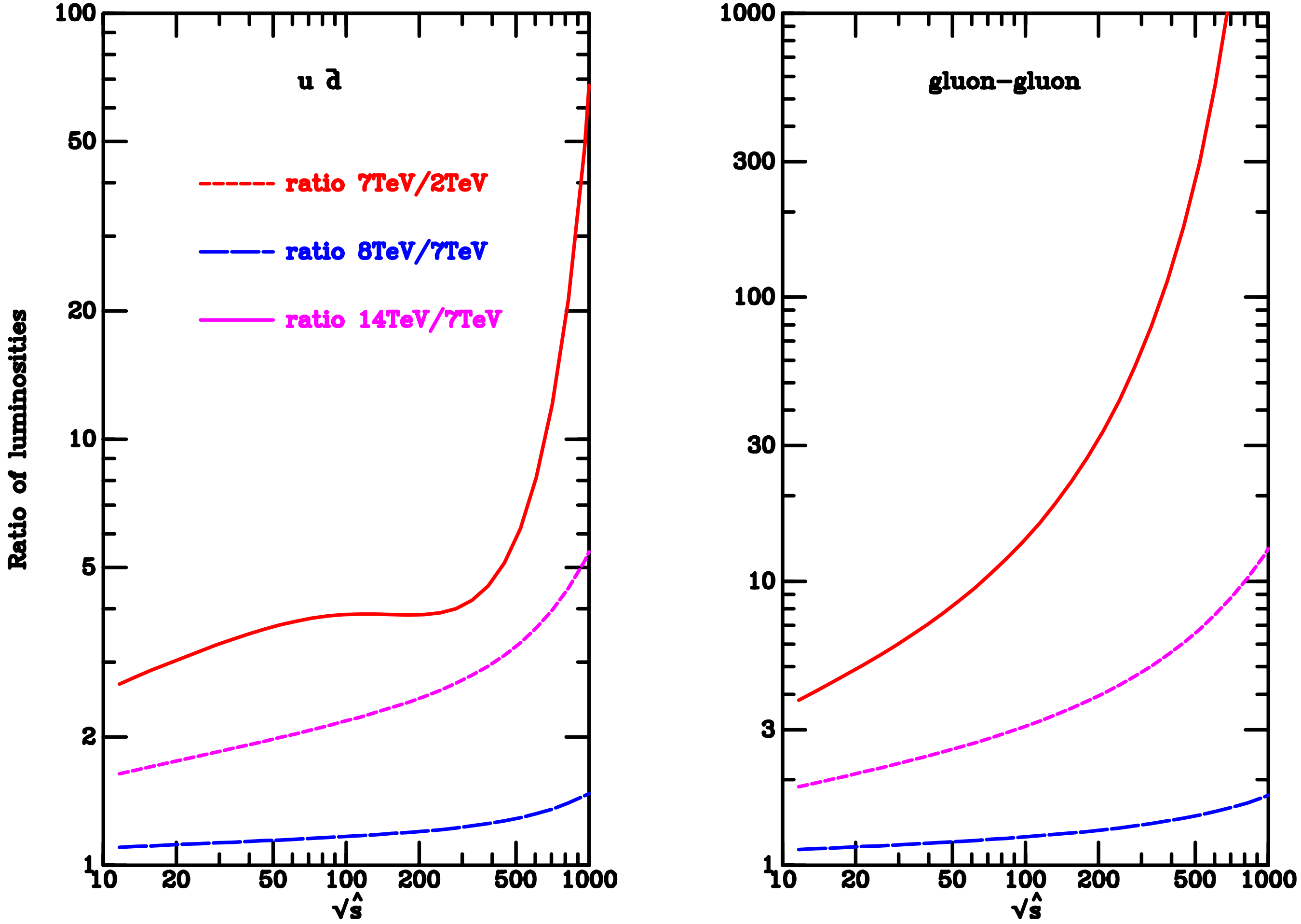
- The collider luminosity is quite distinct from the parton luminosity. The former is a property of a machine, whereas the latter is a property of the proton.
- We now assume that $\hat{\sigma}$ depends only on $\hat{s}$.

$$\sigma(s) = \sum_{\{ij\}} \int_{\tau_0}^1 \frac{d\tau}{\tau} \left[\frac{1}{s} \frac{dL_{ij}}{d\tau} \right] \left[\hat{s} \hat{\sigma}_{ij} \right],$$

Parton luminosity

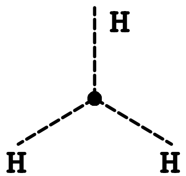


Ratios of luminosities

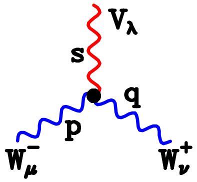


Boson rules

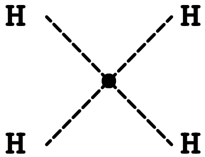
- 3 and 4 point vertices determined by the non-abelian term in the field strength.



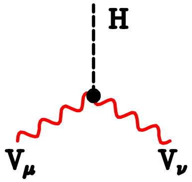
$-6i\lambda v \qquad (\lambda = \frac{M_H^2}{2v^2})$



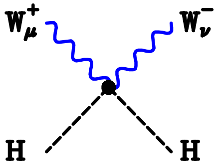
$-ig_V\ [(p-q)_\lambda g_{\mu\nu}+(q-s)_\mu g_{\nu\lambda}+(s-p)_\nu g_{\lambda\mu}]$
(all momenta incoming,
$g_A=e,\ g_Z=g_W\ \cos\theta_W$)



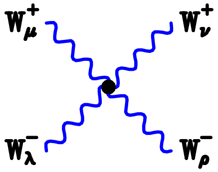
$-6i\lambda$



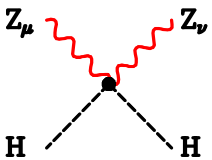
$+ig_{VH}M_W g_{\mu\nu}$
($g_{WH}=g_W,\ g_{ZH}=g_W/\cos^2\theta_W$)



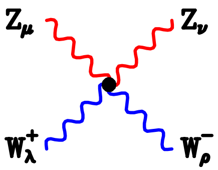
$\frac{ig_W^2}{2}\ g_{\mu\nu}$



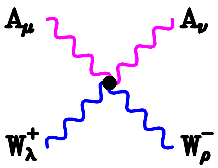
$+ig_W^2[2g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}]$



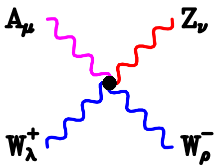
$\frac{ig_W^2}{2\cos^2\theta_W}\ g_{\mu\nu}$



$-ig_W^2\cos^2\theta_W[2g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}]$



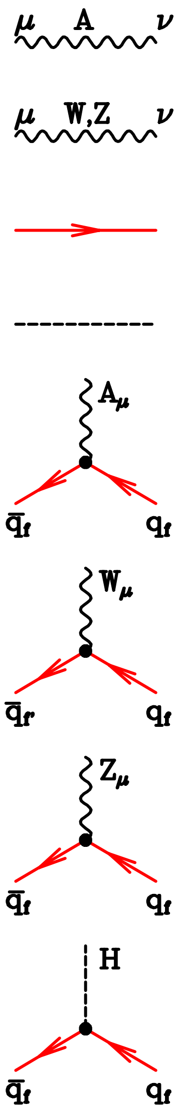
$-ie^2[2g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}]$



$-ieg_W\cos\theta_W[2g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}]$

Fermion rules

- The propagators are shown in the Unitary gauge.
- This gauge eliminates fields that do not correspond to physical particles.
- In this gauge the propagators have worse ultra-violet behaviour.
- The Weinberg angle fixes the coupling to the Z boson.
- Measurements of the Weinberg angle fix the ratio of the Z and W masses



$$[-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}]\frac{i}{q^2}$$

$$[-g^{\mu\nu}+\frac{q^\mu q^\nu}{M^2}]\frac{i}{(q^2-M^2)}$$

$$\frac{i}{\not{q}-m}$$

$$\frac{i}{(q^2-M_H^2)}$$

$$-ieQ_f\gamma_\mu$$

$$\frac{-ig_W}{2\sqrt{2}}\gamma^\mu(1-\gamma_5)(T^+)_{f'f} \qquad g_W=\frac{e}{\sin\theta_W}$$

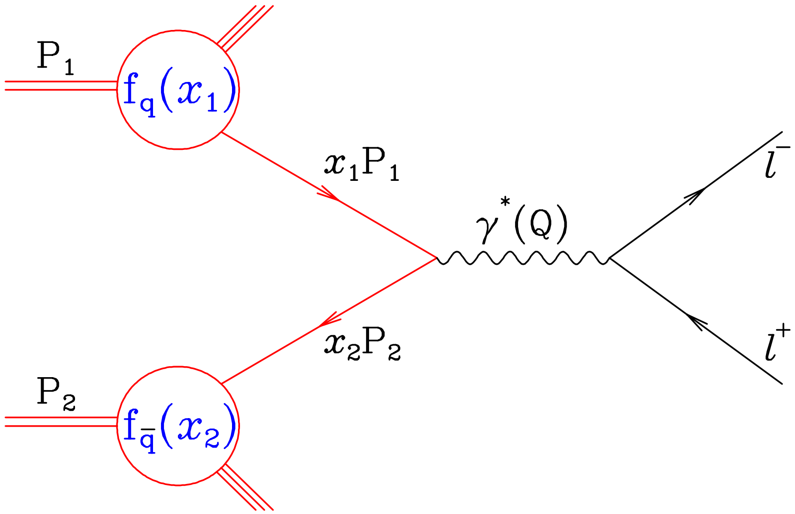
$$\frac{-ig_W}{2\cos\theta_W}\gamma^\mu(V_f-A_f\gamma_5)=\frac{-ig_W}{\cos\theta_W}\gamma^\mu(R_f\gamma_R+L_f\gamma_L)$$

$$(V_f=T_f^3-2Q_f\sin^2\theta_W,\ A_f=T_f^3\ ),\ T_f^3=\pm 1/2$$

$$(R_f=-Q_f\sin^2\theta_W,\ L_f=T_f^3-Q_f\sin^2\theta_W)$$

$$\frac{-ig_W m_f}{2M_W}$$

Lepton-pair production



- Mechanism for Lepton pair production, W-production, Z-production, Vector-boson pairs, ...
- Collectively known as the Drell-Yan process.
- Colour average $1/N$.

$$\frac{d\hat{\sigma}}{dQ^2} = \frac{\sigma_0}{N}\, Q_q^2\, \delta(\hat{s} - Q^2), \qquad \sigma_0 = \frac{4\pi\alpha^2}{3Q^2}, \quad \text{cf } e^+e^- \text{ annihilation.}$$

In the CM frame of the two hadrons, the momenta of the incoming partons are

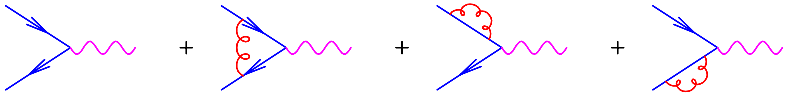
$$p_1 = \frac{\sqrt{s}}{2}(x_1, 0, 0, x_1), \quad p_2 = \frac{\sqrt{s}}{2}(x_2, 0, 0, -x_2)\ .$$

The square of the $q\bar{q}$ collision energy $\hat{s}$ is related to the overall hadron-hadron collision energy by $\hat{s} = (p_1 + p_2)^2 = x_1 x_2 s$. The parton-model cross section for this process is:

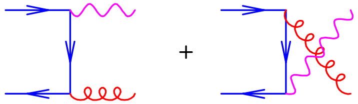
$$
\begin{aligned}
\frac{d\sigma}{dM^2} &= \int_0^1 dx_1 dx_2 \sum_q \{f_q(x_1) f_{\bar{q}}(x_2) + (q \leftrightarrow \bar{q})\} \frac{d\hat{\sigma}}{dM^2}(q\bar{q} \to l^+ l^-) \\
&= \frac{\sigma_0}{Ns} \int_0^1 \frac{dx_1}{x_1} \frac{dx_2}{x_2}\, \delta(1-z) \left[\sum_q Q_q^2 \{f_q(x_1) f_{\bar{q}}(x_2) + (q \leftrightarrow \bar{q})\} \right] .
\end{aligned}
$$

- For later convenience we have introduced the variable $z = \frac{Q^2}{\hat{s}} = \frac{Q^2}{x_1 x_2 s}$.
- The sum here is over quarks only and the $\bar{q}q$ contributions are indicated explicitly.

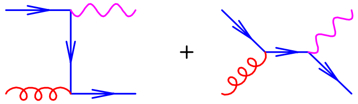
Next-to-leading order



(a)



(b)



(c)

- The contribution of the real diagrams (in four dimensions) is

$$|M|^2 \sim g^2 C_F \left[\frac{u}{t} + \frac{t}{u} + \frac{2Q^2 s}{ut} \right] = g^2 C_F \left[\left(\frac{1+z^2}{1-z} \right) \left(\frac{-s}{t} + \frac{-s}{u} \right) - 2 \right]$$

  where $z = Q^2/s, s + t + u = Q^2$.

- Note that the real diagrams contain collinear singularities, $u \to 0, t \to 0$ and soft singularities, $z \to 1$.
- The coefficient of the divergence is the unregulated branching probability $\hat{P}_{qq}(z)$.
- Ignore for simplicity the diagrams with incoming gluons.

- Control the divergences by continuing the dimensionality of space-time, $d = 4 - 2\epsilon$, (technically this is dimensional reduction). Performing the phase space integration, the total contribution of the real diagrams is

$$\begin{aligned} \sigma_R &= \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} c_\Gamma \Bigg[\Big(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} - \frac{\pi^2}{3}\Big) \delta(1-z) - \frac{2}{\epsilon} P_{qq}(z) \\ &- 2(1-z) + 4(1+z^2) \left[\frac{\ln(1-z)}{1-z}\right]_+ - 2 \frac{1+z^2}{(1-z)} \ln z \Bigg] \end{aligned}$$

with $c_\Gamma = (4\pi)^\epsilon / \Gamma(1-\epsilon)$.

- The contribution of the virtual diagrams is

$$\sigma_V = \delta(1-z) \left[1 + \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} c'_\Gamma \Big(- \frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 6 + \pi^2 \Big) \right]$$

$c'_\Gamma = c_\Gamma + O(\epsilon^3)$

- Adding it up we get in dim-reduction

$$\sigma_{R+V} = \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} c_\Gamma \left[\left(\frac{2\pi^2}{3} - 6\right)\delta(1-z) - \frac{2}{\epsilon} P_{qq}(z) - 2(1-z)\right.$$
$$\left. + \quad 4(1+z^2)\left[\frac{\ln(1-z)}{1-z}\right]_+ - 2\frac{1+z^2}{(1-z)}\ln z\right]$$

- The divergences, proportional to the branching probability , are universal.
- We will factorize them into the parton distributions. We perform the mass factorization by subtracting the counterterm

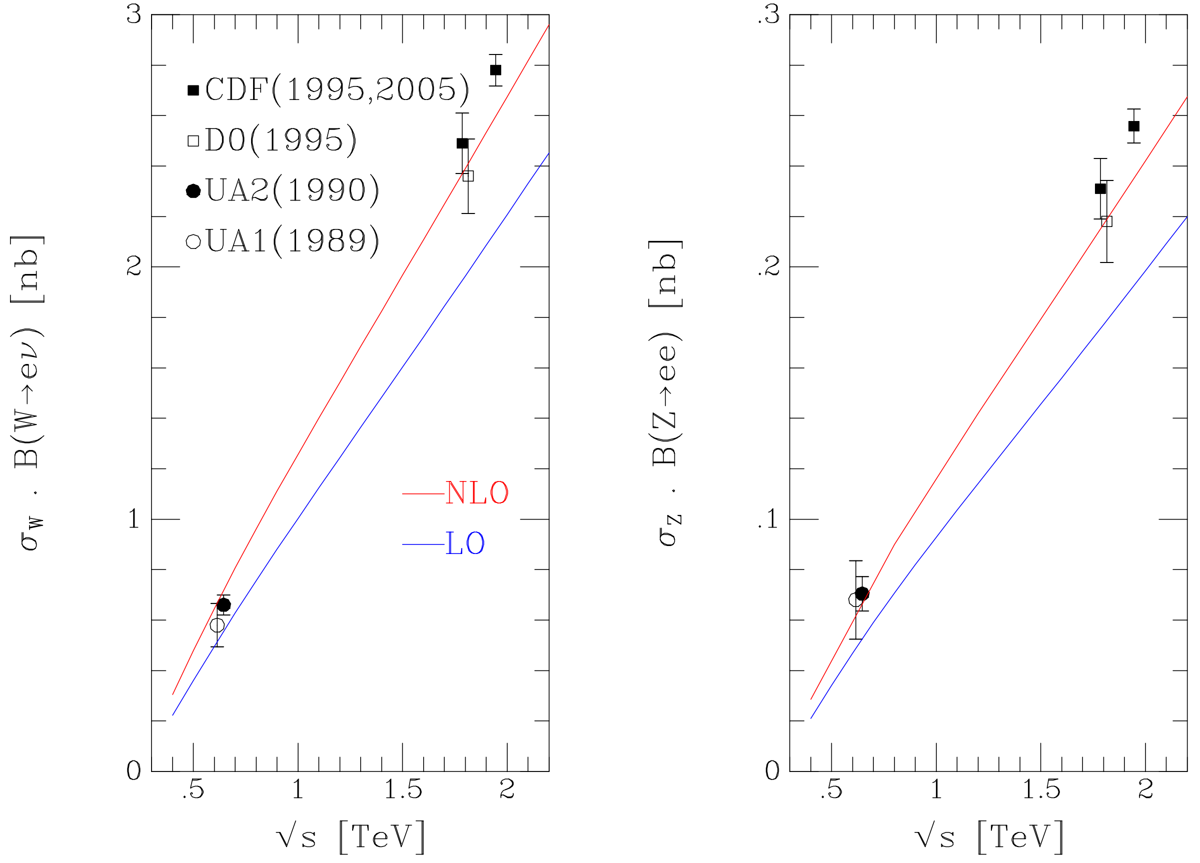
$$2\frac{\alpha_s}{2\pi} C_F \left[\frac{-c_\Gamma}{\epsilon} P_{qq}(z) - (1-z) + \delta(1-z)\right]$$

(The finite terms are necessary to get us to the $\overline{MS}$-scheme).

$$\hat{\sigma} = \frac{\alpha_s}{2\pi} C_F \left[\left(\frac{2\pi^2}{3} - 8\right)\delta(1-z) + 4(1+z^2)\left[\frac{\ln(1-z)}{1-z}\right]_+ - 2\frac{1+z^2}{(1-z)}\ln z + 2P_{qq}(z)\ln\frac{Q^2}{\mu^2}\right]$$

- Similar correction for incoming gluons.

Application to W, Z production



- Agreement with NLO theory is good.
- LO curves lie about 25% too low.
- NNLO results are also known and lead to a further modest (4%) increase at the Tevatron.

Why top?

- The top quark cross section is large at LHC energies, one event in 10^7
- Since $m_t > M_W + m_b$ a top quark decays predominantly into a b quark and an on-shell W boson

$$\begin{aligned} t &\to W^+ + b \\ & \quad\ \ \hookrightarrow l^+ + \nu \end{aligned}$$

$$\begin{aligned} t &\to W^+ + b \\ & \quad\ \ \hookrightarrow q + \bar{q} \end{aligned}$$

- In the limit $m_t \gg M_W$ the result for the total width is

$$\Gamma(t \to bW) = \frac{G_F m_t^3}{8\pi\sqrt{2}} |V_{tb}|^2 \approx 1.76 \ \ \mathrm{GeV} \left(\frac{m_t}{175 \ \ \mathrm{GeV}} \right)^3 .$$

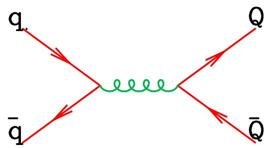
  $V_{tb} \approx 1$ as suggested by the unitarity relation $|V_{tb}|^2 + |V_{cb}|^2 + |V_{ub}|^2 = 1$.
- The top quark decays before it has time to hadronize.
- The top is a copious source of b's and W's

LO Top production

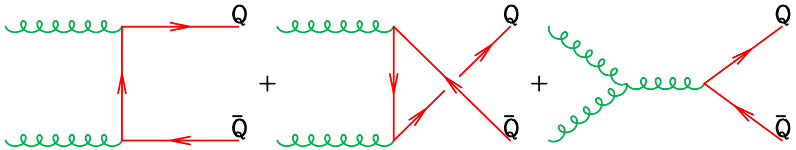
- The leading-order processes for the production of a heavy quark Q of mass m in hadron-hadron collisions

$$
\begin{array}{ll}
(a) & q(p_1) + \overline{q}(p_2) \to Q(p_3) + \overline{Q}(p_4) \\
(b) & g(p_1) + g(p_2) \to Q(p_3) + \overline{Q}(p_4)
\end{array}
$$

where the four-momenta of the partons are given in brackets ($\rho = 4m^2/s$).



(a)



(b)

Process	$\overline{\sum}\vert\mathcal{M}\vert^2/g^4$
$q\,\overline{q} \to Q\,\overline{Q}$	$\frac{4}{9}\left(\tau_1^2 + \tau_2^2 + \frac{\rho}{2}\right)$
$g\,g \to Q\,\overline{Q}$	$\left(\frac{1}{6\tau_1\tau_2} - \frac{3}{8}\right)\left(\tau_1^2 + \tau_2^2 + \rho - \frac{\rho^2}{4\tau_1\tau_2}\right)$

- The matrix elements squared have been averaged (summed) over initial (final) colours and spins, as indicated by $\overline{\sum}$.
- Notation for the ratios of scalar products:

$$
\tau_1 = \frac{2p_1.p_3}{\hat{s}}, \quad \tau_2 = \frac{2p_2.p_3}{\hat{s}}
$$

$$
\rho = \frac{4m^2}{\hat{s}}, \quad \hat{s} = (p_1 + p_2)^2 .
$$

Differential distributions

- The short-distance cross section is obtained from the invariant matrix element in the usual way:

$$d\hat{\sigma}_{ij} = \frac{1}{2\hat{s}} \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \overline{\sum} |\mathcal{M}_{ij}|^2 .$$

  The first factor is the flux factor for massless incoming particles. The other terms come from the phase space for $2 \to 2$ scattering.

- In terms of the rapidity $y = \frac{1}{2} \ln((E + p_z)/(E - p_z))$ and transverse momentum, p_T, the relativistically invariant phase space volume element of the final-state heavy quarks is

$$\frac{d^3p}{E} = dy \; d^2p_T \; .$$

  The result for the invariant cross section may be written as

$$\frac{d\sigma}{dy_3 dy_4 d^2p_T} = \frac{1}{16\pi^2 \hat{s}^2} \sum_{ij} x_1 f_i(x_1, \mu^2) \; x_2 f_j(x_2, \mu^2) \overline{\sum} |\mathcal{M}_{ij}|^2 .$$

  x_1 and x_2 are fixed if we know the transverse momenta and rapidity of the outgoing heavy quarks.

Differential distributions

- In the centre-of-mass system of the incoming hadrons we may write

$$
\begin{aligned}
p_1 &= \frac{1}{2}\sqrt{s}(x_1, 0, 0, x_1) \\
p_2 &= \frac{1}{2}\sqrt{s}(x_2, 0, 0, -x_2) \\
p_3 &= (m_T \cosh y_3, p_T, 0, m_T \sinh y_3) \\
p_4 &= (m_T \cosh y_4, -p_T, 0, m_T \sinh y_4).
\end{aligned}
$$

Applying energy and momentum conservation, we obtain

$$
\begin{aligned}
x_1 &= \frac{m_T}{\sqrt{s}}\left(e^{y_3} + e^{y_4}\right) \\
x_2 &= \frac{m_T}{\sqrt{s}}\left(e^{-y_3} + e^{-y_4}\right) \\
\hat{s} &= 2m_T^2(1 + \cosh \Delta y).
\end{aligned}
$$

The quantity $m_T = \sqrt{(m^2 + p_T^2)}$ is the transverse mass of the heavy quarks and $\Delta y = y_3 - y_4$ is the rapidity difference between them.

Differential distributions

- In these variables the leading order cross section is

$$\frac{d\sigma}{dy_3 dy_4 d^2 p_T} = \frac{1}{64\pi^2 m_T^4 (1+\cosh(\Delta y))^2} \times \sum_{ij} x_1 f_i(x_1,\mu^2)\, x_2 f_j(x_2,\mu^2)\, \overline{\sum} |\mathcal{M}_{ij}|^2 .$$

Expressed in terms of m, m_T and Δy, the matrix elements for the two processes are

$$\overline{\sum} |\mathcal{M}_{q\bar{q}}|^2 = \frac{4g^4}{9} \Big(\frac{1}{1+\cosh(\Delta y)}\Big) \Big(\cosh(\Delta y) + \frac{m^2}{m_T^2}\Big) ,$$

$$\overline{\sum} |\mathcal{M}_{gg}|^2 = \frac{g^4}{24} \Big(\frac{8\cosh(\Delta y) - 1}{1+\cosh(\Delta y)}\Big) \Big(\cosh(\Delta y) + 2\frac{m^2}{m_T^2} - 2\frac{m^4}{m_T^4}\Big) .$$

- As the rapidity separation Δy between the two heavy quarks becomes large

$$\overline{\sum} |\mathcal{M}_{q\bar{q}}|^2 \sim \text{ constant}, \quad \overline{\sum} |\mathcal{M}_{gg}|^2 \sim \exp \Delta y \ .$$

- The cross section is damped at large Δy and heavy quarks produced by $q\bar{q}$ annihilation are more closely correlated in rapidity those produced by gg fusion.

NLO Heavy quark production

In NLO heavy quark production m is the heavy quark mass.

$$\sigma(S) = \sum_{i,j} \int dx_1 dx_2 \; \hat{\sigma}_{ij}(x_1 x_2 S, m^2, \mu^2) f_i(x_1, \mu^2) f_j(x_2, \mu^2), \quad \hat{\sigma}_{i,j}(\hat{s}, m^2, \mu^2) = \sigma_0 c_{ij}(\hat{\rho}, \mu^2)$$

where $\hat{\rho} = 4m^2/\hat{s}, \bar{\mu}^2 = \mu^2/m^2, \sigma_0 = \alpha_s^2(\mu^2)/m^2$ and $\hat{s}$ in the parton total c-of-m energy squared. The coupling satisfies

$$\frac{d\alpha_s}{d\ln\mu^2} = -b_0 \frac{\alpha_s^2}{2\pi} + O(\alpha_s^3), \; b_0 = \frac{11N - 2n_f}{6}$$

$$c_{ij}\left(\rho, \frac{\mu^2}{m^2}\right) = c_{ij}^{(0)}(\rho) + 4\pi\alpha_s(\mu^2)\left[c_{ij}^{(1)}(\rho) + \bar{c}_{ij}^{(1)}(\rho)\ln(\frac{\mu^2}{m^2})\right] + O(\alpha_s^2)$$

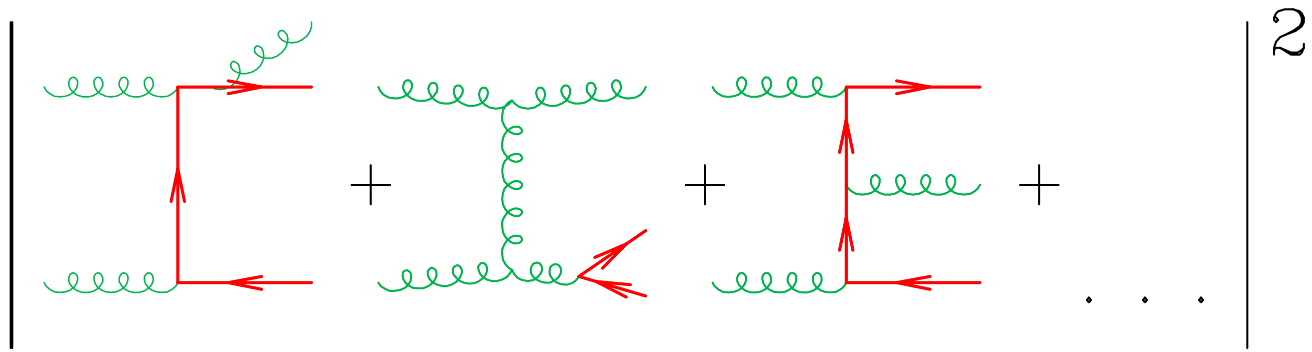
The lowest-order functions $c_{ij}^{(0)}$ are obtained by integrating the lowest order matrix elements

$$c_{gg}^{(0)}(\rho) = \frac{\pi\beta\rho}{192}\left[\frac{1}{\beta}\left[\rho^2 + 16\rho + 16\right]\ln\left(\frac{1+\beta}{1-\beta}\right) - 28 - 31\rho\right],$$

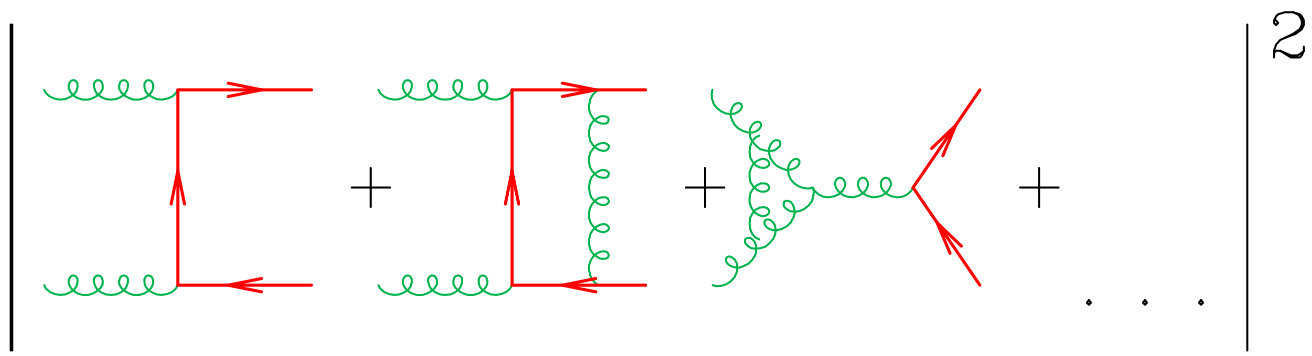
$$c_{q\bar{q}}^{(0)}(\rho) = \frac{\pi\beta\rho}{27}\left[(2+\rho)\right], \quad c_{gq}^{(0)}(\rho) = c_{g\bar{q}}^{(0)}(\rho) = 0$$

NLO Heavy quark production

- The functions $c_{ij}^{(1)}$ are also known
- In order to calculate the c_{ij} in perturbation theory we must perform both renormalization and factorization of mass singularities. The subtractions required for renormalization and factorization are done at mass scale μ.



Real emission diagrams



Virtual emission diagrams

Scale dependence

The scale μ is an unphysical parameter. The physical predictions should be invariant under changes of μ at the appropriate order in perturbation theory. If we have performed a calculation to $O(\alpha_s^3)$, variations of the scale μ will lead to corrections of $O(\alpha_s^4)$,

$$\mu^2 \frac{d}{d\mu^2}\sigma = O(\alpha_s^4).$$

- The term $\overline{c}^{(1)}$, which controls the μ dependence of the higher-order perturbative contributions, is fixed in terms of the lower-order result $c^{(0)}$:

$$\begin{aligned} \overline{c}_{ij}^{(1)}(\rho) &= \frac{1}{8\pi^2}\Bigg[4\pi b c_{ij}^{(0)}(\rho) - \int_\rho^1 dz_1 \sum_k c_{kj}^{(0)}(\frac{\rho}{z_1}) P_{ki}^{(0)}(z_1) \\ &- \int_\rho^1 dz_2 \sum_k c_{ik}^{(0)}(\frac{\rho}{z_2}) P_{kj}^{(0)}(z_2)\Bigg]. \end{aligned}$$

Scale dependence

- In obtaining this result we have used the renormalization group equation for the running coupling

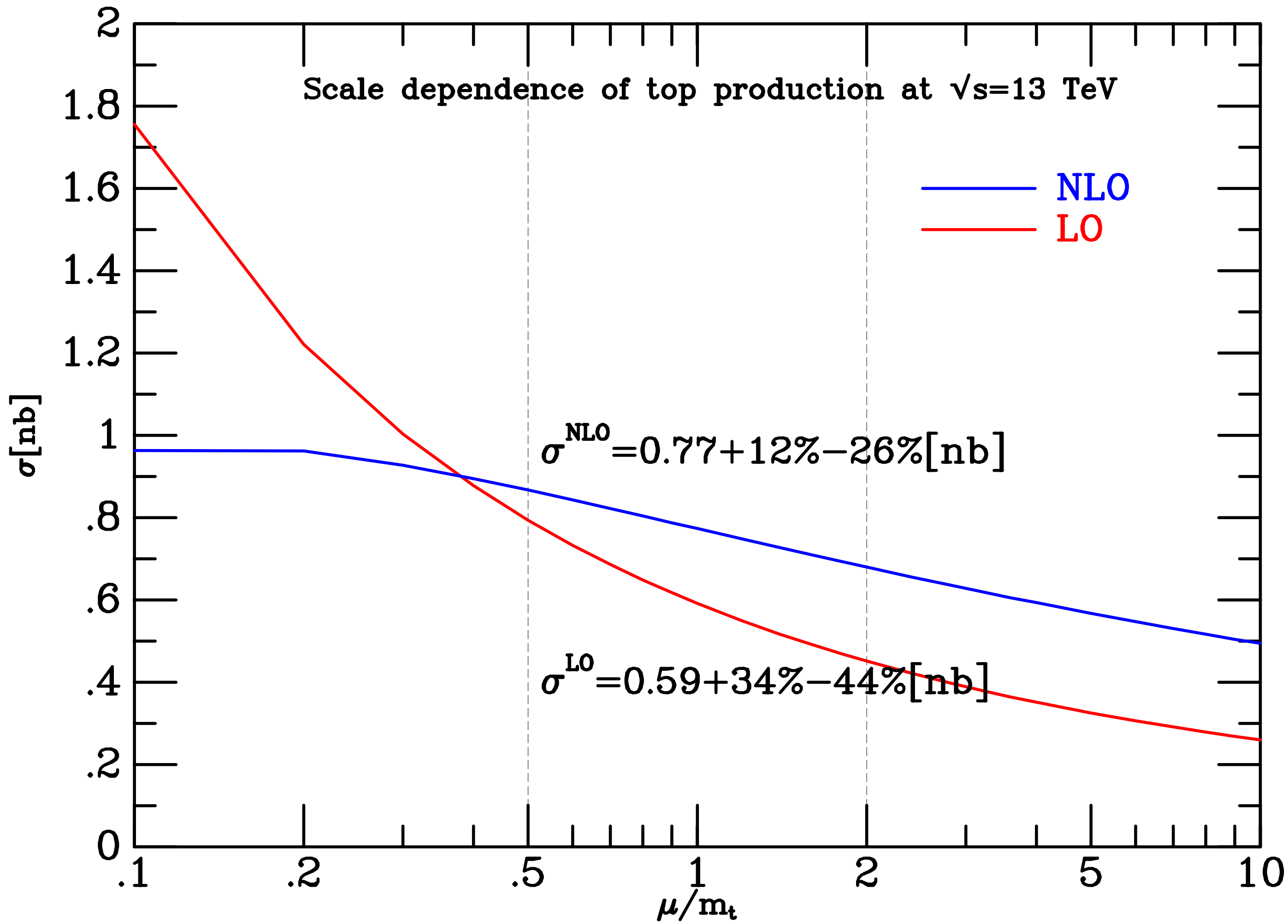
$$\mu^2 \frac{d}{d\mu^2} \alpha_s(\mu^2) = -b\alpha_s^2 + \ldots$$

and the lowest-order form of the DGLAP equation

$$\mu^2 \frac{d}{d\mu^2} f_i(x, \mu^2) = \frac{\alpha_s(\mu^2)}{2\pi} \sum_k \int_x^1 \frac{dz}{z} P_{ik}^{(0)}(z) f_k(\frac{x}{z}, \mu^2) + \ldots .$$

- This illustrates an important point which is a general feature of renormalization group improved perturbation series in QCD.
- The coefficient of the perturbative correction depends on the choice made for the scale μ, but the scale dependence changes the result in such a way that the physical result is independent of that choice.
- Thus the scale dependence is formally small because it is of higher order in α_s.
- This does not assure us that the scale dependence is actually numerically small for all series.
- A pronounced dependence on the scale μ is a signal of an untrustworthy perturbation series.

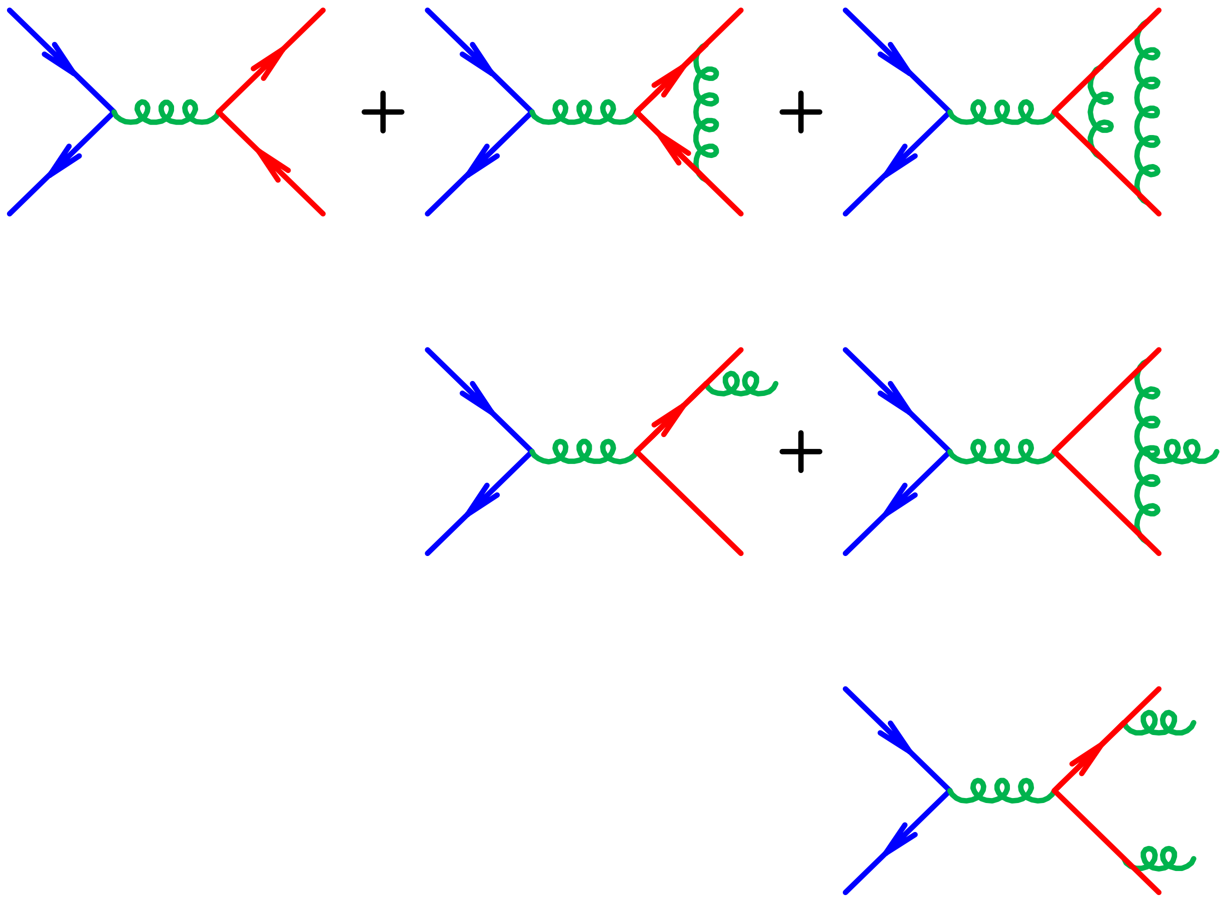
Scale dependence of top cross section



- Note that despite the fact that α_S is of order 10%, we do not obtain 10% predictions at NLO.
- This is 'feature' of renormalization group improved perturbation theory.

Top at NNLO

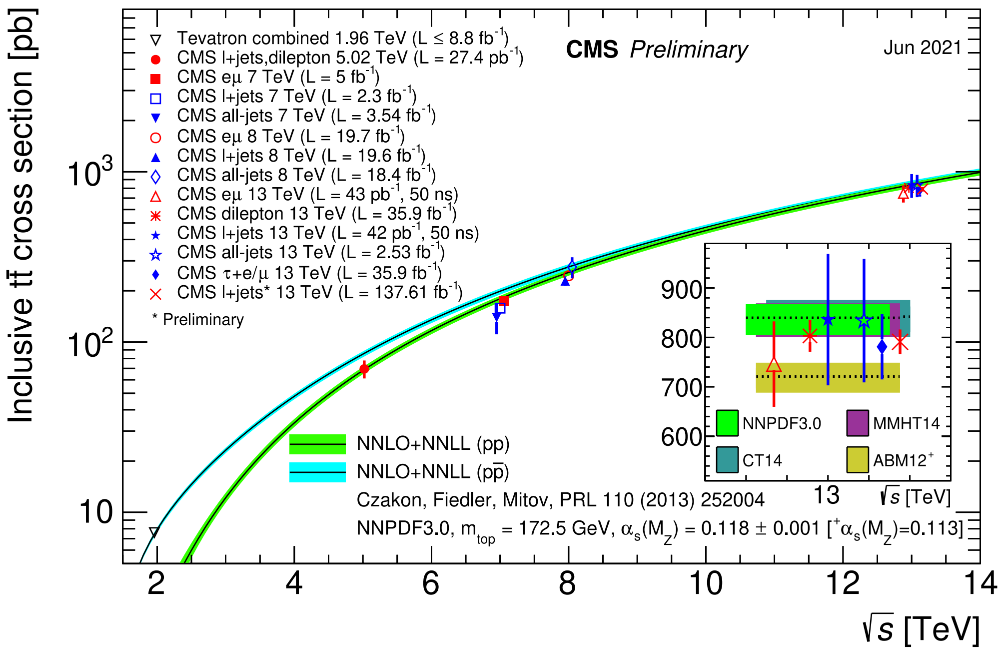
- Challenge is not the calculation of the individual diagrams, but rather the assembly of pieces that individually contain infrared divergences
- Tension between the need to cancel infra-red divergences, which for the higher multiplicity processes are only manifest after integration and the desire to have a fully differential prediction.



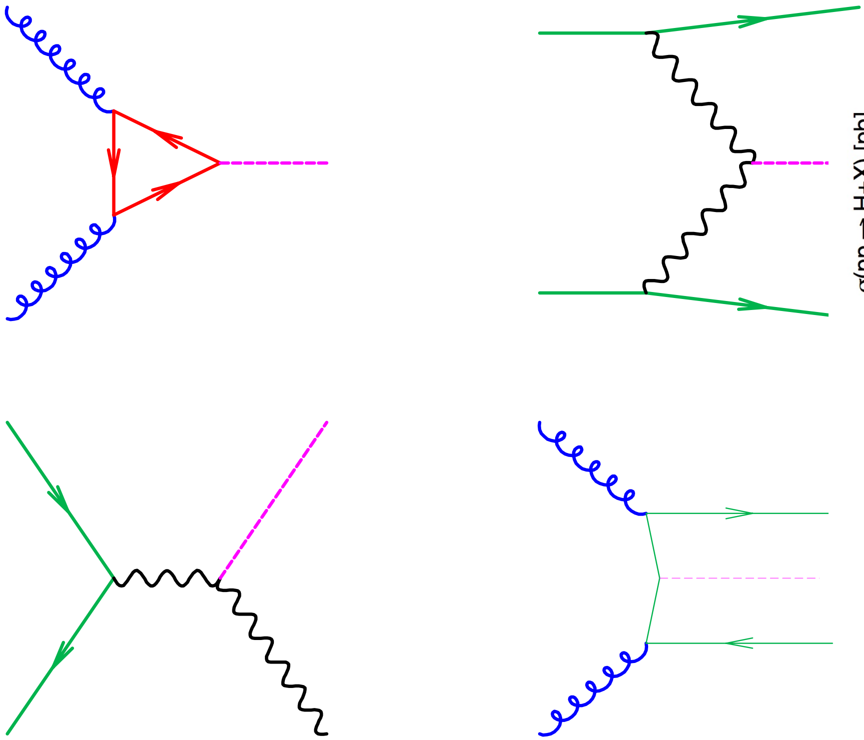
Top at NNLO

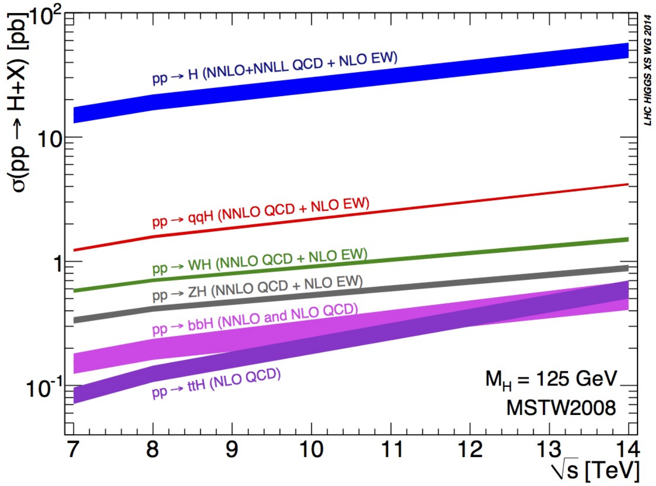
Collider	$\sigma_{\rm tot}$ [pb]	scales [pb]	pdf [pb]
Tevatron	7.164	$+0.110(1.5\%)$ $-0.200(2.8\%)$	$+0.169(2.4\%)$ $-0.122(1.7\%)$
LHC 7 TeV	172.0	$+4.4(2.6\%)$ $-5.8(3.4\%)$	$+4.7(2.7\%)$ $-4.8(2.8\%)$
LHC 8 TeV	245.8	$+6.2(2.5\%)$ $-8.4(3.4\%)$	$+6.2(2.5\%)$ $-6.4(2.6\%)$
LHC 14 TeV	953.6	$+22.7(2.4\%)$ $-33.9(3.6\%)$	$+16.2(1.7\%)$ $-17.8(1.9\%)$

- Best NNLO+NNLL theoretical predictions for various colliders and c.m. energies.
- c.f. scale uncertainty at NLO $+12\% - 26\%$



Higgs production at LHC





- The dominant production mechanism is $gg \to H$, via an intermediate top quark loop;
- Averaged over spins and colours of the initial gluons we have

$$\overline{\sum}|\mathcal{M}(H \to gg)|^2 = \frac{G_F \alpha_s^2 M_H^4}{512\sqrt{2}\pi^2} \mid \sum_q F_{1/2}(\tau_q) \mid^2 ,$$

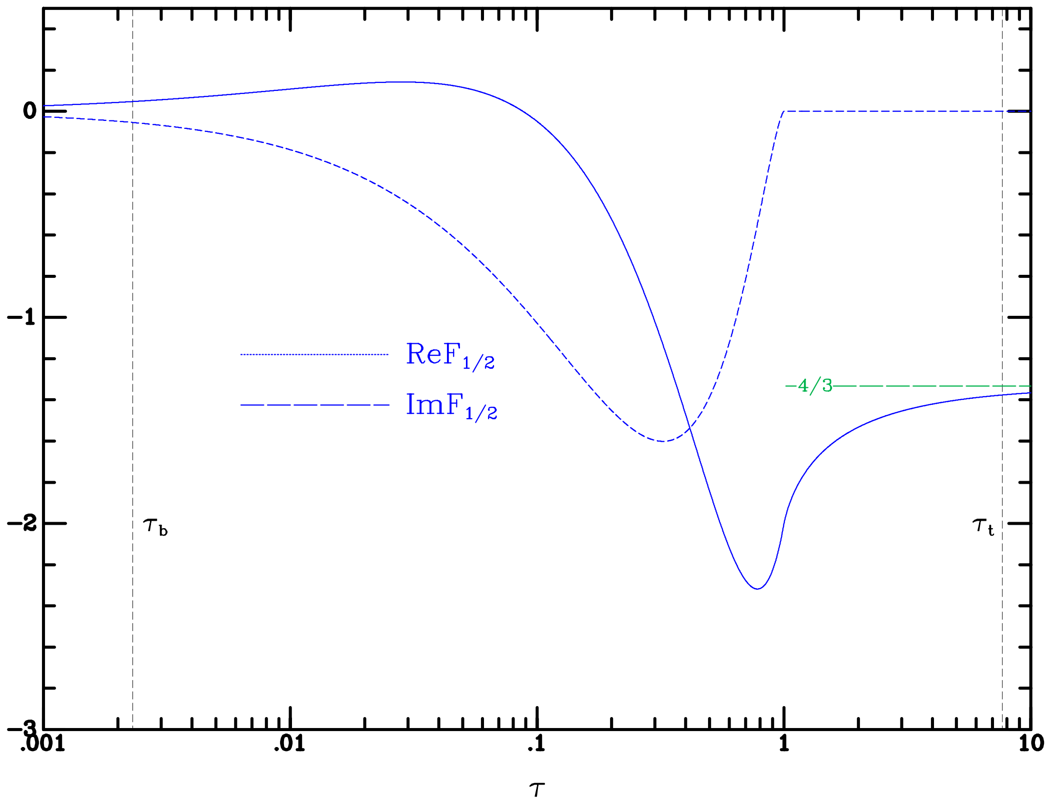
where $\tau_f = 4m_f^2/M_H^2$

Gluon-fusion production of Higgs

- The function $F_{1/2}$ is a dimensionless functions given by

$$F_{1/2}(\tau) = -2\tau\left[1+(1-\tau)f(\tau)\right] \quad (1)$$

$$f(\tau) = -\frac{1}{4}\theta(1-\tau)\left[\ln\left(\frac{1+\sqrt{1-\tau}}{1-\sqrt{1-\tau}}\right)-i\pi\right]^2 + \theta(\tau-1)\left[\sin^{-1}(1/\sqrt{\tau})\right]^2 . \quad (2)$$



- $\tau_t = 4m_t^2/m_H^2, \tau_b = 4m_b^2/m_H^2$.

Higgs effective field theory

- The function f is related to the scalar triangle integral

$$
\begin{aligned}
C_0(p_1,p_2,m,m,m) &= \frac{1}{i\pi^2}\int d^4l \, \frac{1}{(l-m^2)\,((l+p_1)^2-m^2)\,((l\;+p_1+p_2)^2-m^2)} \\
&= -\int da_1 \int da_2 \int da_3 \, \frac{\delta(1-a_1-a_2-a_3)}{(m^2-a_1a_2s)}
\end{aligned}
$$

Exploiting the exact relation between C_0 and f allows us to perform a large mass (=large τ) expansion

$$
\begin{aligned}
f(\tau) &= -\frac{s}{2}C_0(p_1,p_2,m,m,m)\,, \qquad \tau = \frac{4m^2}{s} \\
&= \frac{s}{2}\int_0^1 da_1 \int_0^{1-a_1} da_2 \, \frac{1}{(m^2-a_1a_2s)} \\
&= \frac{2}{\tau}\int_0^1 da_1 \int_0^{1-a_1} da_2 \, \frac{1}{(1-\frac{4a_1a_2}{\tau})} \\
&= \frac{1}{\tau} + \frac{1}{3\,\tau^2} + O(\frac{1}{\tau^3})
\end{aligned}
$$

- For a heavy quark, $\tau_q \to \infty$, and $F_{1/2}(\tau_q) \to -\frac{4}{3}$.

Higgs effective field theory

- In the large top quark limit, the effective Lagrangian is,

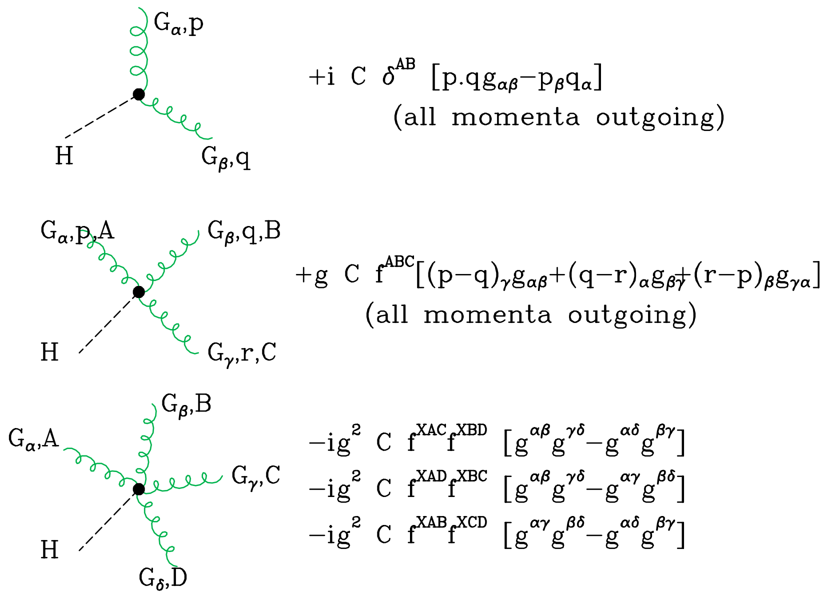
$$\mathcal{L}_{gg} = -\frac{1}{4}\frac{C}{v} G_a^{\mu\nu} G_{\mu\nu\, a}\; h$$

  This effective Lagrangian can be used to derive an approximate result for Higgs pair production

- Using the Feynman rules of we see that that matrix element squared for $H \to g(p_1) + g(p_2)$ (summed over polarisations and colours) is $V = 8$,

$$\sum |\mathcal{M}|^2 = V C^2 \left(\frac{M_H^2}{2}\right)^2$$

$$C^2 = \frac{\alpha_s^2}{9\pi^2}\frac{1}{v^2} = \frac{\alpha_s^2}{9\pi^2}\frac{G_F}{2\sqrt{2}}$$



- for physical top mass, corrctions to heavy top limit are of order 3% in amplitude.

Cross section

- The lowest order partonic cross section is given by

$$\sigma^0_{gg\to h} = \frac{1}{2\hat{s}}\,\frac{1}{2^2(N^2-1)^2}\sum |\mathcal{M}|^2 dPS_{2\to 1}$$

- In the centre of mass frame of the gluons, ($\hat{s} = (p_1+p_2)^2$)

$$p_1 = \frac{\sqrt{\hat{s}}}{2}(1,0,0,+1), \quad p_2 = \frac{\sqrt{\hat{s}}}{2}(1,0,0,-1)$$

so that

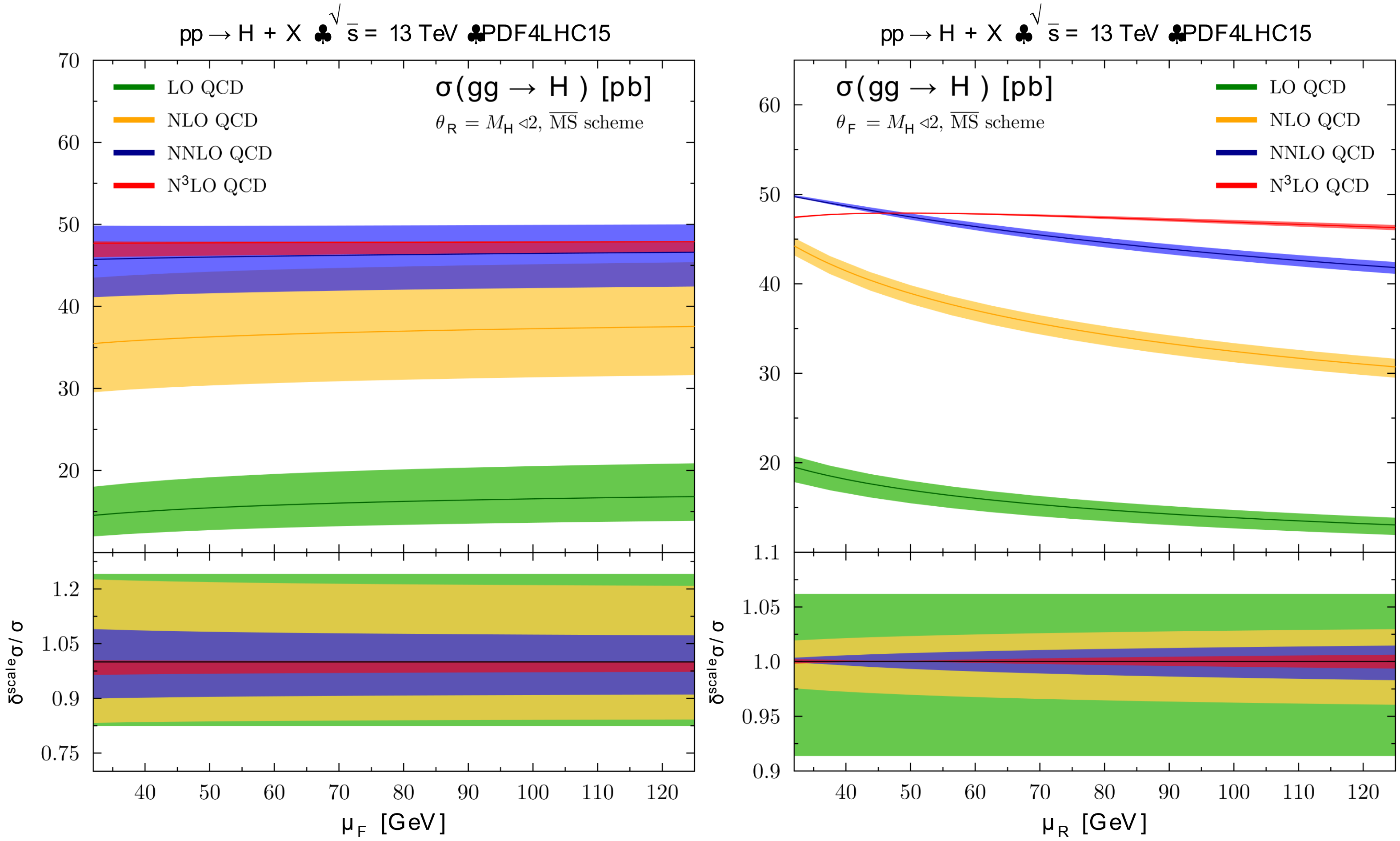
- The $2 \to 1$ phase space factor is

$$dPS_{2\to 1} = \frac{d^4p_h}{(2\pi)^3}\,(2\pi)^4\delta(p_1+p_2-p_h)\delta(p_h^2-m_h^2) = \frac{2\pi}{M_H^2}\delta(1-z),\ \text{for } z = M_H^2/\hat{s}$$

The result for the Higgs production cross section is,

$$\begin{aligned}\sigma^0_{gg\to h} &= \frac{1}{2\hat{s}}\,\frac{1}{16^2}\,4C^2M_H^4\,\frac{2\pi}{M_H^2}\delta(1-z)\\ &= \frac{\alpha_s^2}{576\pi v^2}\,\frac{M_H^2}{\hat{s}}\delta(1-z) = \sigma^0\delta(1-z)\end{aligned}$$

Higher order results



- Left plot shows factorization scale dependence in LO,NLO,NNLO,NNNLO.
- Right plot shows renormalization scale dependence in LO,NLO,NNLO,NNNLO.
- From 2209.06138

Recap 1

- The factorization formula of the QCD parton model provides a systematically improvable framework to calculate hard scattering cross sections in QCD.
- Renormalization and scale dependence is formally of higher order, but not always numerically insignificant.
- The Drell Yan process is the simplest process to examine in QCD parton model
- Top quark production has been calculated through to NNLO
- Higgs production can be calculated using an effective lagrangian, valid in the $m_t \to \infty$ limit. Correction for physical top mass are of order 6%. This calculation has been performed at N^{3}LO in the effective theory.

Spinor helicity methods

Z. Xu, Da-Hua Zhang and L. Chang, Tsinghua University Preprints, Beijing, The People's Republic of China, TUTP–84/4, TUTP–84/5, TUTP–84/6 and Nucl. Phys. B291 (1987), 392.

Dirac equation

- The Lagrangian for a free massive 4-component Dirac field Ψ is

$$\mathcal{L} = i\overline{\Psi}\gamma^\mu\partial_\mu\Psi - m\overline{\Psi}\Psi\,.$$

- The equation of motion for $\overline{\Psi}$ gives the Dirac equation

$$(i\,\partial\!\!\!/ - m)\Psi = 0\,.$$

- Multiplying the Dirac equation by $(i\,\partial\!\!\!/ + m)$ gives the Klein-Gordon equation, $(\partial_\mu\partial^\mu + m^2)\Psi = 0.$, if $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbf{1}$. It is solved by a plane-wave expansion

$$\Psi(x) \;\sim\; u(p)\,e^{ip.x} + v(p)\,e^{-ip.x}$$

provided $p^2 \equiv p^\mu p_\mu = m^2$. This $\Psi(x)$ will also solve the Dirac equation if

$$(\,p\!\!\!/ - m)u(p) = 0 \quad \text{and} \quad (p\!\!\!/ + m)v(p) = 0\,.$$

- These equations give the momentum space form of the Dirac equation. Each of the equations has two independent solutions.

Definition of Gamma Matrices

- We choose an explicit representation for the gamma matrices. The Bjorken and Drell representation is,

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}, \gamma_5 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix},$$

The Weyl representation is more suitable at high energy. This is Peskin's definition of the Weyl representation.

$$\gamma^0 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}, \gamma_5 = \begin{pmatrix} -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix},$$

In the Weyl representation upper and lower components have different chiralities ($\equiv$ helicity for massless particles).

$$\gamma_L = \frac{1}{2}(1-\gamma_5) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \gamma_R = \frac{1}{2}(1+\gamma_5) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}.$$

Definition of Gamma Matrices

- Both representations satisfy the same commutation relations.

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbf{1}$$

- in the Weyl representation

$$\gamma^0\gamma^i = \begin{pmatrix} -\sigma^i & \mathbf{0} \\ \mathbf{0} & \sigma^i \end{pmatrix}$$

- σ are the Pauli matrices.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Solution to massless Dirac equation

- We now specialize to the $m = 0$ case appropriate at high energy.
- In order to derive an explicit solution for the massless Dirac equation $\not{p}u(p) = 0$ it is useful to write out an explicit expression for $\not{p} = \gamma^0 p^0 - \gamma^1 p^1 - \gamma^2 p^2 - \gamma^3 p^3$ in the Weyl representation. To keep this compact we introduce the notation

$$\sigma = (1, \sigma_1, \sigma_2, \sigma_3), \qquad \bar{\sigma} = (1, -\sigma_1, -\sigma_2, -\sigma_3)$$

$$\not{p} = \begin{pmatrix} \mathbf{0} & E\mathbf{I} - \vec{\sigma}.p \\ E\mathbf{I} + \vec{\sigma}.p & \mathbf{0} \end{pmatrix}, = \begin{pmatrix} \mathbf{0} & \bar{\sigma}.p \\ \sigma.p & \mathbf{0} \end{pmatrix},$$

$$\not{p} = \begin{pmatrix} 0 & 0 & p^- & -p^1 + ip^2 \\ 0 & 0 & -p^1 - ip^2 & p^+ \\ p^+ & p^1 - ip^2 & 0 & 0 \\ p^1 + ip^2 & p^- & 0 & 0 \end{pmatrix},$$

where $p^{\pm} = p^0 \pm p^3$.

Dirac equation massless solutions

- The massless spinors solns of Dirac eqn are

$$u_-(p) = \frac{1}{\sqrt{p^+}} \begin{bmatrix} (-p^1 + ip^2) \\ p^+ \\ 0 \\ 0 \end{bmatrix}, \quad u_+(p) = \frac{1}{\sqrt{p^+}} \begin{bmatrix} 0 \\ 0 \\ p^+ \\ (p^1 + ip^2) \end{bmatrix},$$

$$u_-(p) = \begin{bmatrix} -\sqrt{p^-}e^{-i\varphi_p} \\ \sqrt{p^+} \\ 0 \\ 0 \end{bmatrix}, \quad u_+(p) = \begin{bmatrix} 0 \\ 0 \\ \sqrt{p^+} \\ \sqrt{p^-}e^{i\varphi_p} \end{bmatrix},$$

where

$$e^{\pm i\varphi_p} \equiv \frac{p^1 \pm ip^2}{\sqrt{(p^1)^2 + (p^2)^2}} = \frac{p^1 \pm ip^2}{\sqrt{p^+p^-}}, \qquad p^\pm = p^0 \pm p^3.$$

Dirac conjugate spinors

- In this representation the Dirac conjugate spinors are

$$\overline{u}_-(p) \equiv u_-^\dagger(p)\gamma^0 \equiv \langle p| = \frac{1}{\sqrt{p^+}}\left[0, 0, -p^1 - ip^2, p^+\right]$$

$$\overline{u}_+(p) \equiv u_+^\dagger(p)\gamma^0 \equiv [p| = \frac{1}{\sqrt{p^+}}\left[p^+, p^1 - ip^2, 0, 0\right]$$

- With these in hand, we define spinor products,

$$\langle p\, q\rangle = \overline{u}_-(p)u_+(q) = \overline{u}_-(p)v_-(q) = \sqrt{\frac{p^+}{q^+}}(q^1 + iq^2) - \sqrt{\frac{q^+}{p^+}}(p^1 + ip^2)$$

$$[q\, p] = \overline{u}_+(q)u_-(p) = \overline{u}_+(q)v_+(p) = \sqrt{\frac{p^+}{q^+}}(q^1 - iq^2) - \sqrt{\frac{q^+}{p^+}}(p^1 - ip^2)$$

$$\langle p\, q\rangle\, [q\, p] = 2p\cdot q, \quad \langle p\, q\rangle \sim \sqrt{2p.q}, \quad [q\, p] \sim \sqrt{2p.q}$$

- The spinors are normalized so that,

$$u_\pm^\dagger u_\pm = 2p^0$$

Spinor notation

- It is customary in amplitude calculations to consider all particles as outgoing
- Feynman rules for outgoing massless (anti)fermions:
  - ⋆ Outgoing fermion with $h = +1/2$: $\overline{u}_+ \longleftrightarrow \left([p|^a \, , \, 0 \right)$
  - ⋆ Outgoing fermion with $h = -1/2$: $\overline{u}_- \longleftrightarrow \left(0 \, , \, \langle p|_{\dot{a}} \right)$
  - ⋆ Outgoing anti-fermion with $h = +1/2$: $v_+ \longleftrightarrow \begin{pmatrix} |p]_a \\ 0 \end{pmatrix}$
  - ⋆ Outgoing anti-fermion with $h = -1/2$: $v_- \longleftrightarrow \begin{pmatrix} 0 \\ |p\rangle^{\dot{a}} \end{pmatrix}$
- Since the left-handed and right-handed spinors occupy different subspaces, we write them in terms of two-index (Weyl) spinors, (also called holomorphic and anti-holomorphic spinors), with dotted and undotted indices.
  $|q\rangle^{\dot{\alpha}} = (\tilde{\lambda}_q)^{\dot{\alpha}}, \quad |q]_\alpha = (\lambda_q)_\alpha, \quad \langle p|_{\dot{\alpha}} = \tilde{\lambda}_{\dot{\alpha}}, \quad [p|^\alpha = \lambda^\alpha$
- The spinor products in terms of these spinors are, ($\epsilon^{\alpha\beta}$ is the antisymmetric tensor in two dimensions).

$$\langle p\, q\rangle = \epsilon_{\dot{\alpha}\dot{\beta}}(\tilde{\lambda}_p)^{\dot{\beta}}(\tilde{\lambda}_q)^{\dot{\alpha}} = (\tilde{\lambda}_p)_{\dot{\alpha}}(\tilde{\lambda}_q)^{\dot{\alpha}}, \quad [p\, q] = \epsilon^{\alpha\beta}(\lambda_p)_\beta(\lambda_q)_\alpha = (\lambda_p)^\alpha(\lambda_q)_\alpha$$

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot{\alpha}\dot{\beta}}$$

Charge conjugation

- In QED we have $\Big((+i\partial^\mu + eA^\mu)\gamma_\mu - m\Big)\psi = 0$
- Taking the complex conjugate $\Big((-i\partial^\mu + eA^\mu)\gamma_\mu^* - m\Big)\psi^* = 0$
- The equation satisfied by the charge conjugate state is

$$\Big((+i\partial^\mu - eA^\mu)\gamma_\mu - m\Big)\psi_c = 0$$

The operation of charge conjugation is therefore given by $\psi_c = C\gamma^0\psi^*$ where the matrix C is determined up to a phase by the condition $(C\gamma^0)\gamma^{\mu *}(C\gamma^0)^{-1} = -\gamma^\mu$. Since for our representation $\gamma^0\gamma^{\mu\ *}\gamma^0 = \gamma^{\mu\ T}$ the defining condition on matrix C can be written $C^{-1}\gamma^\mu C = -\gamma^{\mu T}$

- We choose the phase such that

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} i\sigma^2 & \mathbf{0} \\ \mathbf{0} & -i\sigma^2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

so that $C^T = C^{-1} = -C$.

Antiparticle spinors

- For free particle spinors we have that

$$v_{\pm}(p) = C\overline{u}^{T}_{\pm}(p),$$

- Thus in the massless case we get

$$v_{+}(p) \equiv |p\rangle = \begin{bmatrix} -\sqrt{p^{-}}e^{-i\varphi_p} \\ \sqrt{p^{+}} \\ 0 \\ 0 \end{bmatrix}, \quad v_{-}(p) \equiv |p] = \begin{bmatrix} 0 \\ 0 \\ \sqrt{p^{+}} \\ \sqrt{p^{-}}e^{i\varphi_p} \end{bmatrix},$$

- We note that for the case of massless spinors $v_{\pm}(p) = u_{\mp}(p)$

Schouten identity

- The Schouten identity exploits the fact that in two dimensions, the tensor antisymmetric in three indices is equal to zero

$$\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta} - \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta} - \varepsilon^{\gamma\beta}\varepsilon^{\alpha\delta} = 0$$

- Thus using the forms for the spinor products in terms of Weyl spinors we can show that

$$\langle A\, B\rangle\, \langle C\, D\rangle - \langle A\, C\rangle\, \langle B\, D\rangle - \langle C\, B\rangle\, \langle A\, D\rangle = 0$$

  We can check this relation by setting $\langle C| = \alpha\langle A| + \beta\langle B|$

- Alternatively by explicit construction we can show that

$$|B+\rangle\langle C-| - |C+\rangle\langle B-| = \langle C - |B+\rangle\gamma_R$$

- Thus we get the Schouten identity

$$\langle A - |B+\rangle\langle C - |D+\rangle - \langle A - |C+\rangle\langle B - |D+\rangle = \langle C - |B+\rangle\langle A - |D+\rangle$$

  or written more concisely,

$$\langle A\, B\rangle\, \langle C\, D\rangle - \langle A\, C\rangle\, \langle B\, D\rangle = \langle C\, B\rangle\, \langle A\, D\rangle\,, \quad [A\, B]\, [C\, D] - [A\, C]\, [B\, D] = [C\, B]\, [A\, D]$$

Charge conjugation identity

- We want to show the charge conjugation identity,

$$\langle A|\gamma_\mu|B] = [B|\gamma_\mu|A\rangle$$

$$\langle A-,k|(\gamma_\mu\gamma_L)_{kl}|B-,l\rangle = \langle B+,k|(\gamma_\mu\gamma_R)_{kl}|A+,l\rangle$$

- We shall now show that this equation follows as a consequence of the relations obeyed under charge conjugation by massless spinors.

$$u_- = C\bar{u}_+^T, \quad u_+ = C\bar{u}_-^T$$

and hence that

$$\bar{u}_+^T = -Cu_-$$

since $C^{-1} = -C = C^T$. Thus for massless spinors we have that

$$\bar{u}_-(p_A)\gamma^\mu u_-(p_B) = -u_+^T(p_A)C^{-1}\gamma^\mu C\bar{u}_+^T(p_B)$$

The defining equation for charge conjugation is $C^{-1}\gamma^\mu C = -(\gamma^\mu)^T$

- This allows us to prove that

$$\langle A|\gamma_\mu|B] = [B|\gamma_\mu|A\rangle$$

Fierz transformation

- By forming products of the γ matrices, we can construct 16 linearly independent 4×4 matrices, Λ_i where $\Lambda_i = (1, \gamma_\mu, \sigma_{\mu\nu}/\sqrt{2}, \gamma_\mu\gamma_5, \gamma_5)$ and $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$.
- These matrices obey a crossing relation

$$\Lambda_{32}^{(i)} \otimes \Lambda_{14}^{(i)} = \sum_{j=1}^{5} \lambda_{ij}\, \Lambda_{12}^{(j)} \otimes \Lambda_{34}^{(j)}$$

$$\lambda_{ij} = \frac{1}{4} \begin{pmatrix} +1 & +1 & +1 & -1 & +1 \\ +4 & -2 & 0 & -2 & -4 \\ +6 & 0 & -2 & 0 & +6 \\ -4 & -2 & 0 & -2 & +4 \\ +1 & -1 & +1 & +1 & +1 \end{pmatrix},$$

- Using this relation it is easy to show that

$$(\gamma_\mu\gamma_L)_{32} \otimes (\gamma_\mu\gamma_L)_{14} = -(\gamma_\mu\gamma_L)_{12} \otimes (\gamma_\mu\gamma_L)_{34}$$

and that

$$(\gamma_\mu\gamma_R)_{32} \otimes (\gamma_\mu\gamma_L)_{14} = 2(\gamma_R)_{12} \otimes (\gamma_L)_{34}$$

Fierz+Charge conjugation identity

We want to show the identity, $\gamma^\mu \otimes \langle C - |\gamma_\mu|D-\rangle = 2\Bigg[|C+\rangle\langle D+| + |D-\rangle\langle C-|\Bigg]$ it is helpful to make the indices explicit, so we can rewrite this as

$$\gamma^\mu_{ij}\ \langle C-, k|(\gamma_\mu\gamma_L)_{kl}|D-, l\rangle = 2\Bigg[|C+, i\rangle\langle D+, j| + |D-, i\rangle\langle C-, j|\Bigg]$$

The indices have been added to the bras and kets which remind us that these are four component objects. The relation can be written as two separate equations, where in the second we exploit the charge conjugation identity,

$$(\gamma^\mu\gamma_R)_{ij}\ \langle C-, k|(\gamma_\mu\gamma_L)_{kl}|D-, l\rangle = 2\Bigg[|D-, i\rangle\langle C-, j|\Bigg]$$

$$(\gamma^\mu\gamma_L)_{ij}\ \langle D+, k|(\gamma_\mu\gamma_R)_{kl}|C+, l\rangle = 2\Bigg[|C+, i\rangle\langle D+, j|\Bigg]$$

Both of these relations then follow as a consequence of the Fierz identity

$$(\gamma_\mu\gamma_R)_{ij} \otimes (\gamma_\mu\gamma_L)_{kl} = 2(\gamma_L)_{il} \otimes (\gamma_R)_{kj}$$

Conventions for massless spinor products

$$\langle pq \rangle = \langle p - | q + \rangle, \quad [pq] = \langle p + | q - \rangle$$

$$\langle p | = \langle p - |, \quad [p| = \langle p + |, \quad |q+\rangle = |q\rangle, \quad |q-\rangle = |q]$$

$$\langle pq \rangle = \langle p - | q + \rangle, \quad [pq] = \langle p + | q - \rangle$$

$$\langle p|\gamma_\mu|p] = [p|\gamma_\mu|p\rangle = 2p_\mu$$

$$\langle p + | q + \rangle = \langle p - | q - \rangle = \langle pp \rangle = [pp] = 0$$

$$\langle pq \rangle = -\langle qp \rangle, \quad [pq] = -[qp]$$

$$2|p\pm\rangle\langle q \pm | = \tfrac{1}{2}(1 \pm \gamma_5)\gamma^\mu \langle q \pm |\gamma_\mu|p\pm\rangle$$

$$\langle pq \rangle^* = -\mathrm{sign}(p \cdot q)[pq] = \mathrm{sign}(p \cdot q)[qp]$$

$$|\langle pq \rangle|^2 = \langle pq \rangle \langle pq \rangle^* = 2|p \cdot q| \equiv |s_{pq}|$$

$$\langle pq \rangle [qp] = 2p \cdot q \equiv s_{pq}$$

$$\langle p \pm |\gamma_{\mu_1} \ldots \gamma_{\mu_{2n+1}}|q\pm\rangle = \langle q \mp |\gamma_{\mu_{2n+1}} \ldots \gamma_{\mu_1}|p\mp\rangle$$

$$\langle p \pm |\gamma_{\mu_1} \ldots \gamma_{\mu_{2n}}|q\mp\rangle = -\langle q \pm |\gamma_{\mu_{2n}} \ldots \gamma_{\mu_1}|p\mp\rangle$$

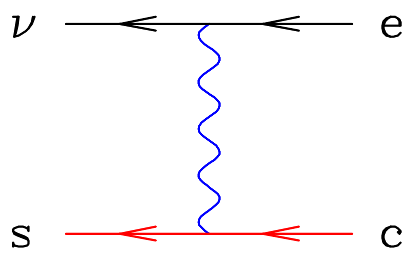
$$\langle AB \rangle \langle CD \rangle = \langle AD \rangle \langle CB \rangle + \langle AC \rangle \langle BD \rangle$$

$$\langle A + |\gamma_\mu|B+\rangle \langle C - |\gamma^\mu|D-\rangle = 2[AD]\langle CB \rangle$$

$c\bar{s} \to W^+ \to \nu e^+$

- Consider the crossed process

$$\nu + s \leftarrow e^- + c$$



- The matrix element is given by

$$\mathcal{M} = \frac{(-ig_W)^2}{2} \frac{(-i)}{P_W(s_{e\nu})} \langle \nu|\gamma^\alpha|e] \, \langle s|\gamma_\alpha|c] \equiv \frac{ig_W^2}{P_W(s_{e\nu})} \langle \nu s \rangle \, [ce]$$

where $P_X(p) = p^2 - m_X^2 + im_X\Gamma_X$.

- We see that the answer is immediate.
- We have assumed that all fermions, including the charmed quark are massless.

$$|\mathcal{M}|^2 = \frac{g_W^4}{|P_W(s_{e\nu})|^2} \; 2\nu \cdot s \; 2e \cdot c$$

since $\langle ij \rangle [ji] = 2i \cdot j$.

- Matrix element is largest when e and c (ν and s) are antiparallel

$c\bar{s} \rightarrow W^+ \rightarrow \nu e^+$

- Compare the calculation performed using the traditional method with the traces, setting $u(s)\bar{u}(s) = \not{s}$ etc. The matrix element is given by

$$\mathcal{M} \propto \bar{u}(\nu)\gamma^\alpha \gamma_L u(e)\ \bar{u}(s)\gamma_\alpha \gamma_L u(c)$$

$$\begin{aligned} |\mathcal{M}|^2 &= \mathrm{Tr}\{\not{\nu}\gamma^\alpha \gamma_L \not{e}\gamma^\beta \gamma_L\}\{\not{s}\gamma_\alpha \gamma_L \not{c}\gamma_\beta \gamma_L\} \\ &= 4\ \{\nu^\alpha e^\beta + \nu^\beta e^\alpha - g^{\alpha\beta} e\cdot\nu + i\epsilon^{\alpha\beta\gamma\delta}\nu_\gamma e_\delta\} \\ &\times\ \{s_\alpha c_\beta + s_\beta c_\alpha - g_{\alpha\beta} c\cdot s + i\epsilon_{\alpha\beta\rho\sigma} s^\rho c^\sigma\} \\ &= 4\ e\cdot c\ s\cdot\nu \end{aligned}$$

where we have used

$$\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\rho\sigma} = -2\Big[g^\gamma_\rho g^\delta_\sigma - g^\gamma_\sigma g^\delta_\rho\Big]$$

- Using the spinor method the γ-matrix algebra simply disappears.

Definition of gluon (photon) polarization

- We can define the photon polarization in terms of massless spinors
- For polarization with momentum k and gauge vector b

$$\varepsilon_\mu^\pm(k,b) = \pm \frac{\langle k \pm | \gamma_\mu | b \pm \rangle}{\sqrt{2}\langle b \mp | k \pm \rangle}$$

- Hence we have that $\varepsilon_\mu^\pm(k,b) \cdot k = 0$ and $\varepsilon_\mu^\pm(k,b) \cdot b = 0$

$$\varepsilon_\mu^+(k,b) = \frac{\langle k + | \gamma_\mu | b + \rangle}{\sqrt{2}\langle bk \rangle} \equiv \frac{[k|\gamma_\mu|b\rangle}{\sqrt{2}\langle bk \rangle}$$

$$\varepsilon_\mu^-(k,b) = \frac{\langle k - | \gamma_\mu | b - \rangle}{\sqrt{2}[kb]} \equiv \frac{\langle k|\gamma_\mu|b]}{\sqrt{2}[kb]}$$

and

$$\gamma^\mu \varepsilon_\mu^+(k,b) = \frac{\sqrt{2}\Big[|k-\rangle\langle b-| + |b+\rangle\langle k+|\Big]}{\langle bk \rangle} \equiv \frac{\sqrt{2}\Big[|k]\langle b| + |b\rangle[k|\Big]}{\langle bk \rangle}$$

$$\gamma^\mu \varepsilon_\mu^-(k,b) = \frac{\sqrt{2}\Big[|k+\rangle\langle b+| + |b-\rangle\langle k-|\Big]}{[kb]} \equiv \frac{\sqrt{2}\Big[|k\rangle[b| + |b]\langle k|\Big]}{[kb]}$$

Auxiliary vector

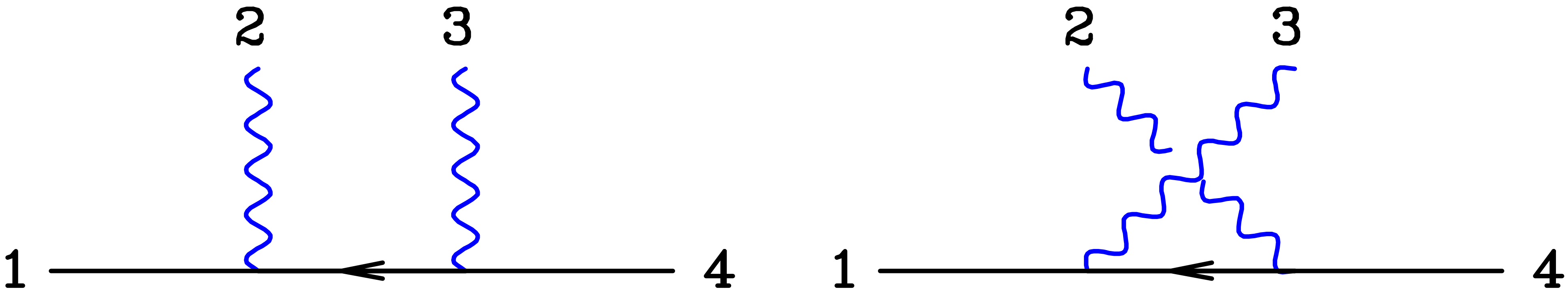
Different choices of the auxiliary vector b correspond to different choices of gauge. Thus

$$\begin{aligned}\varepsilon_\mu^+(k,b) - \varepsilon_\mu^+(k,c) &= \frac{[k|\gamma_\mu|b\rangle}{\sqrt{2}\langle bk\rangle} - \frac{[k|\gamma_\mu|c\rangle}{\sqrt{2}\langle ck\rangle} \\ &= \frac{1}{\sqrt{2}\langle bk\rangle\langle ck\rangle}\Big[[k|\gamma_\mu|b\rangle\langle ck\rangle - [k|\gamma_\mu|c\rangle\langle bk\rangle\Big] \\ &= \frac{1}{\sqrt{2}\langle bk\rangle\langle ck\rangle}\Big[[k|\gamma_\mu|k\rangle\langle cb\rangle\Big] = \frac{\sqrt{2}\langle cb\rangle}{\langle bk\rangle\langle ck\rangle}k_\mu \end{aligned} \tag{3}$$

where we have used the Schouten identity

$$[k|\gamma_\mu|b\rangle\langle ck\rangle = [k|\gamma_\mu|k\rangle\langle cb\rangle + [k|\gamma_\mu|c\rangle\langle bk\rangle$$

$1_e^- 2_\gamma^+ 3_\gamma^+ 4_{\bar{e}}^+$



The relevant diagram is shown. We shall consider only the case of a left-handed electron line.

$$\mathcal{M} = (-ie)^2 i \Big[\frac{\langle 1 - |\not\epsilon_2 (\not 1 + \not 2) \not\epsilon_3 | 4 - \rangle}{\langle 1\, 2 \rangle \, [2\, 1]} + \frac{\langle 1 - |\not\epsilon_3 (\not 1 + \not 3) \not\epsilon_2 | 4 - \rangle}{\langle 1\, 3 \rangle \, [3\, 1]} \Big]$$

Using our form for the polarization vectors we obtain the result where both polarizations are positive

$$\mathcal{M}(1_e^-, 2_\gamma^+, 3_\gamma^+, 4_{\bar{e}}^+) = \frac{-2ie^2}{\langle b_2\, 2 \rangle \, \langle b_3\, 3 \rangle} \Big[\frac{\langle 1\, b_2 \rangle \, [2\, 1] \, \langle 1\, b_3 \rangle \, [3\, 4]}{\langle 1\, 2 \rangle \, [2\, 1]} + \frac{\langle 1\, b_3 \rangle \, [3\, 1] \, \langle 1\, b_2 \rangle \, [2\, 4]}{\langle 1\, 3 \rangle \, [3\, 1]} \Big]$$

Making the gauge choice $b_2 = b_3 = 1$ this gives zero.

$1_e^- 2_\gamma^+ 3_\gamma^- 4_{\bar{e}}^+$

Inserting the case where the polarizations are $(+-)$ we get

$$\mathcal{M}(1_e^-, 2_\gamma^+, 3_\gamma^-, 4_{\bar{e}}^+) = \frac{-2ie^2}{\langle b_2\, 2\rangle\, [3\, b_3]} \left[\frac{\langle 1\, b_2\rangle\, [2\, 1]\, \langle 1\, 3\rangle\, [b_3\, 4]}{\langle 1\, 2\rangle\, [2\, 1]} + \frac{\langle 1\, 3\rangle\, \langle b_3 + |(\not{1} + \not{3})| b_2 + \rangle\, [2\, 4]}{\langle 1\, 3\rangle\, [3\, 1]} \right]$$

Making the gauge choice $b_2 = 1, b_3 = 4$, the first diagram gives no contribution this gives

$$\mathcal{M}(1_e^-, 2_\gamma^+, 3_\gamma^-, 4_{\bar{e}}^+) = \frac{-2ie^2}{\langle 1\, 2\rangle\, [3\, 4]} \frac{\langle 1\, 3\rangle\, [4\, 3]\, \langle 3\, 1\rangle\, [2\, 4]}{\langle 1\, 3\rangle\, [3\, 1]} = 2ie^2 \frac{\langle 3\, 1\rangle\, [2\, 4]}{\langle 1\, 2\rangle\, [3\, 1]} \equiv 2ie^2 \frac{[2\, 4]^2}{[3\, 4]\, [3\, 1]}$$

In deriving the latter formula we have used momentum conservation. Note that the result is of second degree in $|2+\rangle$ and $|3-\rangle$ as it must be for a (+-) amplitude, and also of first degree in $|1-\rangle$ and $|4+\rangle$.
In summary we find

$$\mathcal{M}(1_e^-, 2_\gamma^-, 3_\gamma^+, 4_{\bar{e}}^+) = 2ie^2 \frac{[2\, 4]^2}{[3\, 4]\, [3\, 1]}$$

$$\mathcal{M}(1_e^-, 2_\gamma^+, 3_\gamma^-, 4_{\bar{e}}^+) = 2ie^2 \frac{[3\, 4]^2}{[2\, 4]\, [2\, 1]}$$

$$\mathcal{M}(1_e^-, 2_\gamma^+, 3_\gamma^+, 4_{\bar{e}}^+) = 0$$

$$\mathcal{M}(1_e^-, 2_\gamma^-, 3_\gamma^-, 4_{\bar{e}}^+) = 0$$

Color decomposition of gluon amplitudes

- In the amplitude literature it is usual to define the normalization of the color matrices as

$$\mathrm{Tr}\{\tau^{a_1}\tau^{a_2}\} = \delta^{a_1 a_2}$$

so that $\tau^a = \sqrt{2}T^a$.

- We can eliminate the structure constants F^{abc} in favor of the τ^a's, using

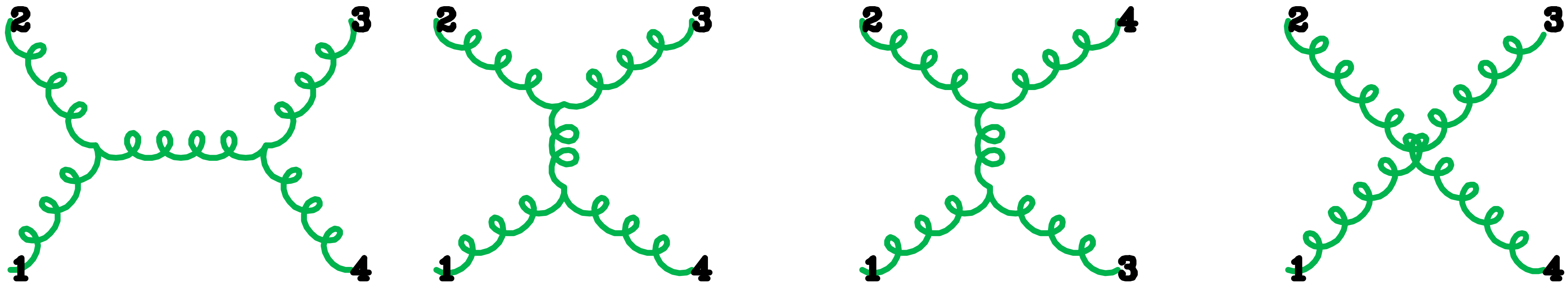
$$F^{abc} = -\frac{i}{\sqrt{2}}\Big(\mathrm{Tr}\big(\tau^a\tau^b\tau^c\big) - \big(\tau^a\tau^c\tau^b\big)\Big),$$

- This leads to the color decomposition of the the n-gluon tree amplitude,

$$\mathcal{A}_n(\{k_i,\lambda_i,a_i\}) = g^{n-2}\sum_{\sigma \mathrm{in} S_n/Z_n} \mathrm{Tr}(\tau^{a_{\sigma(1)}}\cdots\tau^{a_{\sigma(n)}})\; A_n(\sigma(1^{\lambda_1}),\ldots,\sigma(n^{\lambda_n})).$$

- The color ordered amplitudes are gauge invariant,

$A_4(1_g^{\pm} 2_g^+ 3_g^+ 4_g^+)$



- $A_4(1_g^+, 2_g^+, 3_g^+, 4_g^+) = 0$, This is demonstrated by choosing common auxiliary vector b for all polarizations $\varepsilon_n \cdot \varepsilon_m = 0$ for all m and n.
- $A_4(1_g^-, 2_g^+, 3_g^+, 4_g^+) = 0$ choose $b_1 = p_4$ and $b_2 = b_3 = b_4 = p_1$, so that we $\varepsilon_n \cdot \varepsilon_m = 0$ for all m and n.
- These results generalize to all gluon multiplicities, since a n multiplicity diagram has at most $n-2$ vertices linear in the momenta, leaving at least two polarization vectors which must be contracted.
- $A_4(1_g^-, 2_g^-, 3_g^+, 4_g^+) \neq 0$: This is the maximally helicity violating amplitude; which we shall now calculate.

$A_4(1^-, 2^-, 3^+, 4^+)$, *MHV*

Next we turn to the (nonzero) helicity amplitude $A_4(1^-, 2^-, 3^+, 4^+)$, choosing the auxiliary momenta $b_1 = b_2 = p_4$, $b_3 = b_4 = p_1$, so that only the contraction $\varepsilon_2^- \cdot \varepsilon_3^+$ is nonzero. Only one of the four potential graphs contributes, to the color-ordering $\mathrm{Tr}\{\tau^{a_1}\tau^{a_2}\tau^{a_3}\tau^{a_4}\}$, the one with a gluon exchange in the s_{12} channel.

$$
\begin{aligned}
A_4(1^-, 2^-, 3^+, 4^+) &= \left(\frac{-i}{\sqrt{2}}\right)^2 \left(\frac{-i}{s_{12}}\right) \\
&\times \left[\varepsilon_1^- \cdot \varepsilon_2^- (p_1 - p_2)^\mu + (\varepsilon_2^-)^\mu \varepsilon_1^- \cdot (2p_2 + p_1) + (\varepsilon_1^-)^\mu \varepsilon_2^- \cdot (-2p_1 - p_2)\right] \\
&\times \left[\varepsilon_3^+ \cdot \varepsilon_4^+ (p_3 - p_4)_\mu + (\varepsilon_4^+)_\mu \varepsilon_3^+ \cdot (2p_4 + p_3) + (\varepsilon_3^+)_\mu \varepsilon_4^+ \cdot (-2p_3 - p_4)\right] \\
&= -\frac{2i}{s_{12}} (\varepsilon_2^- \cdot \varepsilon_3^+)(\varepsilon_1^- \cdot p_2)(\varepsilon_4^+ \cdot p_3) \\
&= -\frac{2i}{s_{12}} \left(-\frac{2}{2}\frac{[4\,3]\,\langle 1\,2\rangle}{[4\,2]\,\langle 1\,3\rangle}\right) \left(-\frac{[4\,2]\,\langle 2\,1\rangle}{\sqrt{2}\,[4\,1]}\right) \left(+\frac{\langle 1\,3\rangle\,[3\,4]}{\sqrt{2}\,\langle 1\,4\rangle}\right) = -i\,\frac{\langle 1\,2\rangle\,[3\,4]^2}{[1\,2]\,\langle 1\,4\rangle\,[1\,4]}\,.
\end{aligned}
$$

The answer can be simplified using antisymmetry, momentum conservation, and $s_{34} = s_{12}$,

$$
A_4(1^-, 2^-, 3^+, 4^+) = -i\,\frac{\langle 1\,2\rangle\,([3\,4]\,\langle 3\,4\rangle)(\langle 2\,3\rangle\,[3\,4])}{[1\,2]\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 1\,4\rangle\,[1\,4]} = i\,\frac{\langle 1\,2\rangle\,([1\,2]\,\langle 1\,2\rangle)(-\langle 2\,1\rangle\,[1\,4])}{[1\,2]\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle\,[1\,4]}
$$

Maximally helicity violating amplitude

- the final result for the four point amplitude is,

$$A_4(1^-, 2^-, 3^+, 4^+) = i\,\frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}\,,\;=\;i\,\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}\,.$$

- An all n expression for the MHV amplitude was found by Parke and Taylor
- The full n-gluon amplitude including the color decomposition is

$$\mathcal{A}_n(1_g^-, 2_g^-, 3_g^+, \ldots, n_g^+) = ig^{n-2} \sum_{\{1,2,\ldots\}'} \mathrm{Tr}\{\tau_1 \tau_2 \cdots \tau_\mathrm{n}\} \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle \mathrm{n}1\rangle}$$

where the sum is taken over the $(n-1)!$ non-cyclic permutations of the indices.

Recap

- The solutions massless Dirac equation can be divided into holomorphic and antiholomorphic solution, compactly represented by angle and square brackets.
- Charge conjugation for massless particles is particularly simple $v_{\pm}(p) = u_{\mp}(p)$
- Polarization vectors for photons (gluons) are expressed in terms of massless spinors of the photon (gluon) momenta and an additional spinor of an auxiliary momentum.
- Different choices of auxiliary momentum correspond to different gauge choices.
- Amplitudes involving massless particles are naturally expressed in terms of spinor products.
- Using gluon polarizations expressed in terms of massless spinors, the gamma matrix algebra often disappears.