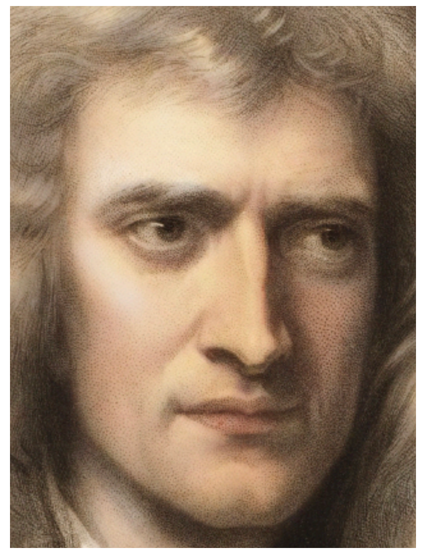
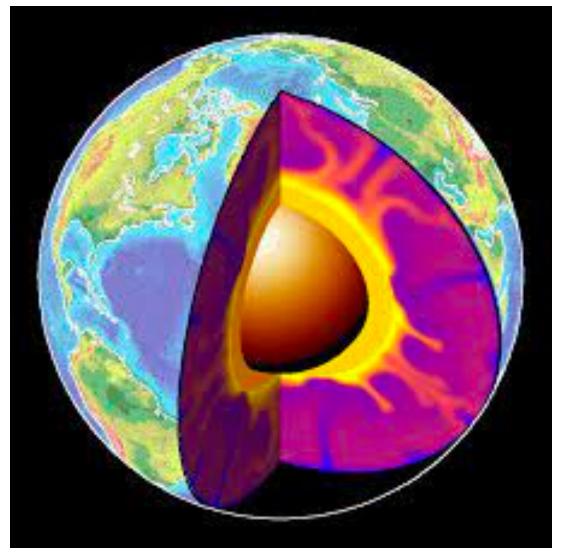
Joint ICTP-EAIFR-IUGG Workshop on Computational Geodynamics: Towards Building a New Expertise Across Africa

Theory from  to  in 3 hours or less

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AMBASSADE DE FRANCE AU RWANDA
Liberté Égalité Fraternité

IUGG

United Nations Educational, Scientific and Cultural Organization

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UNIVERSITY OF OXFORD

RIFT-O-MAT

FOALAB

How to **learn** continuum mechanics

1. Study continuum mechanics;
2. Do problems with continuum mechanics;
3. Go back to 1.

[Caution: this may take your whole life.]

Outline

1. Review of Newton’s laws, displacement & velocity fields
2. Inertia in a continuum
3. Forces in a continuum (stress)
4. Rheology & deformation in 1D
5. Rheology & deformation in 3D
6. Conservation of mass
7. Full equations of motion
8. Scaling analysis & simplification
9. Work, energy & power
10. Plasticity

Newton's laws of mechanics

1. Every body continues in a state of rest or uniform motion in a straight line unless acted upon by a force.
2. **The rate of change of momentum of a body is given by the applied force.**
3. Action and reaction are equal and opposite at all times.

Applies to

- Objects more massive than atoms;
- Speeds slower than the speed of light.

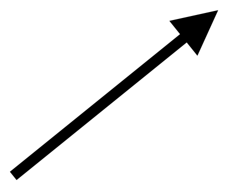
Can be expressed as

$$\sum \mathbf{F} = M\mathbf{a}$$

(And usually, $\frac{dM}{dt} = 0$)

A note about notation

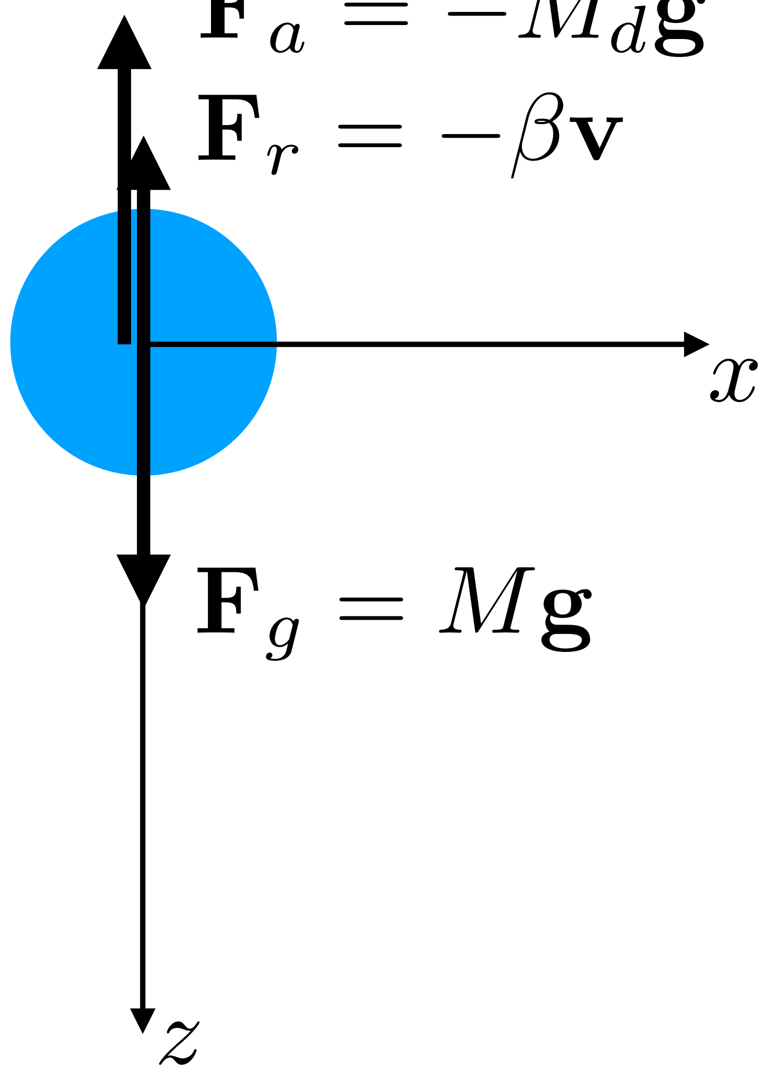
- *Tensors* (of second order or higher) are expressed with italic bold. $\boldsymbol{A}$

- *Vectors* (tensors of first order) are expressed with non-italic bold.  $\mathbf{F}$

- *Scalars* are expressed with italic and have only a magnitude. F

- *Indicies* are integers and are expressed as subscripts. $F_i,\ A_{ij}$

Example: ball falling through fluid



$\mathbf{F}_a = -M_d\mathbf{g}$ Archimedes force

$\mathbf{F}_r = -\beta\mathbf{v}$ drag force

$\mathbf{F}_g = M\mathbf{g}$ gravitational body force

$$\sum \mathbf{F} = M\mathbf{a}$$

$$\mathbf{F}_g + \mathbf{F}_a + \mathbf{F}_r = M\mathbf{a},$$

$$\Delta_M\mathbf{g} - \beta\dot{\mathbf{x}} = M\ddot{\mathbf{x}}$$

where $\Delta_M = M - M_d$

$$\Delta_M g - \beta\dot{z} = M\ddot{z}$$

if inertia is negligible, $\text{Case 1: } \ddot{z} \sim 0 \Rightarrow \quad \Delta_M g \sim \beta\dot{z} \Rightarrow \quad \dot{z} = \frac{\Delta_M g}{\beta} \Rightarrow \quad z(t) = \frac{\Delta_M g}{\beta}t + z_0$

if drag is negligible, $\text{Case 2: } \beta \sim 0 \Rightarrow \quad \Delta_M g \sim M\ddot{z} \Rightarrow \quad \ddot{z} = \frac{\Delta_M}{M}g \Rightarrow \quad z(t) = \frac{\Delta_M}{M}\frac{g}{2}t^2 + v_0 t + z_0$

The forces, speed and acceleration “move” with the ball (obviously)

Another note about notation

- *Tensors fields* (of second order or higher) are expressed with italic bold.

$$\boldsymbol{A}(\mathbf{x}, t)$$

- *Vectors fields* (tensors of first order) are expressed with non-italic bold.

$$\mathbf{F}(\mathbf{x}, t)$$

- *Scalars fields* are expressed with italic and have only a magnitude.

$$F(\mathbf{x}, t)$$

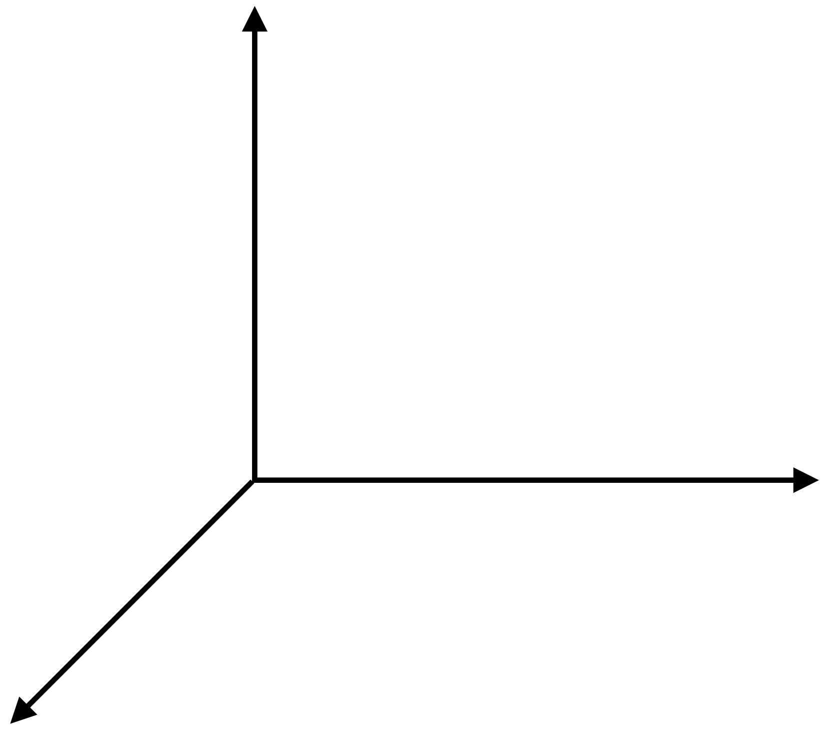
- *Indicies* are integers and are expressed as subscripts.

$$F_i(\mathbf{x}, t), A_{ij}(\mathbf{x}, t)$$

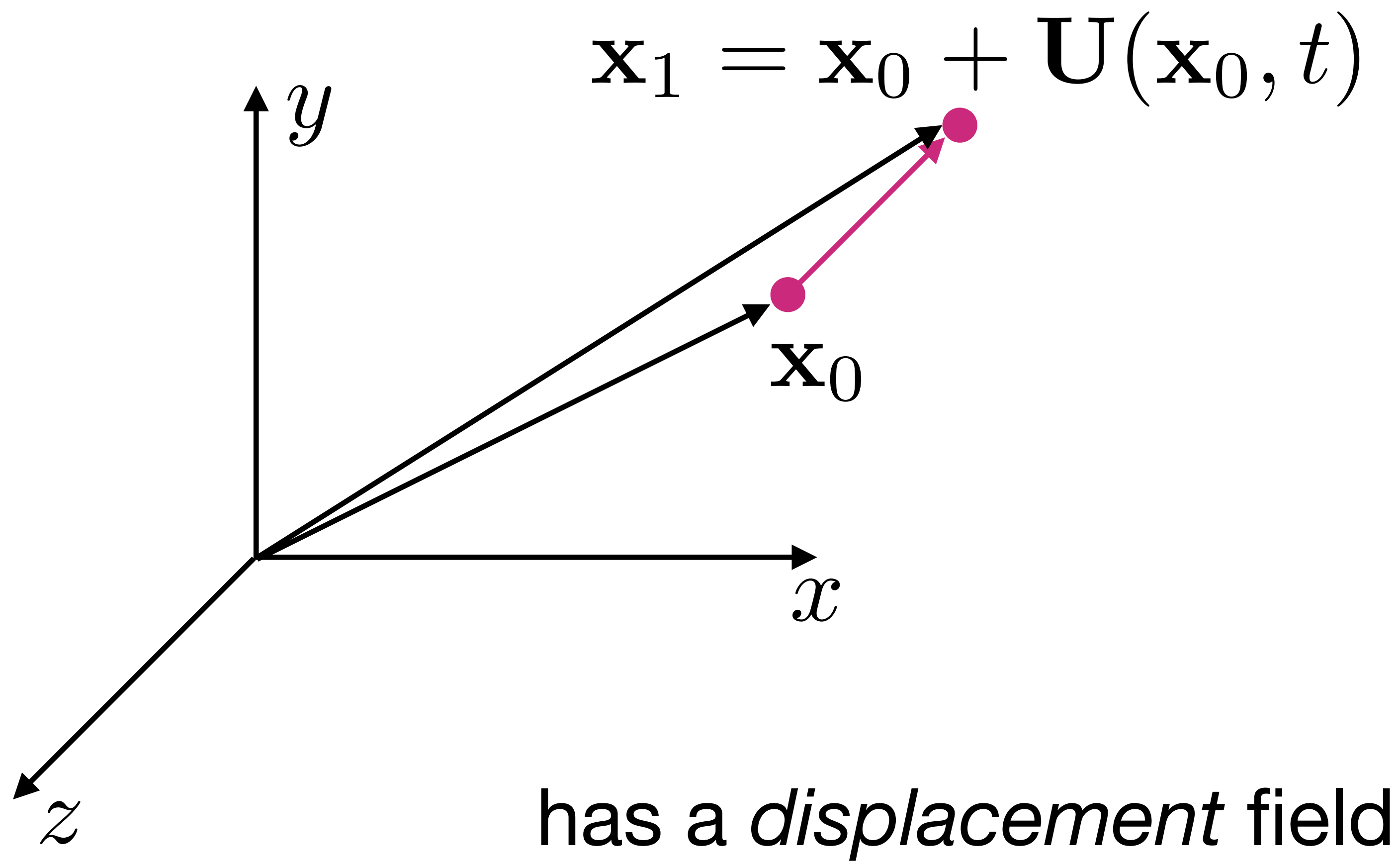

$\mathbf{v}(\mathbf{x})$

Magnetic field around a bar magnet

a continuum



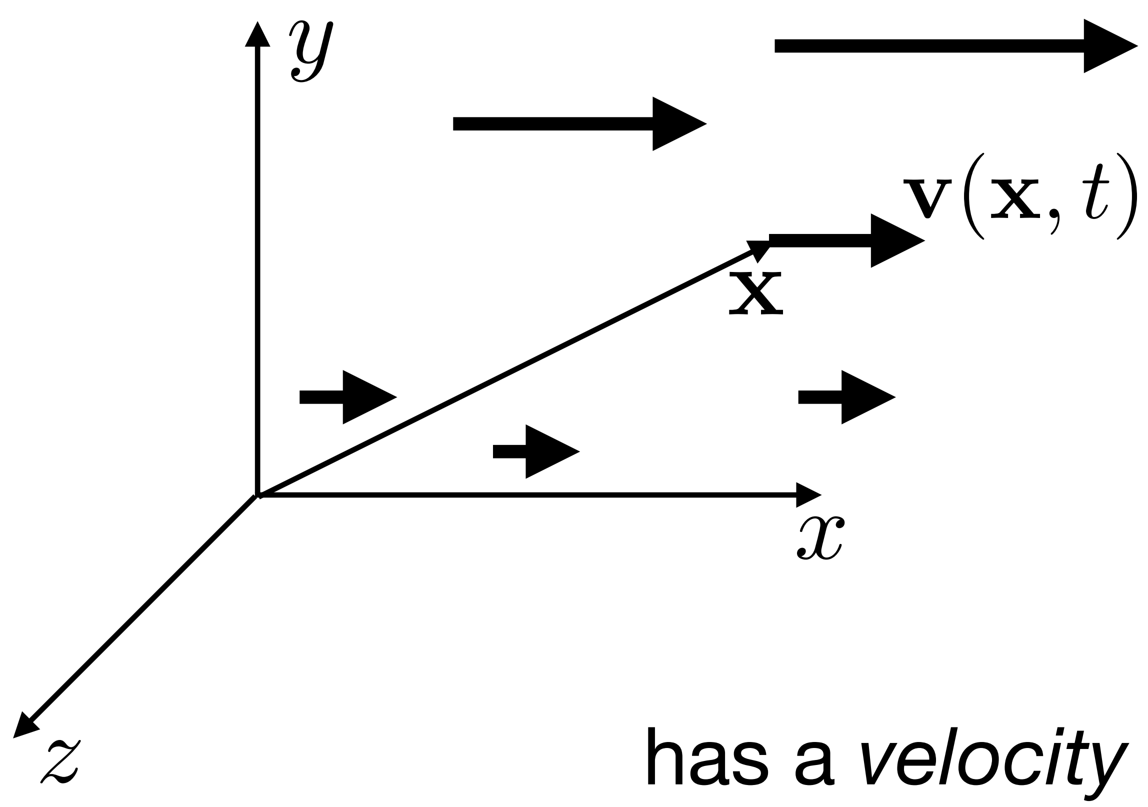
a solid continuum:



has a *displacement* field $\mathbf{U}(\mathbf{x}_0, t)$

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \frac{\partial \mathbf{U}}{\partial t}$$

a fluid continuum:

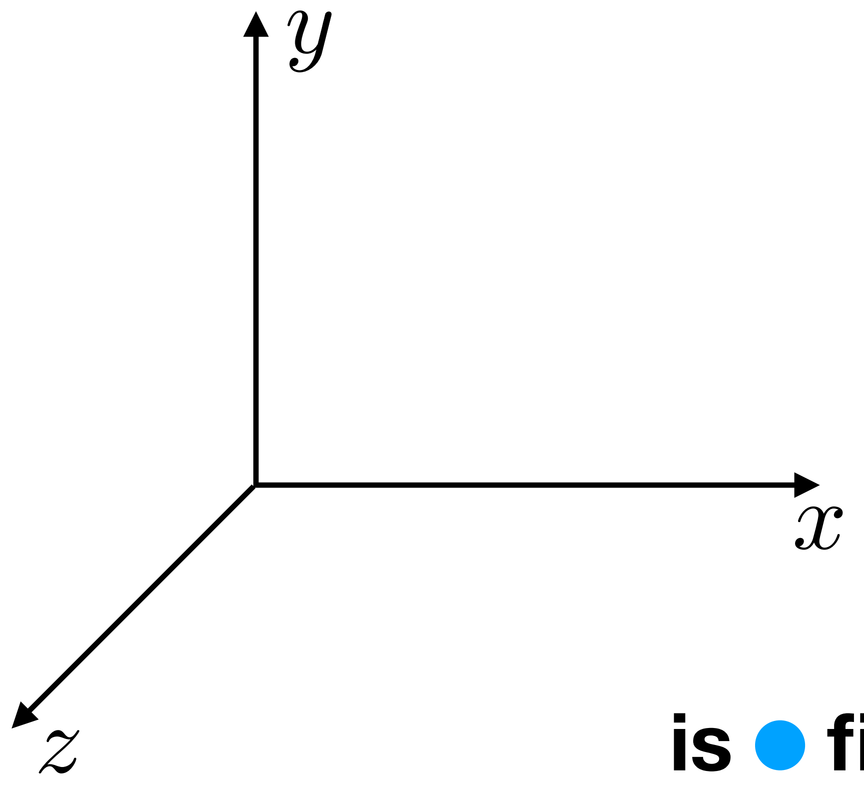


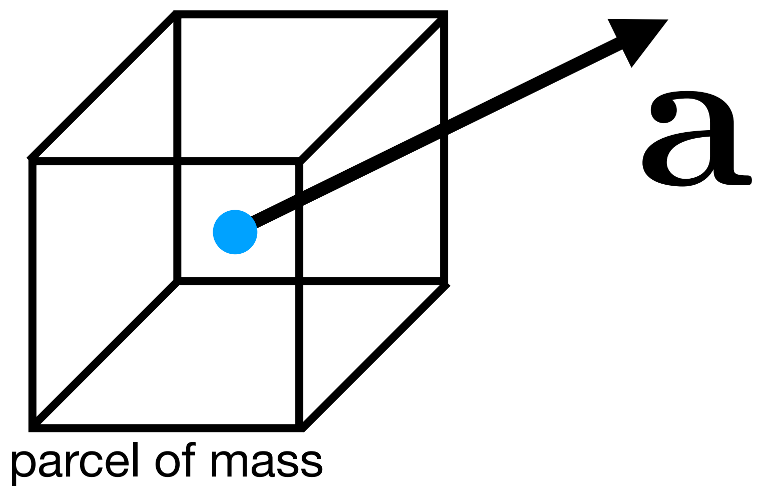
has a *velocity* field $\mathbf{v}(\mathbf{x}, t)$

$$\mathbf{v} = \frac{d\mathbf{x}}{dt}$$

Inertia in a continuum

(very small) parcel of mass in a continuum





is ● fixed in space or fixed to parcel?

$$\sum \mathbf{F} = M\mathbf{a}$$

acceleration of what??

$$\sum \mathbf{F} = M\frac{D\mathbf{v}}{Dt}$$

Lagrangian derivative following the parcel

$$\frac{D\mathbf{v}}{Dt} = \frac{d}{dt}\mathbf{v}(\mathbf{x}(t),t) = \frac{\partial \mathbf{v}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{v}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{v}}{\partial z}\frac{dz}{dt} + \frac{\partial \mathbf{v}}{\partial t}$$

$$= \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \cdot \left(\frac{\partial \mathbf{v}}{\partial x}, \frac{\partial \mathbf{v}}{\partial y}, \frac{\partial \mathbf{v}}{\partial z}\right) + \frac{\partial \mathbf{v}}{\partial t}$$

$$= \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}$$

Material derivative + *Eulerian derivative* at a point in space

The Lagrangian & Eulerian derivatives

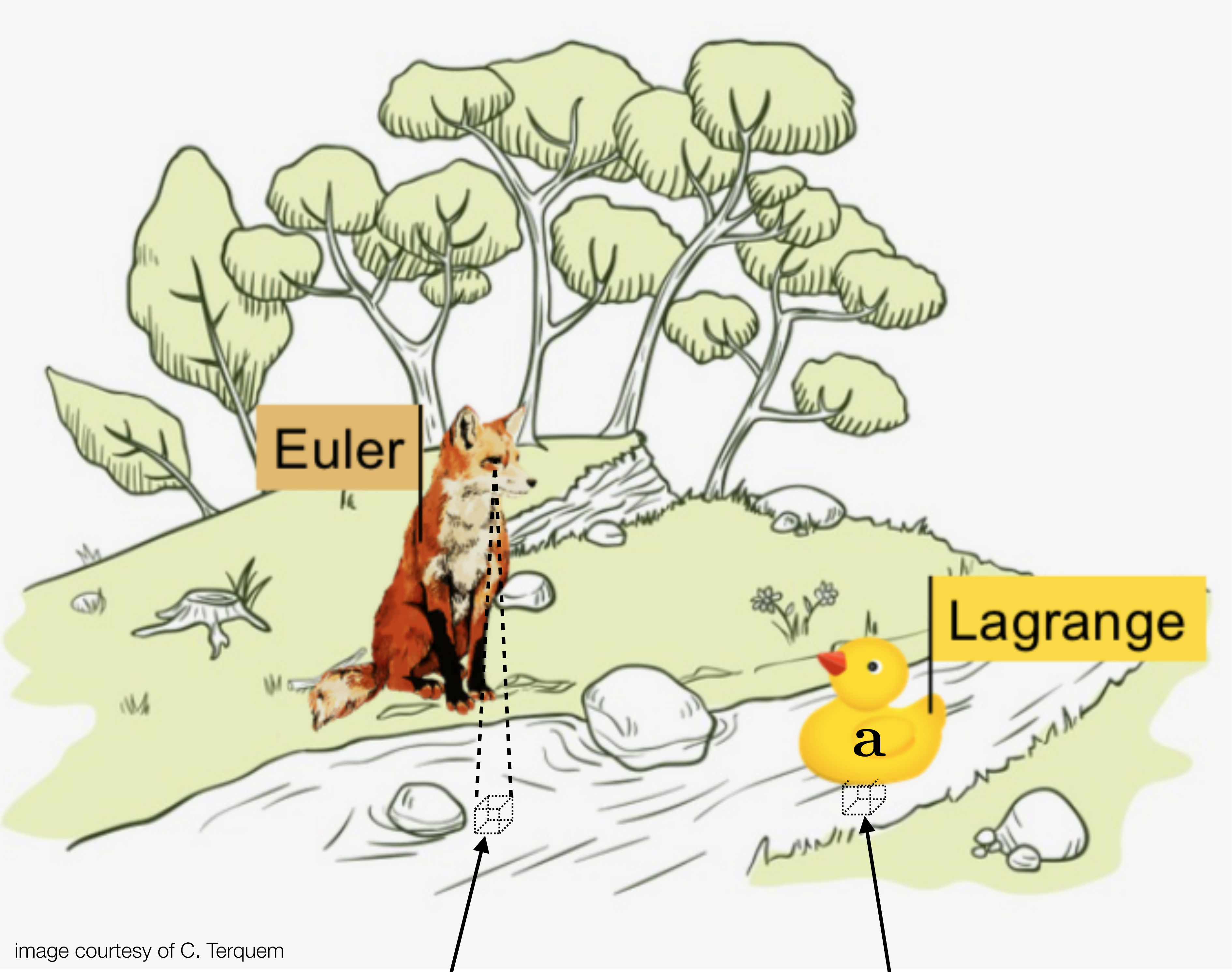


image courtesy of C. Terquem

$$\frac{\partial Q}{\partial t} \qquad \frac{\mathrm{D}Q}{\mathrm{D}t}$$

Performing a Taylor series expansion to first order in δt:

$$Q(\mathbf{r}+\mathbf{v}\delta t, t+\delta t) = Q(\mathbf{r},t) + \delta t \frac{\partial Q(\mathbf{r},t)}{\partial t} + \mathbf{v}\delta t \cdot \boldsymbol{\nabla} Q(\mathbf{r},t),$$

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \lim_{\delta t \to 0} \frac{Q(\mathbf{r}+\mathbf{v}\delta t, t+\delta t) - Q(\mathbf{r},t)}{\delta t}.$$

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = \frac{\partial Q}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} Q.$$

Another note about (index) notation

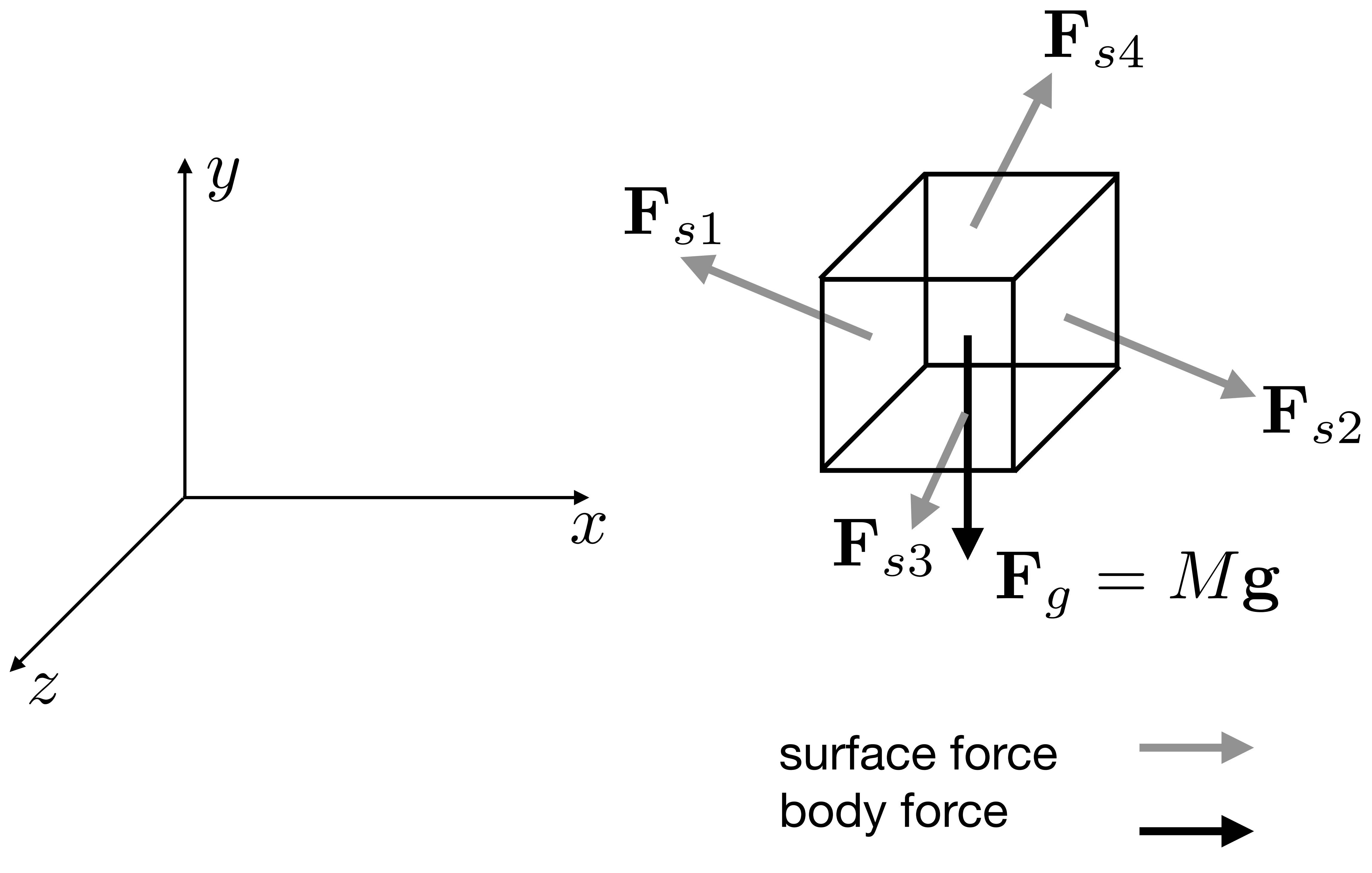
- A vector has one free (unrepeated) index. F_i
- The gradient operator ∇ is a vector. $\frac{\partial}{\partial x_i}$
- A second-rank tensor has two free indicies. A_{ij}
- Summation is indicated by a repeated index (Einstein summation convention).

$$F_i A_{ij} = \sum_i F_i A_{ij}$$

$$\frac{\partial v_i}{\partial x_i} = \sum_i \frac{\partial v_i}{\partial x_i}$$

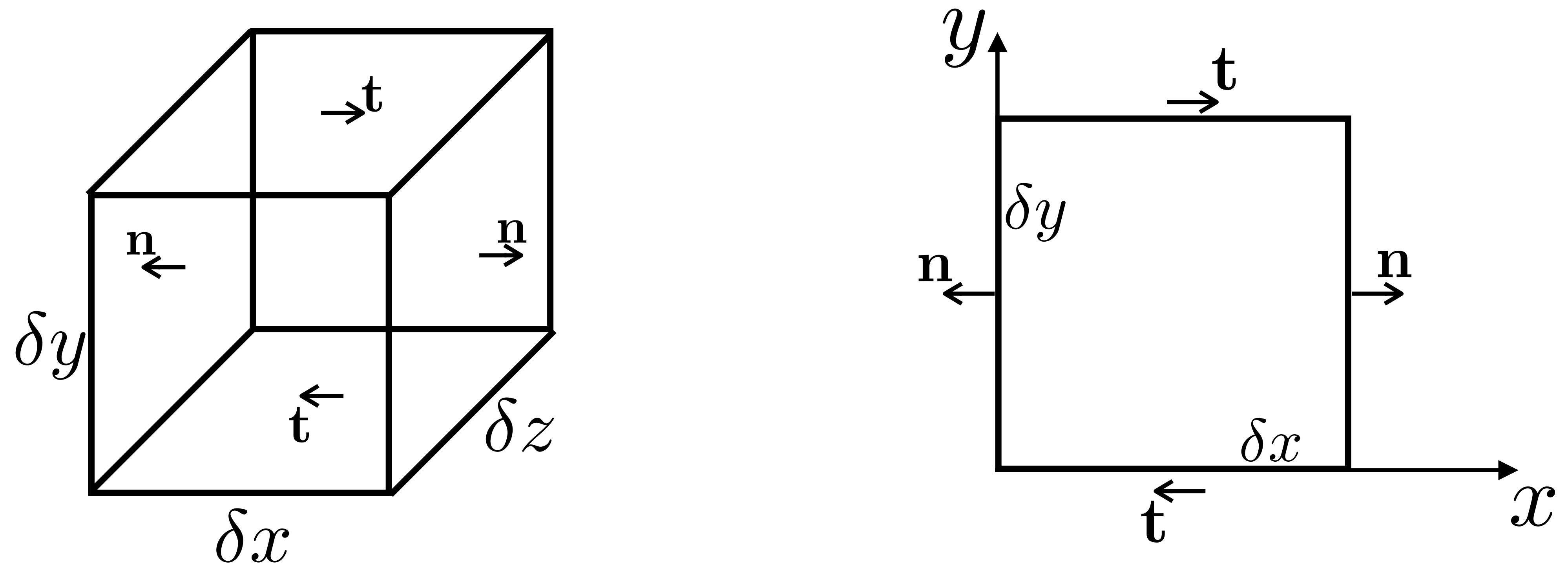
Forces in a continuum

forces on a (very small) parcel of mass in a continuum — in the *x* direction



$$\left(\sum \mathbf{F} = M \frac{D\mathbf{v}}{Dt} \right) \cdot \hat{\mathbf{x}}$$

Define unit normal $\mathbf{n}$ and tangent $\mathbf{t}$ vectors

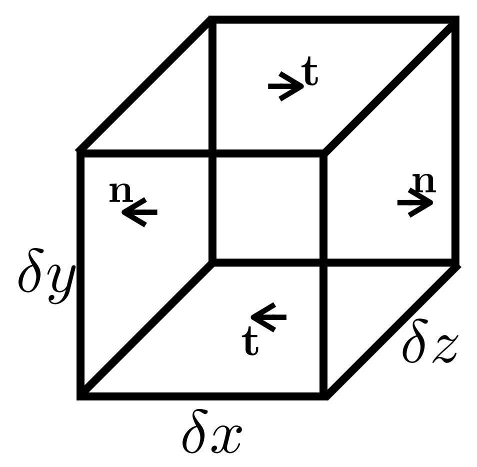


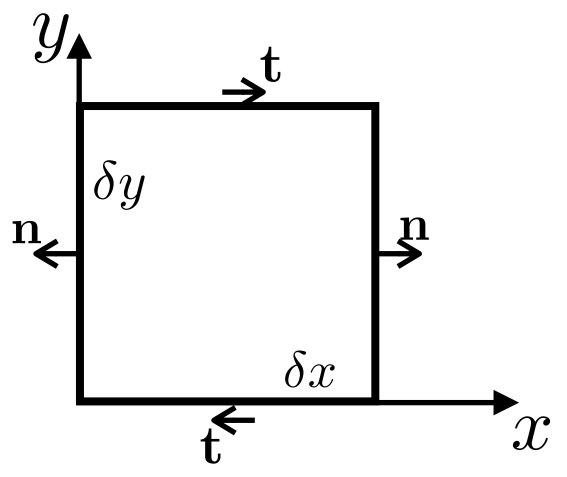
$$\sum \mathbf{F} \cdot \hat{\mathbf{x}} = (\mathbf{F}_{s2} \cdot \mathbf{n})|_{x+\delta x} - (\mathbf{F}_{s1} \cdot \mathbf{n})|_{x} + (\mathbf{F}_{s4} \cdot \mathbf{t})|_{y+\delta y} - (\mathbf{F}_{s3} \cdot \mathbf{t})|_{y} + \ldots$$

forces on a (very small) parcel of mass in a continuum — in the *x* direction

$$\sum \mathbf{F} \cdot \hat{\mathbf{x}} = (\mathbf{F}_{s2} \cdot \mathbf{n})|_{x+\delta x} - (\mathbf{F}_{s1} \cdot \mathbf{n})|_{x} + (\mathbf{F}_{s4} \cdot \mathbf{t})|_{y+\delta y} - (\mathbf{F}_{s3} \cdot \mathbf{t})|_{y} + \ldots$$

$$= [\sigma_{xx}(x+\delta x) - \sigma_{xx}(x)]\delta y \delta z + [\sigma_{yx}(y+\delta y) - \sigma_{yx}(y)]\delta x \delta z + \ldots$$





$$\sigma \equiv \frac{\mathbf{F} \cdot (\mathbf{n}, \mathbf{t})}{\text{area}}$$

define the (normal, shear) *stress* as *force per unit area*!! Units: Pascals = N/m^2

$$\left(\sum \mathbf{F} = M \frac{D\mathbf{v}}{Dt} \right) \cdot \hat{\mathbf{x}}$$

$$[\sigma_{xx}(x+\delta x) - \sigma_{xx}(x)]\delta y \delta z + [\sigma_{yx}(y+\delta y) - \sigma_{yx}(y)]\delta x \delta z + [\sigma_{zx}(z+\delta z) - \sigma_{zx}(z)]\delta x \delta y + M g_x = M \frac{Dv_x}{Dt}$$

Taylor expand: $\sigma_{xx}(x+\delta x) = \sigma_{xx}(x) + \delta x \frac{\partial \sigma_{xx}}{\partial x} + O(\delta x^2)$ for example

Substitute and divide by $V = \delta x \delta y \delta z$ then take $\lim_{\delta x, \delta y, \delta z \to 0}$

$$\Rightarrow \quad \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \frac{M}{V} g_x = \frac{M}{V} \frac{Dv_x}{Dt}$$

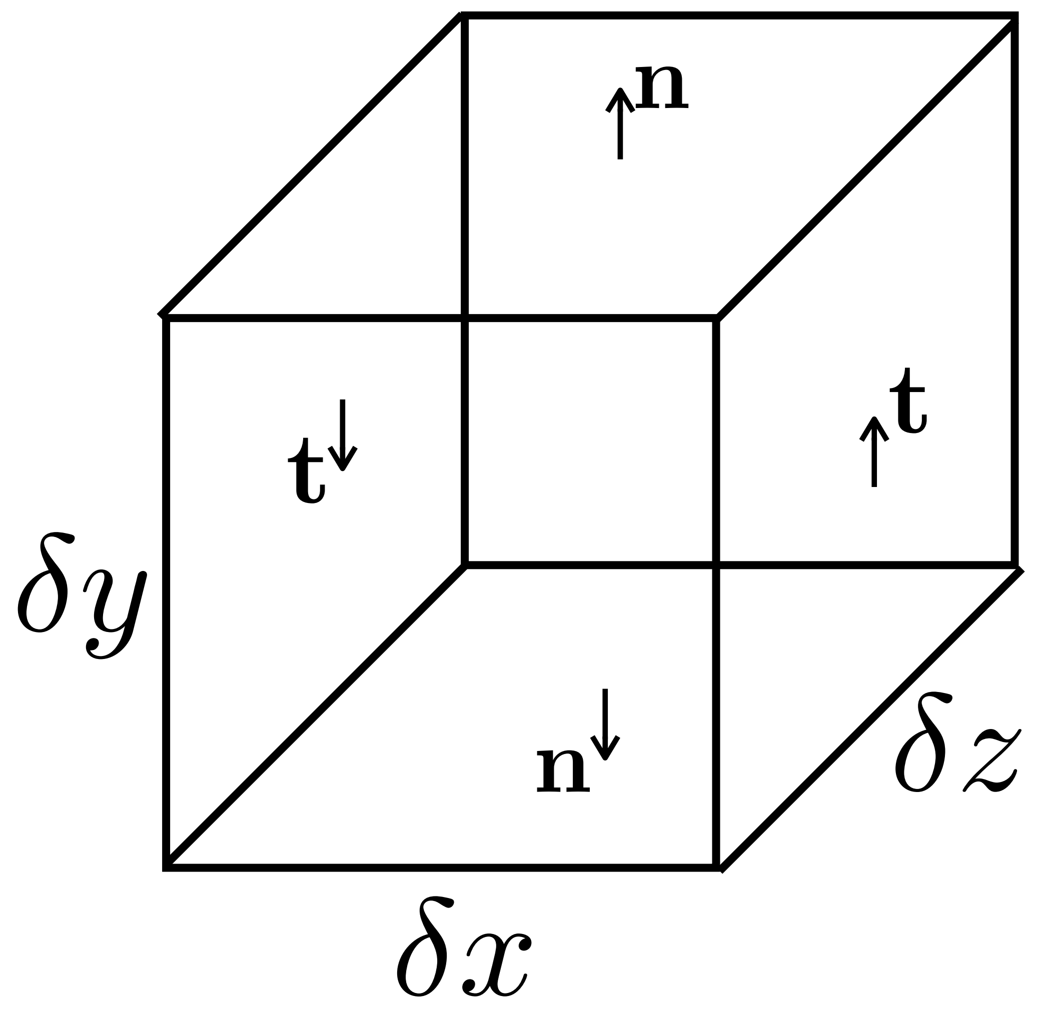
forces on a (very small) parcel of mass in a continuum

In the *x*-direction:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \frac{M}{V} g_x = \frac{M}{V} \frac{D v_x}{Dt}$$

$$\frac{\partial \sigma_{ix}}{\partial x_i} + \rho g_x = \rho \frac{D v_x}{Dt}$$

$$\rho \equiv \frac{M}{V}, \text{ density}$$



In the *y*-direction:

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial z} + \frac{M}{V} g_y = \frac{M}{V} \frac{D v_y}{Dt}$$

$$\frac{\partial \sigma_{iy}}{\partial x_i} + \rho g_y = \rho \frac{D v_y}{Dt}$$

In the j^{th}-direction:

$$\frac{\partial \sigma_{ij}}{\partial x_i} + \rho g_j = \rho \frac{D v_j}{Dt}$$

or

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \rho \mathbf{g} = \rho \frac{D \mathbf{v}}{Dt}$$

The Cauchy momentum equation for any continuum

the Cauchy momentum equation

$$\sum \mathbf{F} = M\mathbf{a} \qquad \Rightarrow \qquad \nabla \cdot \boldsymbol{\sigma} + \mathbf{f}_b = \rho \frac{D\mathbf{v}}{Dt}$$

net surface force	body force (e.g., gravity)	Lagrangian acceleration

Units: Newtons, i.e., kg-m/s^2
[Force]

Units: N/m^3, i.e., kg/m^{2}s^2
[Force per unit volume]

Applies to the centre of mass of a rigid particle

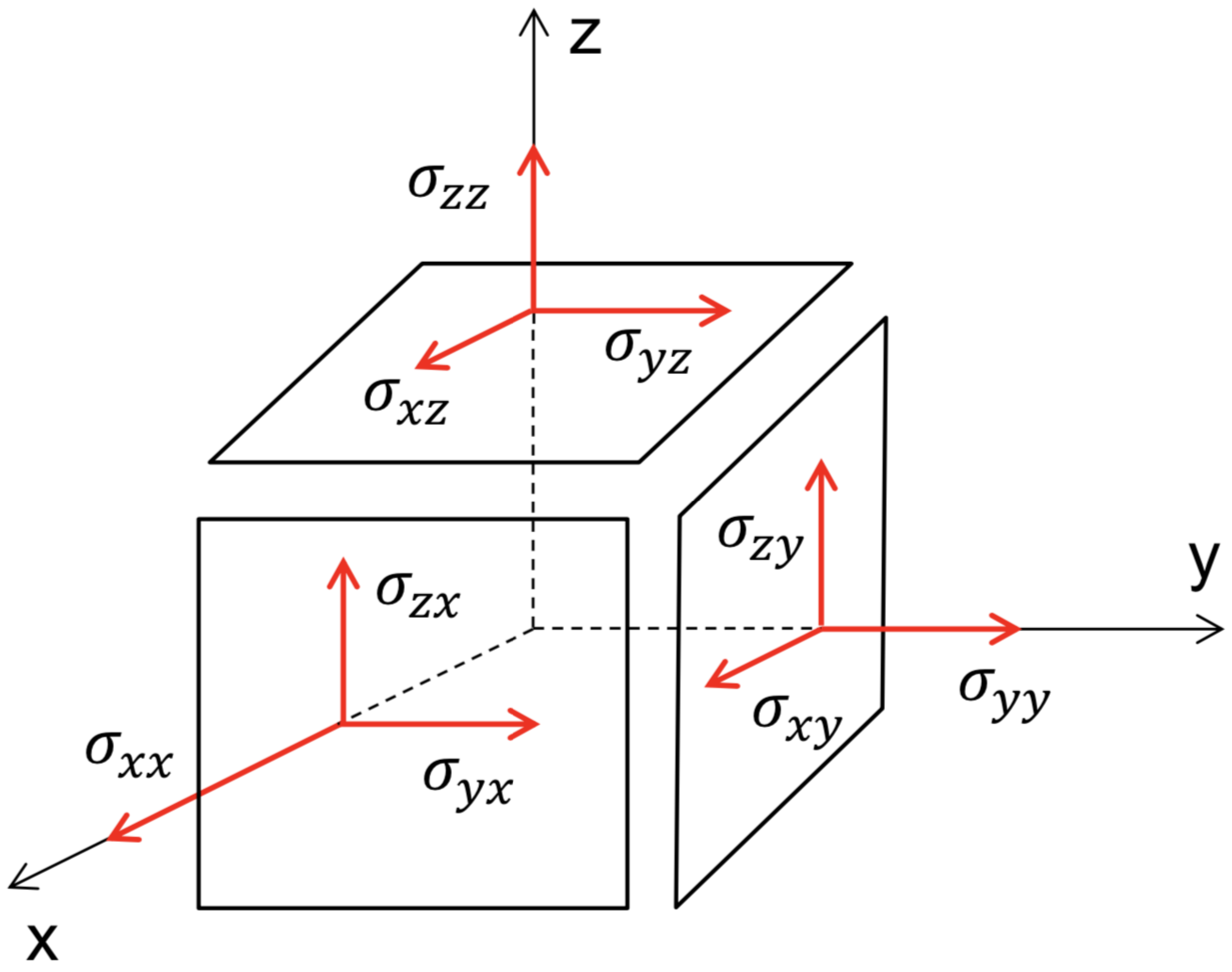
Applies to all continuua (e.g., fluids, solids, complex)

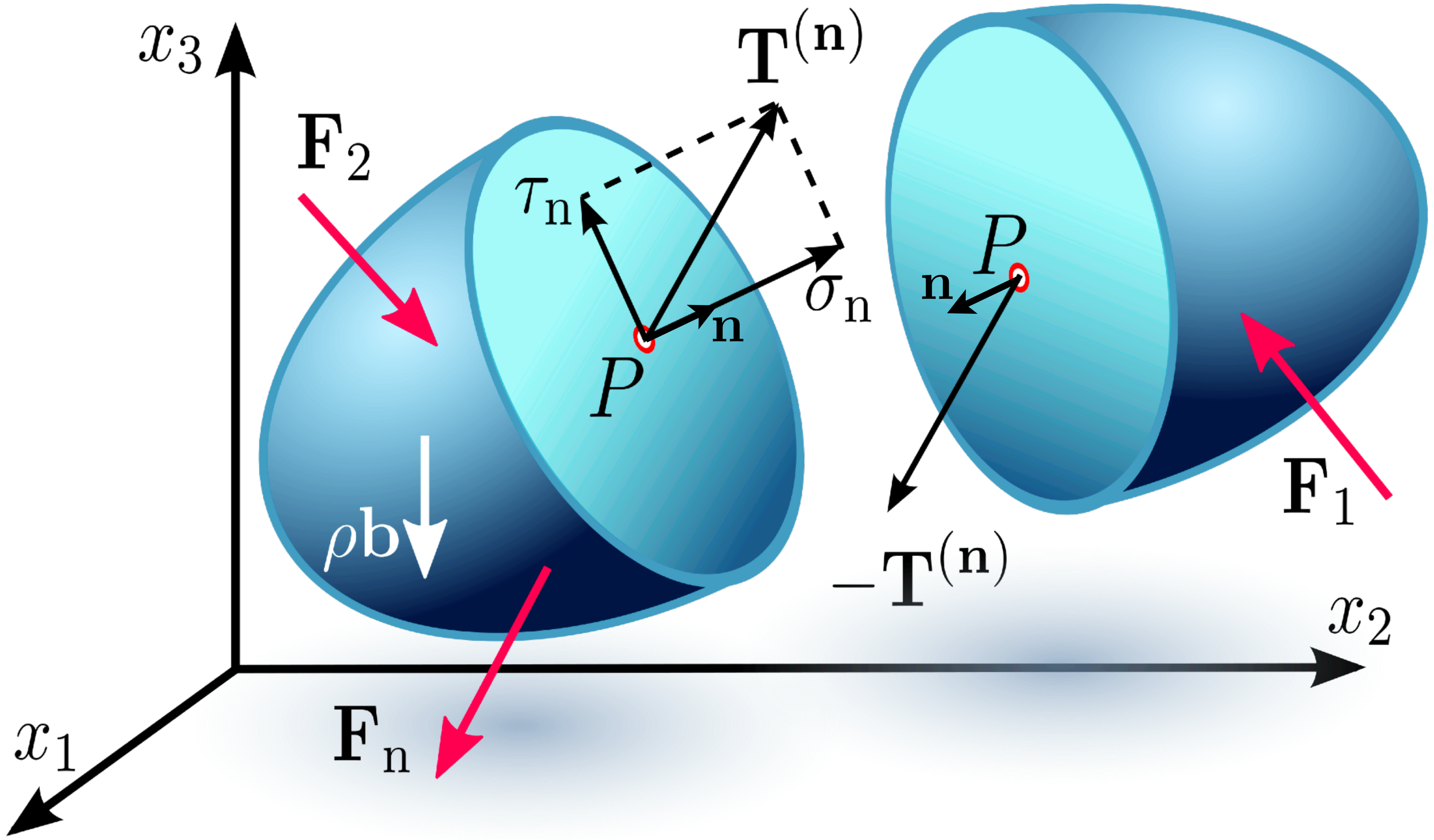
Conservation of angular momentum

We learn that the stress tensor must be symmetrical

$$\sigma_{ij} = \sigma_{ji}$$

the Cauchy stress tensor





The force per unit area at a point P is given by the ***traction vector*** $\mathbf{T}^{(\mathbf{n})}$

Given the stress tensor,
it is easy to compute traction
on any surface in continuum
or on boundary

$$\mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n}$$

$$T_i = \sigma_{ij} n_j$$

$$T_x = \sigma_{xx} n_x + \sigma_{xy} n_y + \sigma_{xz} n_z,$$

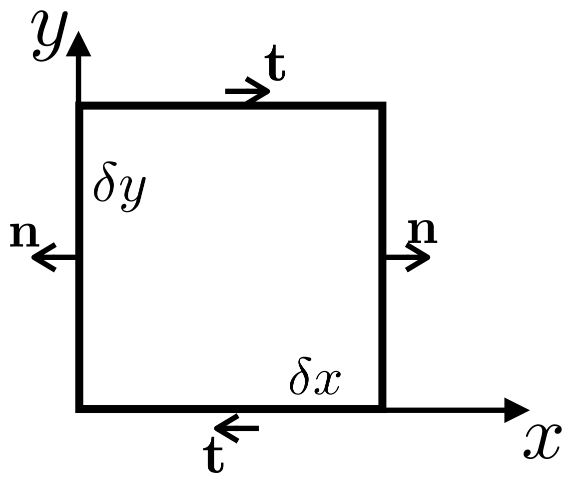
$$T_y = \sigma_{yx} n_x + \sigma_{yy} n_y + \sigma_{yz} n_z,$$

$$T_z = \sigma_{zx} n_x + \sigma_{zy} n_y + \sigma_{zz} n_z.$$

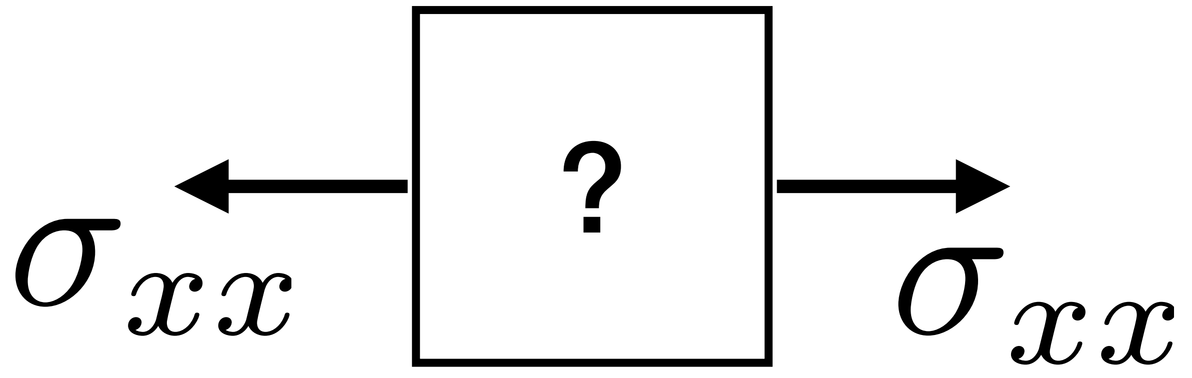
Cauchy reciprocal theorem: $\mathbf{T}^{(\mathbf{n})} = -\mathbf{T}^{(-\mathbf{n})}$ (which is Newton's 3rd law!)

Rheology & deformation (scalar version)

extension in one dimension (no body force, no acceleration)



Cauchy momentum equation becomes $\frac{\partial \sigma_{xx}}{\partial x} = 0$



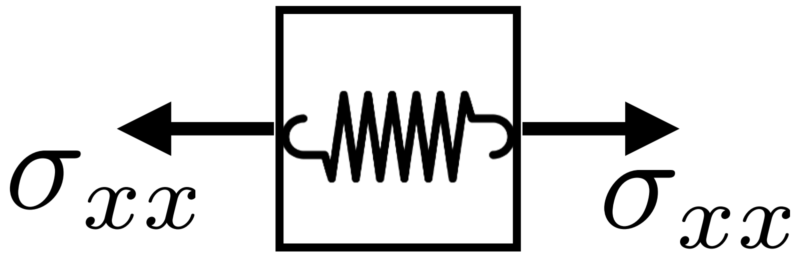
Solid (elastic) response: 

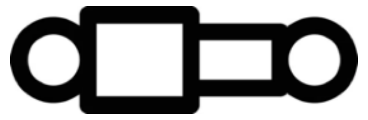
$$\sigma = \mu(L - L_0)/L_0$$

Elastic modulus μ Units: Pa

Define *strain*: $\epsilon \equiv \frac{L - L_0}{L_0} = \frac{\partial U}{\partial x}$

$$\sigma = \mu\epsilon$$



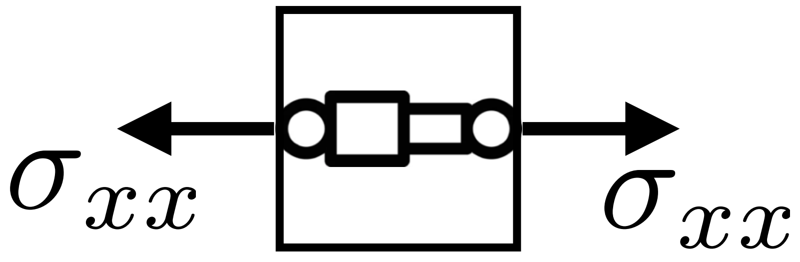
Fluid (viscous) response: 

$$\sigma = \eta \frac{\partial}{\partial t} \frac{L - L_0}{L_0} = \eta(\dot{L}/L_0)$$

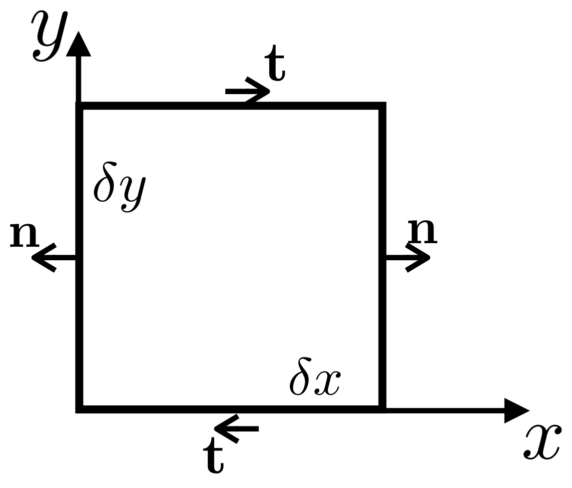
Viscosity η Units: Pa-s

Define *strain rate*: $\dot{\epsilon} \equiv \frac{\partial}{\partial t} \frac{L - L_0}{L_0} = \frac{\partial}{\partial t} \frac{\partial U}{\partial x} = \frac{\partial v}{\partial x}$

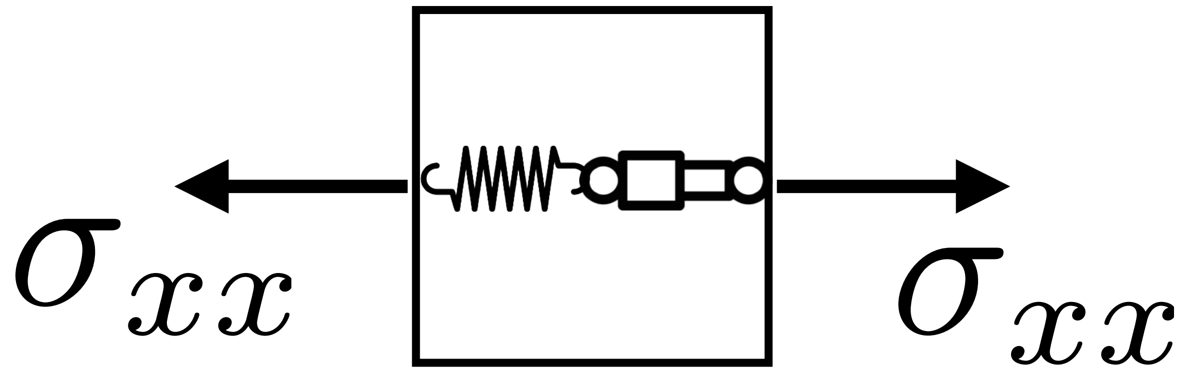
$$\sigma = \eta\dot{\epsilon}$$



extension in one dimension (no body force, no acceleration)



Cauchy momentum equation becomes $\frac{\partial \sigma_{xx}}{\partial x} = 0$



Visco-elastic response 

Strain is additive: $\epsilon = \epsilon^{\mathrm{vi}} + \epsilon^{\mathrm{el}} \quad \Rightarrow \quad \dot{\epsilon} = \dot{\epsilon}^{\mathrm{vi}} + \dot{\epsilon}^{\mathrm{el}}$

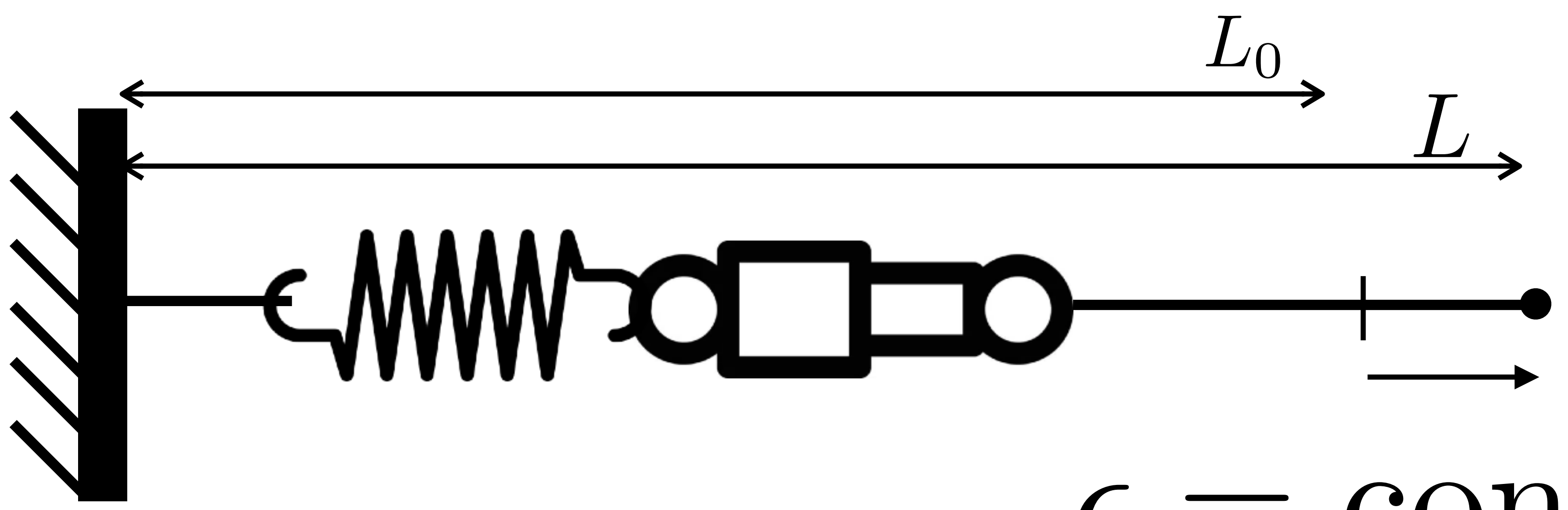
elastic: $\sigma = \mu \, \epsilon^{\mathrm{el}}$

viscous: $\sigma = \eta \dot{\epsilon}^{\mathrm{vi}}$

$$\dot{\epsilon} = \frac{\sigma}{\eta} + \frac{\dot{\sigma}}{\mu},$$

$$= \left(\frac{1}{\eta} + \frac{1}{\mu} \frac{d}{dt} \right) \sigma$$

a viscoelastic problem (no body force, no acceleration)



$$\epsilon = \text{const} = (L - L_0)/L_0$$

$$\dot{\epsilon} = \left(\frac{1}{\eta} + \frac{1}{\mu}\frac{d}{dt}\right)\sigma = 0 \quad \Rightarrow \quad \frac{d\sigma}{dt} = -\frac{\mu}{\eta}\sigma$$

Solution: $\sigma = \sigma_0 \mathrm{e}^{-\mu t/\eta},$

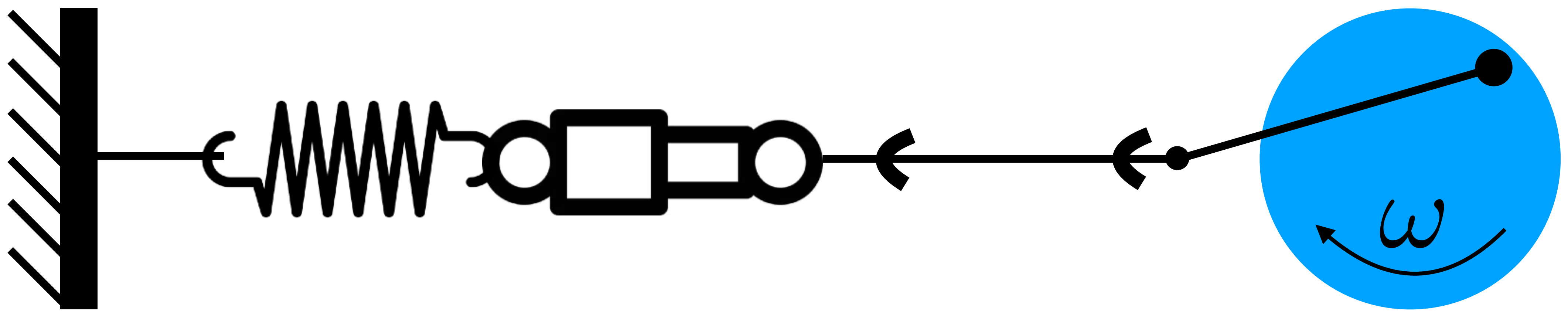
$= \sigma_0 \mathrm{e}^{-t/\tau_{\mathrm{Maxwell}}}$

where: $\sigma_0 = \mu\epsilon_0,$

$\tau_{\mathrm{Maxwell}} \equiv \eta/\mu$

Maxwell time is the timescale for relaxation of elastic stress by viscous flow

a viscoelastic problem (no body force, no acceleration)



$$\epsilon = \mathrm{Re}\left(A\mathrm{e}^{i\omega t}\right) = A\cos\omega t$$

To solve: $\dot{\epsilon} = \left(\frac{1}{\eta} + \frac{1}{\mu}\frac{d}{dt}\right)\sigma$

Guess: $\sigma = \mathrm{Re}\left(a\mathrm{e}^{i\omega t}\right)$ with complex *a*.

Substitute & simplify: $i\omega A = \left(\frac{1}{\eta} + \frac{i\omega}{\mu}\right) a$

Solution: $\sigma = \mathrm{Re}\left(\frac{i\omega A}{1/\eta + i\omega/\mu}\mathrm{e}^{i\omega t}\right)$

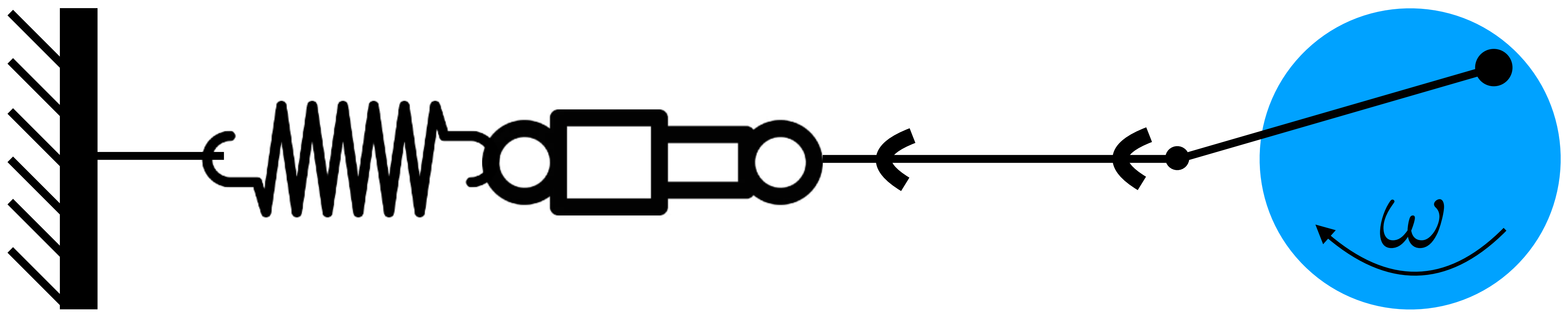
$$\sigma = \frac{A\mu}{\sqrt{1+\mathrm{De}^{-2}}}\sin\left(\omega t + \Phi\right)$$

where $\begin{cases} \Phi \equiv \tan^{-1}\mathrm{De} \\ \mathrm{De} \equiv \dfrac{\eta/\mu}{1/\omega} \equiv 2\pi\dfrac{\tau_{\mathrm{Maxwell}}}{\tau_{\mathrm{forcing}}} \end{cases}$

$$\tau_{\mathrm{Maxwell}} \ll \tau_{\mathrm{forcing}},\ \mathrm{De} \ll 1 \quad \Rightarrow \quad \sigma \sim A\omega\eta\sin\omega t$$

$$\tau_{\mathrm{Maxwell}} \gg \tau_{\mathrm{forcing}},\ \mathrm{De} \gg 1 \quad \Rightarrow \quad \sigma \sim A\mu\cos\omega t$$

a viscoelastic problem (no body force, no acceleration)



$$\epsilon = \mathrm{Re}\left(Ae^{i\omega t}\right) = A\cos\omega t$$

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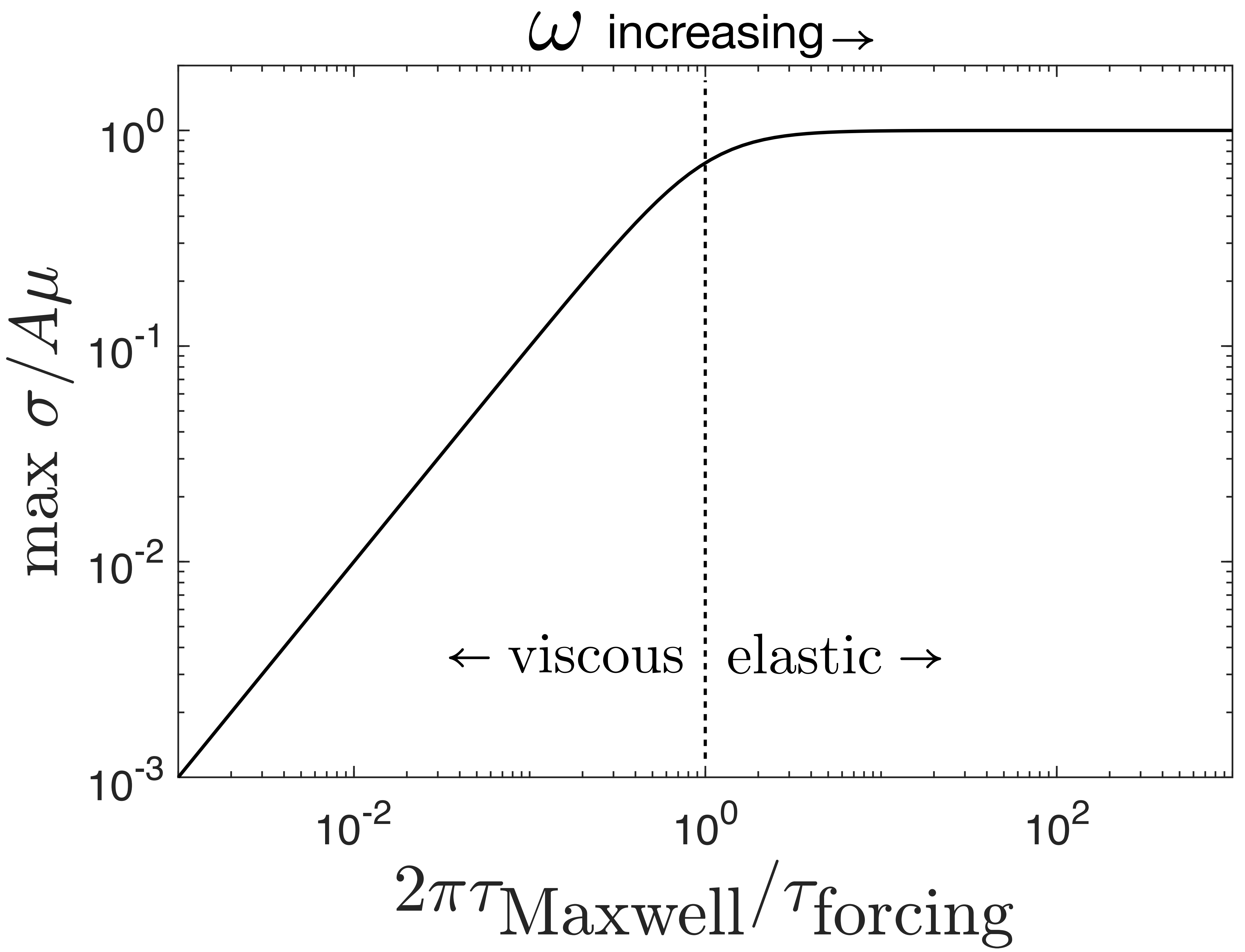
where $\begin{cases} \Phi \equiv \tan^{-1}\mathrm{De} \\ \mathrm{De} \equiv \dfrac{\eta/\mu}{1/\omega} \equiv 2\pi\dfrac{\tau_{\mathrm{Maxwell}}}{\tau_{\mathrm{forcing}}} \end{cases}$

$$\tau_{\mathrm{Maxwell}} \ll \tau_{\mathrm{forcing}},\ \mathrm{De} \ll 1 \quad \Rightarrow \quad \sigma \sim A\omega\eta\sin\omega t$$

$$\tau_{\mathrm{Maxwell}} \gg \tau_{\mathrm{forcing}},\ \mathrm{De} \gg 1 \quad \Rightarrow \quad \sigma \sim A\mu\cos\omega t$$



The Maxwell time & Deborah number of **rocks**

$$\eta = \eta_0 \exp\left[\frac{E^*}{R}\left(\frac{1}{T} - \frac{1}{T_0}\right)\right] \gtrsim 10^{18} \text{ Pa-s}$$

$$\mu \sim \text{const.} \sim 10^{10} \text{ Pa}$$

$$\tau_{\text{Maxwell}} \gtrsim 10^{8} \text{ s}$$

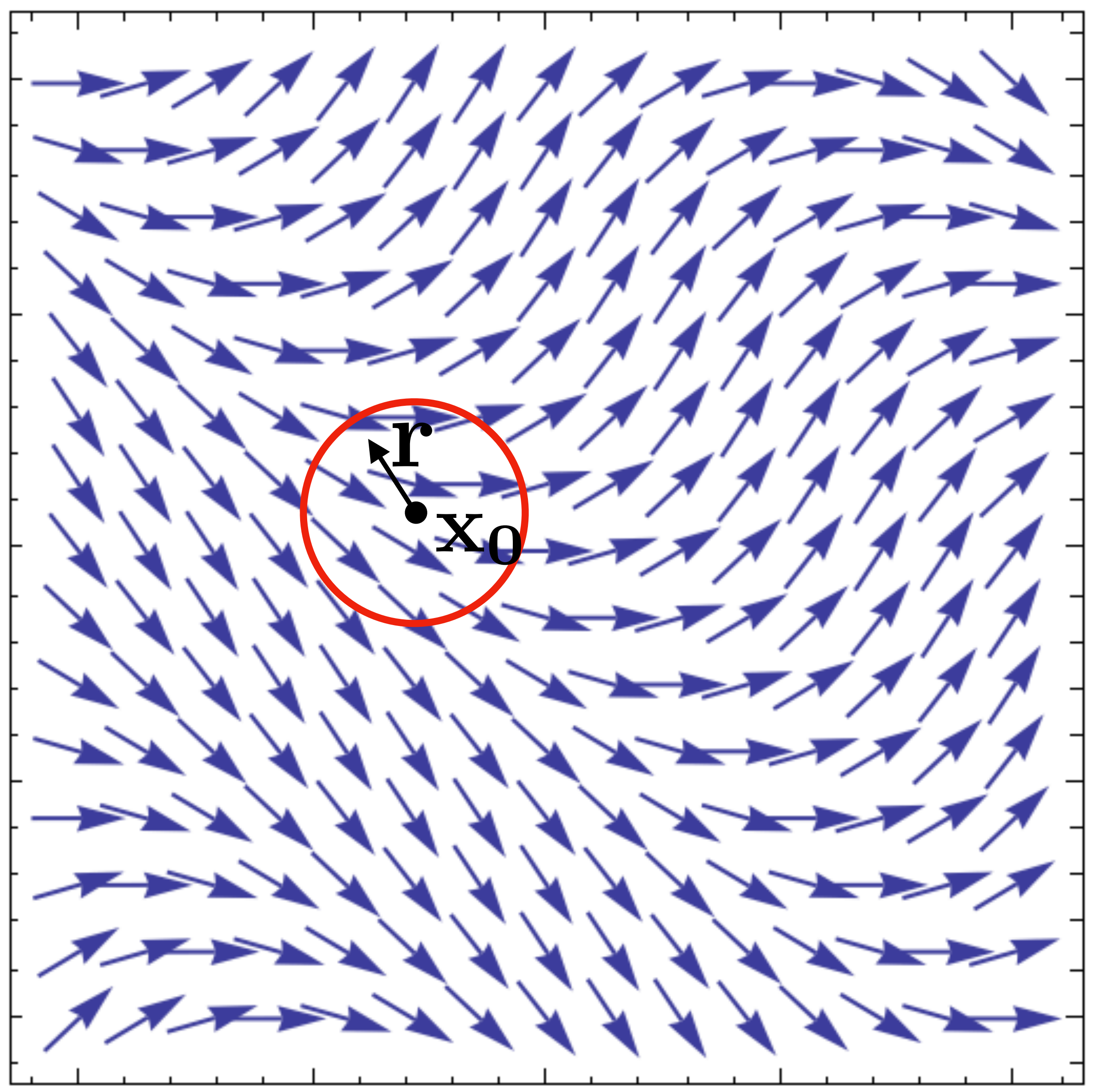
$$\text{De} \sim \frac{\tau_{\text{Maxwell}}}{\tau_{\text{forcing}}}$$

"The mountains flowed before the Lord" (Deborah in Judges 5:5)

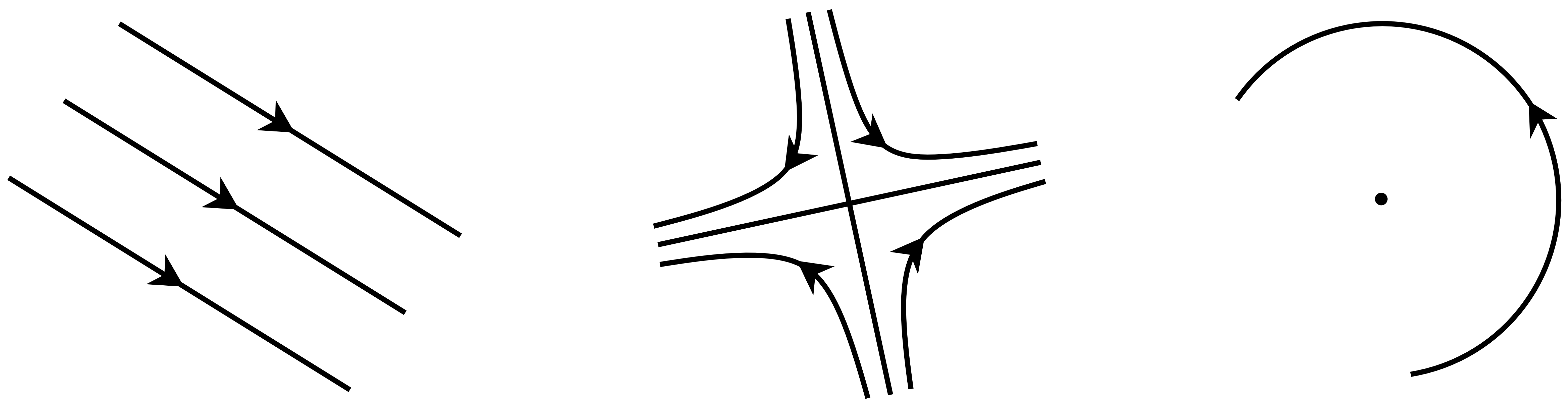
$\tau_{\text{seismic}} \approx 10^{0}$ s $\tau_{\text{tides}} \approx 10^{4}$ s	$\text{De} \gg 1$	elastic
$\tau_{\text{deglaciation}} \approx 10^{10}$ s	$\text{De} > 1$	viscoelastic
$\tau_{\text{mantle convection}} \approx 10^{15}$ s	$\text{De} \ll 1$	viscous

Rheology & deformation (tensor version)

Kinematics of a continuum: the strain (rate) tensor



vectors: displacement (or velocity)



$$\mathbf{u}(\mathbf{x}) \approx \mathbf{u}(\mathbf{x}_0) + \mathbf{r} \cdot \boldsymbol{E} + \mathbf{r} \cdot \boldsymbol{\Omega}$$

Taylor expanding the displacement (or velocity) about a point:

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x}_0) + (\mathbf{x} - \mathbf{x}_0) \cdot \nabla \mathbf{u}|_{\mathbf{x}_0} + O\left(|\mathbf{x} - \mathbf{x}_0|^2\right)$$

$$\approx \mathbf{u}(\mathbf{x}_0) + \mathbf{r} \cdot \nabla \mathbf{u}$$ where $\mathbf{r} = \mathbf{x} - \mathbf{x_0}$

$$\nabla \mathbf{u} = \frac{\partial u_i}{\partial x_j} \equiv D_{ij} = \boldsymbol{D}$$ deformation tensor

$$\boldsymbol{D} = \boldsymbol{E} + \boldsymbol{\Omega}$$ tensor decomposition

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$ symmetric part

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$ antisymmetric part

The strain (rate) tensor: isotropic & deviatoric parts

$$\mathbf{u}(\mathbf{x}) \approx \mathbf{u}(\mathbf{x}_0) + \mathbf{r} \cdot \boldsymbol{E} + \mathbf{r} \cdot \boldsymbol{\Omega}$$

strain (rate) tensor:

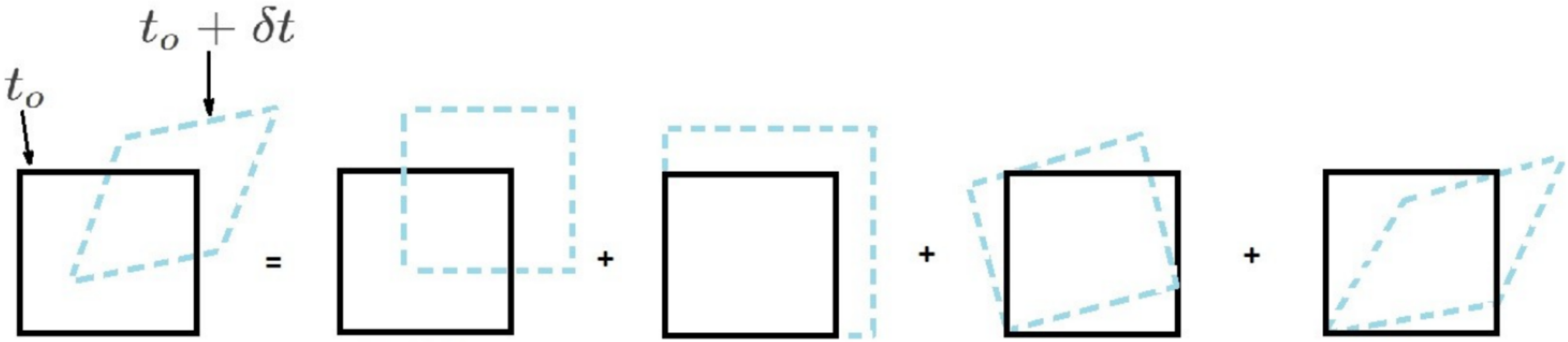
$$\boldsymbol{E} = \frac{1}{2} \begin{pmatrix} 2\partial_x u_x & (\partial_x u_y + \partial_y u_x) & (\partial_x u_z + \partial_z u_x) \\ (\partial_x u_y + \partial_y u_x) & 2\partial_y u_y & (\partial_z u_y + \partial_y u_z) \\ (\partial_x u_z + \partial_z u_x) & (\partial_z u_y + \partial_y u_z) & 2\partial_z u_z \end{pmatrix}$$

isotropic part:

$$\mathrm{Tr}(\boldsymbol{E}) = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \boldsymbol{\nabla} \cdot \mathbf{u}$$ (a scalar)

deviatoric part:

$$\widetilde{\boldsymbol{E}} \equiv \boldsymbol{E} - \tfrac{1}{3}\mathrm{Tr}(\boldsymbol{E})\boldsymbol{I}$$



$\mathbf{u}(\mathbf{x}_0)$ $\quad$ $\mathbf{r} \cdot \mathrm{Tr}(\boldsymbol{E})\boldsymbol{I}$ $\quad$ $\mathbf{r} \cdot \boldsymbol{\Omega}$ $\quad$ $\mathbf{r} \cdot \widetilde{\boldsymbol{E}}$

The stress tensor: isotropic & deviatoric parts

stress tensor:

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

isotropic part:

$$\sigma \equiv \frac{1}{3}\mathrm{Tr}(\boldsymbol{\sigma}) = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}$$

deviatoric part:

$$\boldsymbol{\tau} \equiv \boldsymbol{\sigma} - \sigma \boldsymbol{I}$$

(stress decomposition)

We choose to take *tensile* stress as positive (caution: some authors make a different choice!)

Rheology of a continuum

stress tensor

strain (rate) tensor

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \quad \text{“} \propto \text{”} \quad \boldsymbol{E} = \frac{1}{2}\begin{pmatrix} 2\partial_x u_x & (\partial_x u_y + \partial_y u_x) & (\partial_x u_z + \partial_z u_x) \\ (\partial_x u_y + \partial_y u_x) & 2\partial_y u_y & (\partial_z u_y + \partial_y u_z) \\ (\partial_x u_z + \partial_z u_x) & (\partial_z u_y + \partial_y u_z) & 2\partial_z u_z \end{pmatrix}$$

Solid (elastic) response: 

$$\boldsymbol{\epsilon} \equiv \boldsymbol{E}$$

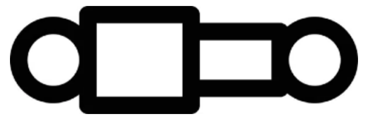
key assumption!

$$\begin{cases} \sigma \propto \mathrm{Tr}(\boldsymbol{\epsilon}) \\ \boldsymbol{\tau} \propto \boldsymbol{\epsilon} - \frac{1}{3}\mathrm{Tr}(\boldsymbol{\epsilon})\boldsymbol{I} \end{cases}$$

$$\boldsymbol{\sigma} = 2\mu\left[\boldsymbol{\epsilon} - \frac{1}{3}\mathrm{Tr}(\boldsymbol{\epsilon})\boldsymbol{I}\right] + K\mathrm{Tr}(\boldsymbol{\epsilon})\boldsymbol{I}$$

shear modulus

bulk modulus

Fluid (viscous) response: 

$$\dot{\boldsymbol{\epsilon}} \equiv \boldsymbol{E}$$

key assumption!

$$\begin{cases} \sigma + p \propto \mathrm{Tr}(\dot{\boldsymbol{\epsilon}}) \\ \boldsymbol{\tau} \propto \dot{\boldsymbol{\epsilon}} - \frac{1}{3}\mathrm{Tr}(\dot{\boldsymbol{\epsilon}})\boldsymbol{I} \end{cases}$$

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\eta\left[\dot{\boldsymbol{\epsilon}} - \frac{1}{3}\mathrm{Tr}(\dot{\boldsymbol{\epsilon}})\boldsymbol{I}\right] + \zeta\mathrm{Tr}(\dot{\boldsymbol{\epsilon}})\boldsymbol{I}$$

thermodynamic pressure

shear viscosity

bulk viscosity

viscoelasticity of a continuum

Viscoelastic response 

Strain is additive: $$\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}^{\mathrm{vi}} + \dot{\boldsymbol{\epsilon}}^{\mathrm{el}}$$

Simplifying assumption of incompressibility: $$\mathrm{Tr}(\dot{\boldsymbol{\epsilon}}^{\mathrm{vi}}) = \mathrm{Tr}(\dot{\boldsymbol{\epsilon}}^{\mathrm{el}}) = 0$$

$$\boldsymbol{\sigma} + \tau_{\mathrm{Maxwell}} \overset{\triangledown}{\boldsymbol{\sigma}} = 2\eta\dot{\boldsymbol{\epsilon}}$$

where $\overset{\triangledown}{\boldsymbol{\sigma}}$ is the *upper convected derivative*
that is something like $\dfrac{D\boldsymbol{\sigma}}{Dt}$

The elastic stress gets transported by the flow but relaxes on the timescale of the Maxwell time

Principle components of stress and strain (rate)

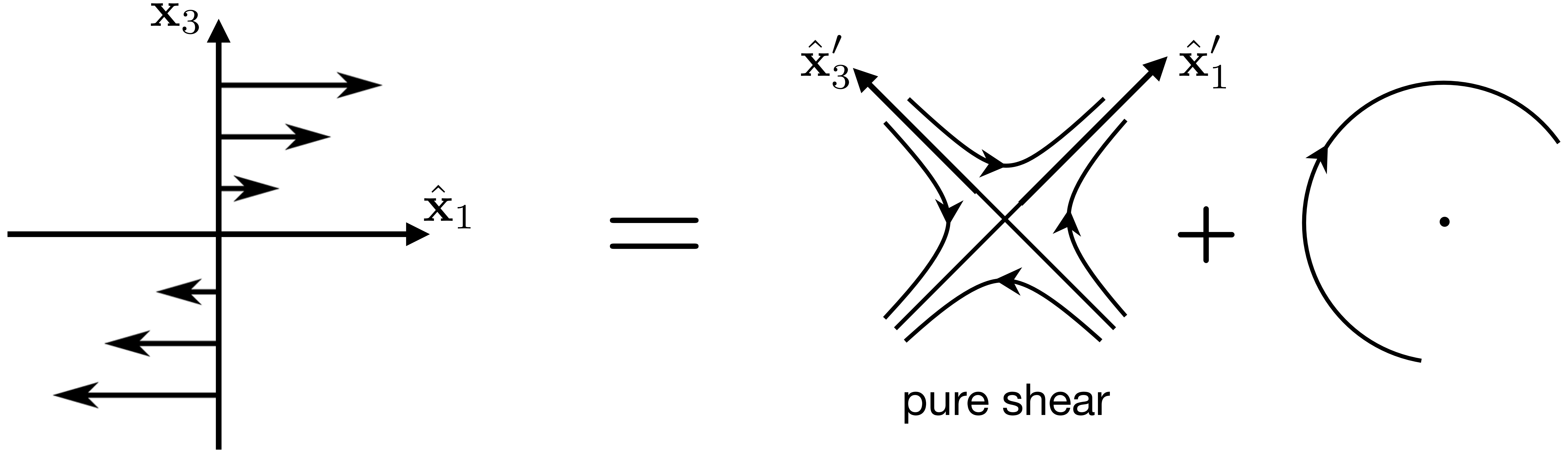
$\boldsymbol{\sigma}, \boldsymbol{\epsilon}, \dot{\boldsymbol{\epsilon}}$ Real, symmetric tensors — can be *diagonalised*

Solve $\det|\boldsymbol{\sigma} - \lambda_j \boldsymbol{I}| = 0$ to determine the eigenvalues λ_j

Solve $(\boldsymbol{\sigma} - \lambda_j \boldsymbol{I})\mathbf{b}_j = 0$ to determine the eigenvectors $\mathbf{b}_j$

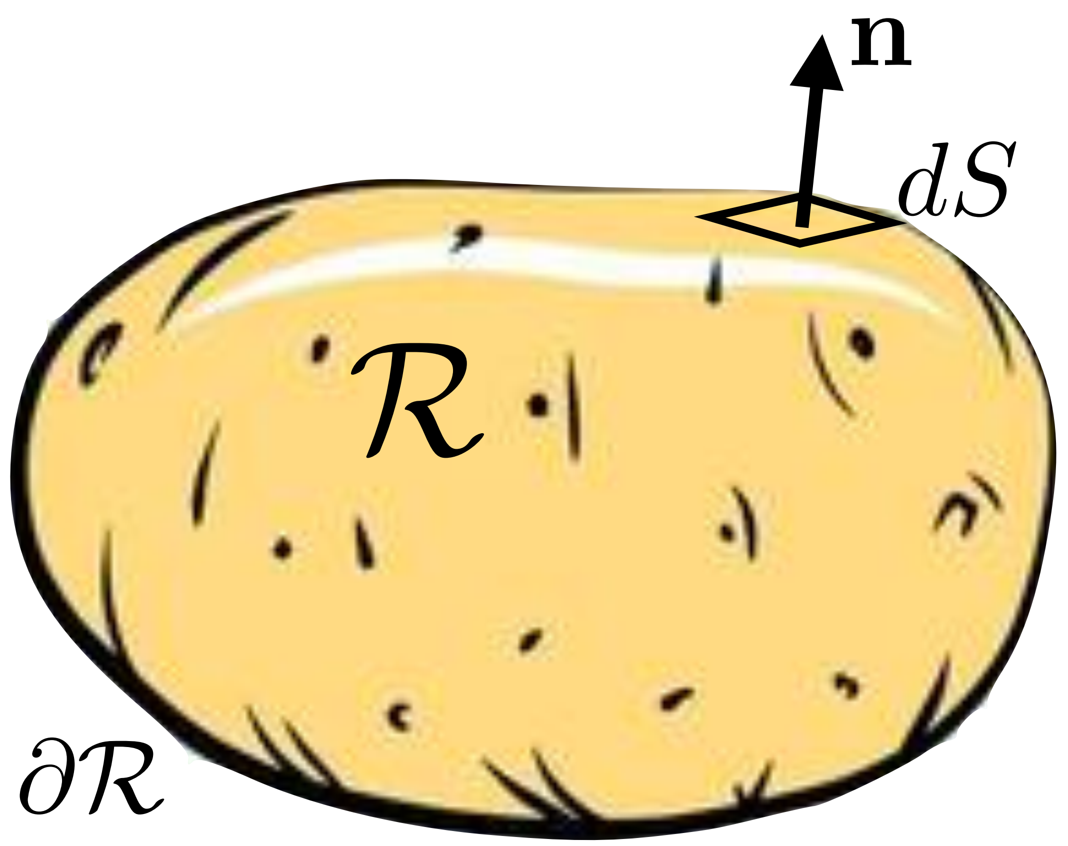
$$\boldsymbol{\sigma}' = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \qquad \dot{\boldsymbol{\epsilon}}' = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \qquad \hat{\mathbf{x}}'_j = \mathbf{b}_j / |\mathbf{b}_j|$$

$$\mathbf{v} = v_x(z)\hat{\mathbf{x}}_1$$



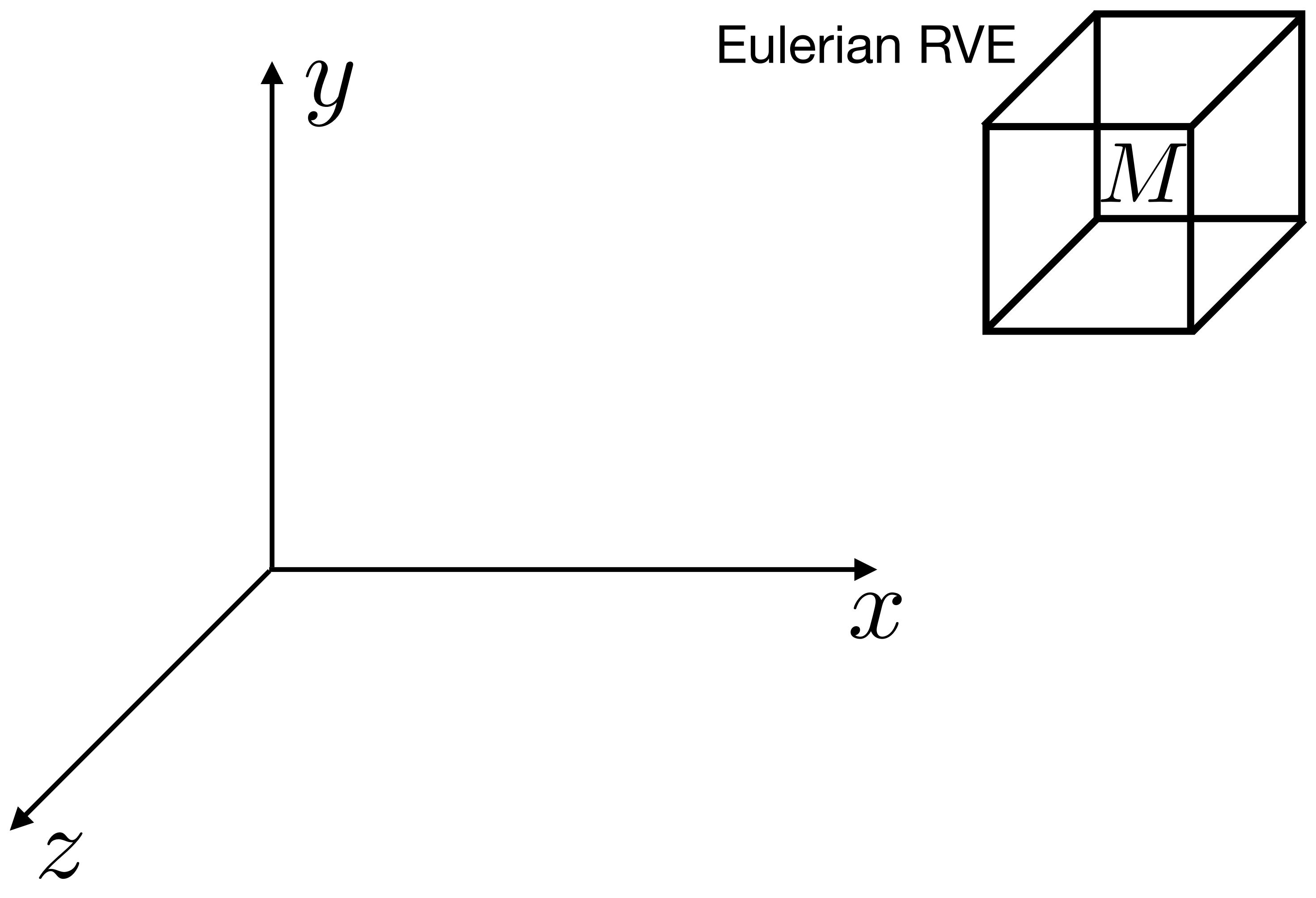
Conservation of mass

The divergence theorem



$$\int_{\partial\mathcal{R}} \mathbf{J}\cdot\mathbf{n}\,dS = \int_{\mathcal{R}} \boldsymbol{\nabla}\cdot\mathbf{J}\,dV$$

Mass conservation



$$\rho = \rho(\mathbf{x}, t)$$

$$\frac{dM}{dt} = (\text{net inward flux of mass})$$

$$\frac{d}{dt}\int_{\text{RVE}} \rho \, dV = -\int_{\partial\text{RVE}} \rho\mathbf{v}\cdot\mathbf{n}\, dS,$$

$$= -\int_{\text{RVE}} \boldsymbol{\nabla}\cdot(\rho\mathbf{v})\, dV$$

$$\int_{\text{RVE}} \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v})\, dV = 0$$

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0$$

The mass conservation equation

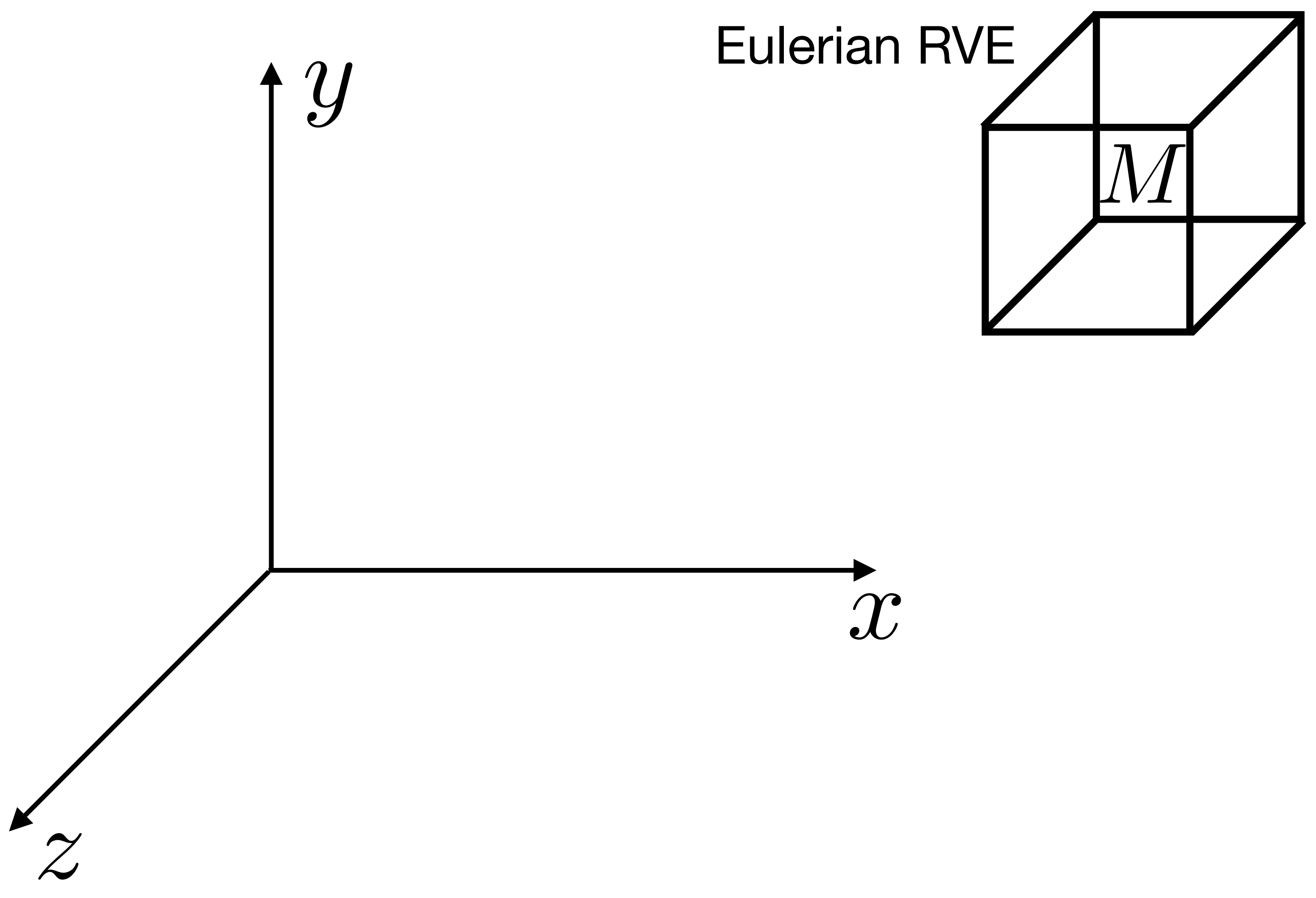
$$\underset{\int dt}{\Longrightarrow}$$

$$\rho \approx \rho_0(1 - \boldsymbol{\nabla}\cdot\mathbf{U})$$

(for small displacements)

Mass conservation with constant density

$$\rho = \text{const}$$



$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0$$

$$\frac{D\rho}{Dt} + \rho \boldsymbol{\nabla} \cdot \mathbf{v} = 0$$

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0 \quad \Rightarrow \quad \boldsymbol{\nabla} \cdot \mathbf{U} = 0$$

The continuity equation

This equation applies for *incompressible flows* of compressible fluids!

Full equations of motion

Elastodynamics and incompressible flow

Recall the Cauchy momentum equation: $$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}+\rho\mathbf{g}=\rho\frac{D\mathbf{v}}{Dt}$$

Solid (elastic) response: 

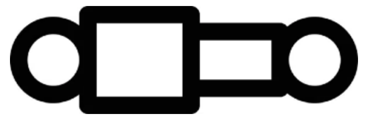
$$\boldsymbol{\sigma}=2\mu\underbrace{\left[\boldsymbol{\epsilon}-\tfrac{1}{3}\mathrm{Tr}(\boldsymbol{\epsilon})\boldsymbol{I}\right]}_{\boldsymbol{\varepsilon}}+K\underbrace{\mathrm{Tr}(\boldsymbol{\epsilon})}_{\boldsymbol{\nabla}\cdot\mathbf{U}}\boldsymbol{I}$$

We will assume uniformity, $K,\mu=\mathrm{const}$

$$\rho\frac{D\mathbf{v}}{Dt}=K\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{U})+\boldsymbol{\nabla}\cdot 2\mu\boldsymbol{\varepsilon}+\rho\mathbf{g}$$

The Navier equation (linear elastodynamics)

$$\rho\frac{\partial^2\mathbf{U}}{\partial t^2}=K\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{U})+\mu\nabla^2\mathbf{U}+\rho\mathbf{g}$$

Fluid (viscous) response: 

$$\boldsymbol{\sigma}=-p\boldsymbol{I}+2\eta\underbrace{\left[\dot{\boldsymbol{\epsilon}}-\tfrac{1}{3}\mathrm{Tr}(\dot{\boldsymbol{\epsilon}})\boldsymbol{I}\right]}_{\dot{\boldsymbol{\varepsilon}}}+\zeta\underbrace{\mathrm{Tr}(\dot{\boldsymbol{\epsilon}})}_{\boldsymbol{\nabla}\cdot\mathbf{v}}\boldsymbol{I}$$

We will assume incompressibility, $\boldsymbol{\nabla}\cdot\mathbf{v}=0$

$$\rho\frac{D\mathbf{v}}{Dt}=-\boldsymbol{\nabla}p+\boldsymbol{\nabla}\cdot 2\eta\dot{\boldsymbol{\varepsilon}}+\rho\mathbf{g}$$

The Navier-Stokes equation (incompressible flow)

$$\rho\frac{D\mathbf{v}}{Dt}=-\boldsymbol{\nabla}p+\boldsymbol{\nabla}\cdot\eta\left[\boldsymbol{\nabla}\mathbf{v}+(\boldsymbol{\nabla}\mathbf{v})^T\right]+\rho\mathbf{g}$$

Isoviscous incompressible flow (Navier-Stokes)

$$\rho \frac{D\mathbf{v}}{Dt} = -\boldsymbol{\nabla} p + \boldsymbol{\nabla} \cdot \eta \left[\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^T\right] + \rho\mathbf{g}$$

$$\eta = \text{const}$$

Inertia | pressure gradient | viscous stress | body force

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}\mathbf{v} \right) = -\boldsymbol{\nabla} p + \eta \nabla^2 \mathbf{v} + \rho\mathbf{g},$$

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0$$

Units: N/m^3

Viscoelastic incompressible flow

$$\begin{cases} \rho \dfrac{D\mathbf{v}}{Dt} = \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \rho \mathbf{g} \\ \overset{\triangledown}{\boldsymbol{\sigma}} = \dfrac{2\eta\dot{\boldsymbol{\epsilon}} - \boldsymbol{\sigma}}{\tau_{\text{Maxwell}}} \\ \boldsymbol{\nabla} \cdot \mathbf{v} = 0 \end{cases}$$

where: $\overset{\triangledown}{\boldsymbol{\sigma}} \equiv \dfrac{\partial \boldsymbol{\sigma}}{\partial t} + \mathbf{v} \cdot \nabla \sigma - \left[\boldsymbol{\sigma} \cdot \nabla \mathbf{v} + (\nabla \mathbf{v})^T \cdot \boldsymbol{\sigma} \right]$

The upper convected time derivative

Scaling analysis & simplification

Choosing characteristic scales to non-dimensionalise the equation

$$\rho \frac{D\mathbf{v}}{Dt} = -\boldsymbol{\nabla} p + \eta \nabla^2 \mathbf{v} + \rho \mathbf{g}$$

Navier-Stokes equation

$$\mathbf{v} = [v]\mathbf{v}',\ \mathbf{x} = [x]\mathbf{x}',\ t = \frac{[x]}{[v]}t',\ p = [p]p'$$

Define *characteristic scales* & non-dimensional variables

$$\frac{\rho [v]^2}{[x]} \frac{D\mathbf{v}'}{Dt'} = -\frac{[p]}{[x]} \boldsymbol{\nabla}' p' + \frac{\eta [v]}{[x]^2} \nabla'^2 \mathbf{v}' + \rho \mathbf{g}$$

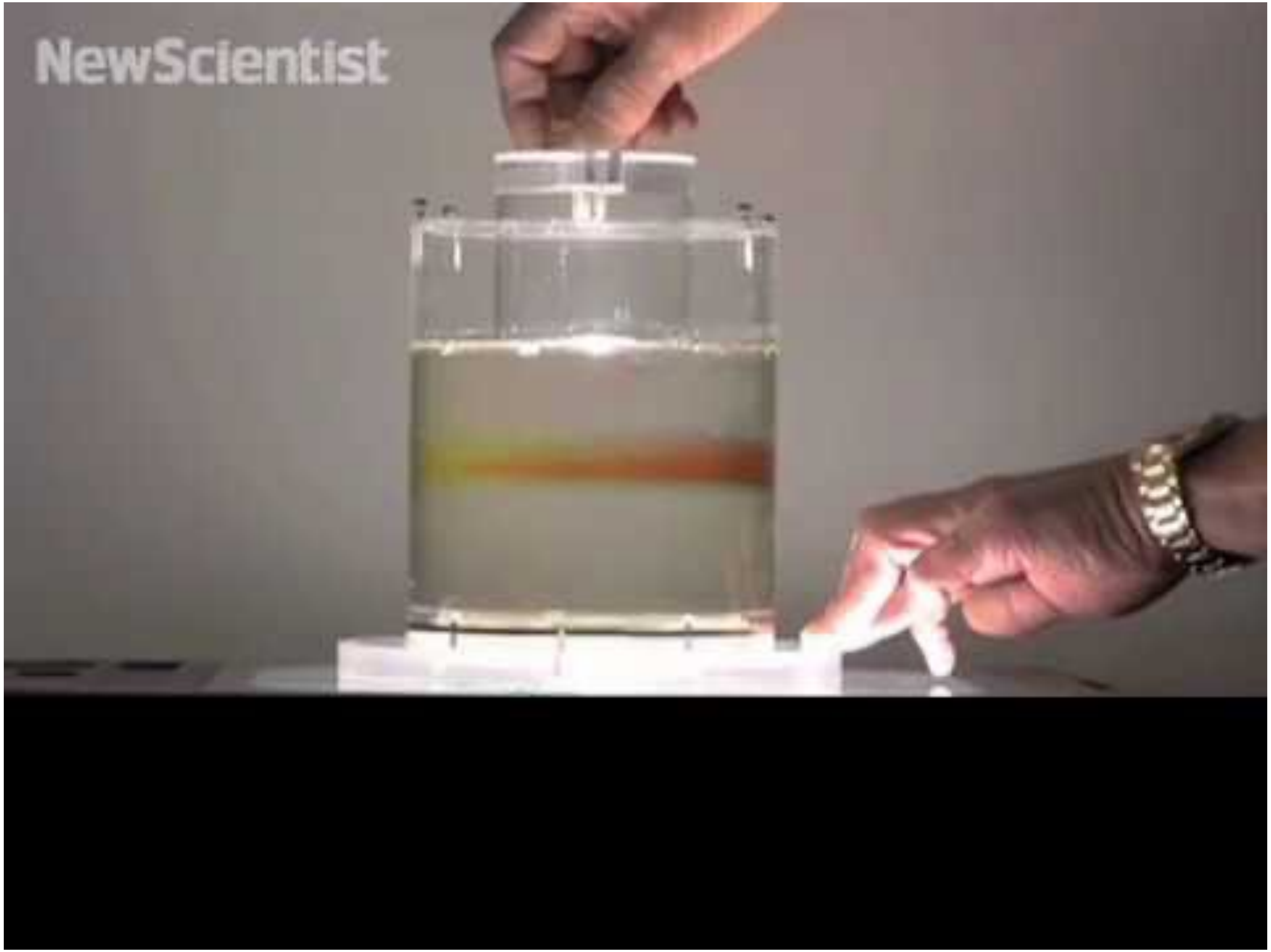
Substitute into Navier-Stokes

$$\frac{\rho [v][x]}{\eta} \frac{D\mathbf{v}'}{Dt'} = -\frac{[p][x]}{\eta [v]} \boldsymbol{\nabla}' p' + \nabla'^2 \mathbf{v}' + \frac{\rho [x]^2}{\eta [v]} \mathbf{g}$$

Rearrange so that the viscous term is of order 1

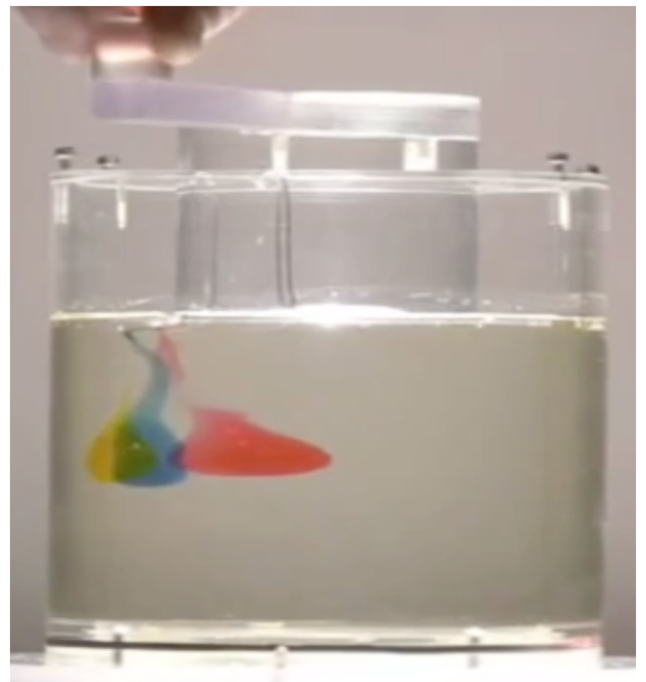
$$\mathrm{Re} \frac{D\mathbf{v}'}{Dt'} = -\boldsymbol{\nabla}' p' + \nabla'^2 \mathbf{v}' + \frac{\rho [x]^2}{\eta [v]} \mathbf{g}$$

Define $\mathrm{Re} \equiv \frac{\rho [v][x]}{\eta}$, choose $[p] = \frac{\eta [v]}{[x]}$


NewScientist

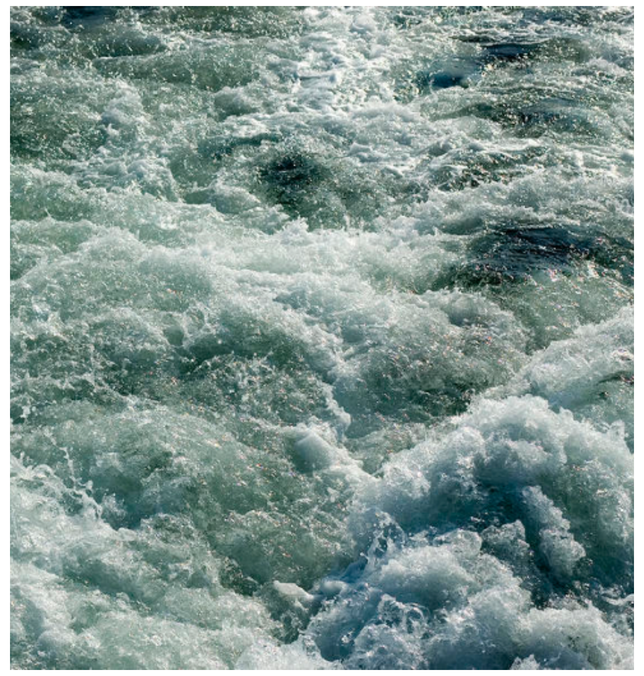
The Reynolds number

$$\mathrm{Re} \equiv \frac{\rho[v][x]}{\eta} = \frac{\rho[v]^2/[x]}{\eta[v]/[x]^2} = \frac{\text{inertia}}{\text{viscous force}},$$



$$\left.\begin{aligned} [x] &\approx 10 \text{ cm} \\ [v] &\approx 10 \text{ cm/sec} \\ \eta &\approx 10 \text{ Pa-sec} \\ \rho &\approx 1000 \text{ kg/m}^3 \end{aligned}\right\} \mathrm{Re} = \frac{1000 \cdot 10^{-1} \cdot 10^{-1}}{10} = 1$$

Laminar!

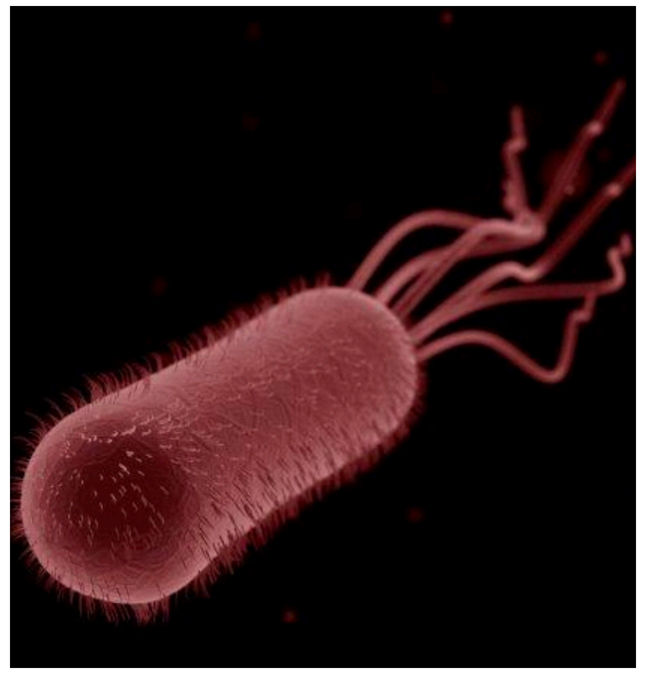


$$\left.\begin{aligned} [x] &\approx 1 \text{ m} \\ [v] &\approx 1 \text{ m/sec} \\ \eta &\approx 10^{-3} \text{ Pa-sec} \\ \rho &\approx 1000 \text{ kg/m}^3 \end{aligned}\right\} \mathrm{Re} = \frac{1000 \cdot 1 \cdot 1}{10^{-3}} = 10^6$$

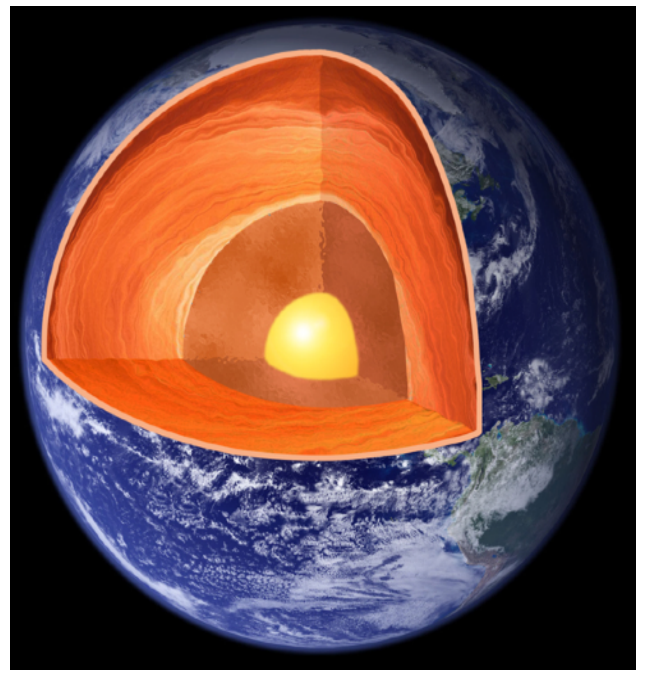
Turbulent!

The Reynolds number

$$\mathrm{Re} \equiv \frac{\rho[v][x]}{\eta} = \frac{\rho[v]^2/[x]}{\eta[v]/[x]^2} = \frac{\text{inertia}}{\text{viscous force}},$$



$$\left.\begin{aligned} [x] &\approx 10^{-6}\ \mathrm{m} \\ [v] &\approx 10^{-5}\ \mathrm{m/s} \\ \eta &\approx 10^{-3}\ \mathrm{Pa\text{-}s} \\ \rho &\approx 1000\ \mathrm{kg/m^3} \end{aligned}\right\} \mathrm{Re} = \frac{100 \cdot 10^{-6} \cdot 10^{-5}}{10^{-3}} = 10^{-6}$$



$$\left.\begin{aligned} [x] &\approx 3000\ \mathrm{km} \\ [v] &\approx 10\ \mathrm{cm/yr} \\ \eta &\approx 10^{20}\ \mathrm{Pa\text{-}sec} \\ \rho &\approx 3000\ \mathrm{kg/m^3} \end{aligned}\right\} \mathrm{Re} \approx \frac{3000 \cdot (3 \times 10^{6}) \cdot (3 \times 10^{-7})}{10^{20}}$$

$$\approx 3 \times 10^{-17}$$

Incompressible, zero-Re flows

$$\rho \frac{D\mathbf{v}}{Dt} = -\boldsymbol{\nabla} p + \eta \nabla^2 \mathbf{v} + \rho \mathbf{g}$$

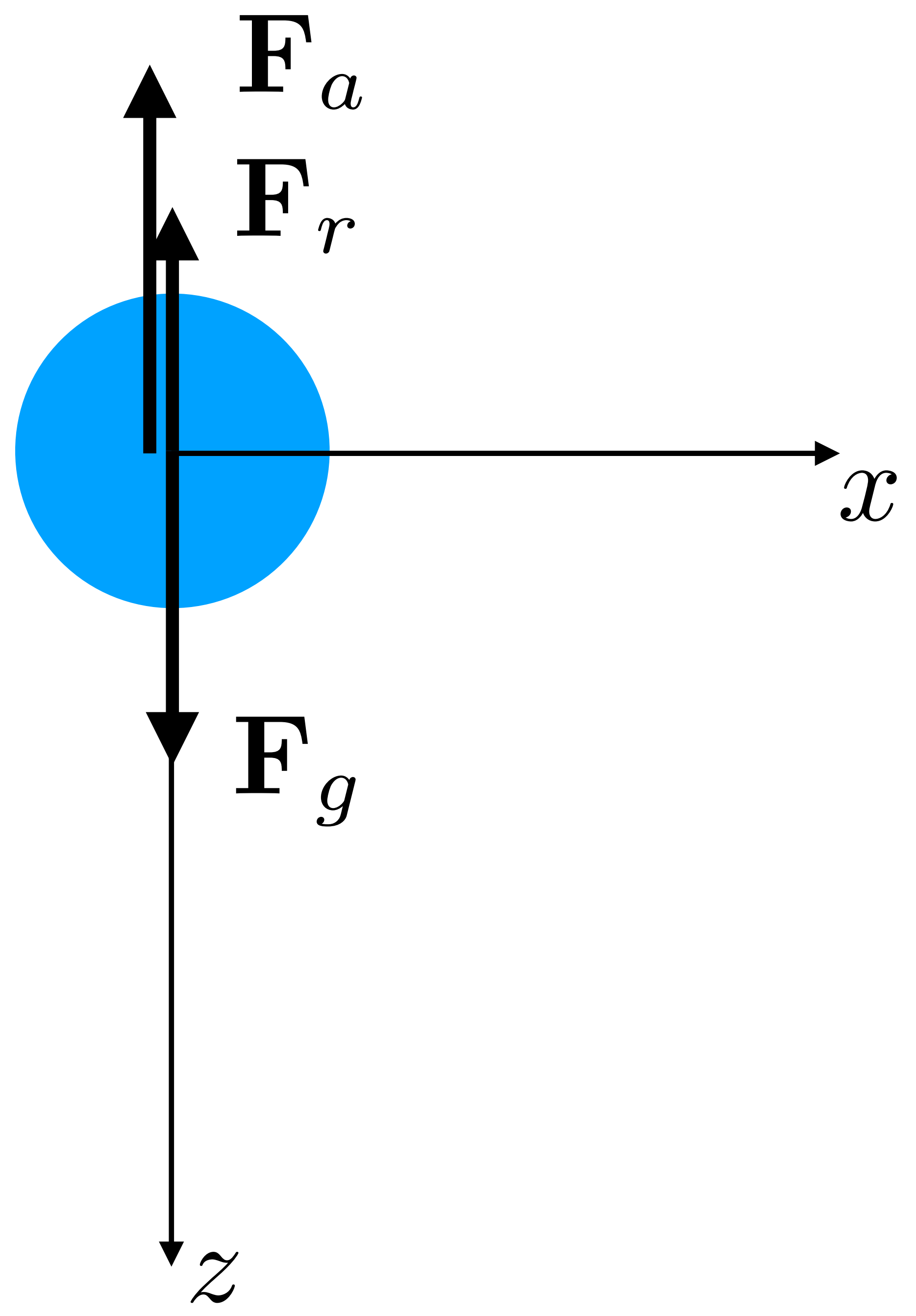
$\frac{D\mathbf{v}}{Dt} \sim 0$

$$\boldsymbol{\nabla} p = \boldsymbol{\nabla} \cdot \eta \left[\boldsymbol{\nabla} \mathbf{v} + (\boldsymbol{\nabla} \mathbf{v})^T \right] + \rho \mathbf{g},$$

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0$$

Stokes equations for viscous, incompressible flows

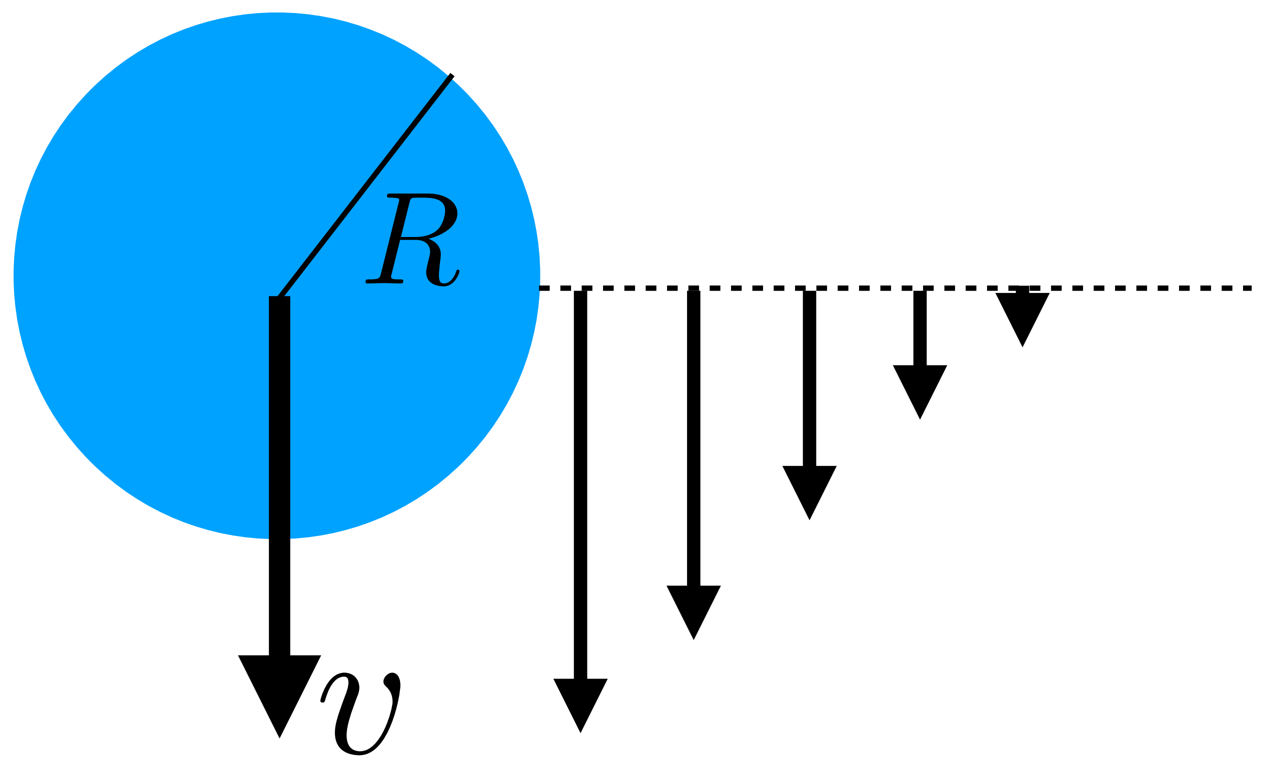
Example: ball falling through fluid



Archimedes force

viscous drag force

gravitational body force



$$\sum \mathbf{F} = M\mathbf{a}$$

$$\Rightarrow \boldsymbol{\nabla} p = \boldsymbol{\nabla} \cdot \eta \left[\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^T \right] + \rho \mathbf{g},$$

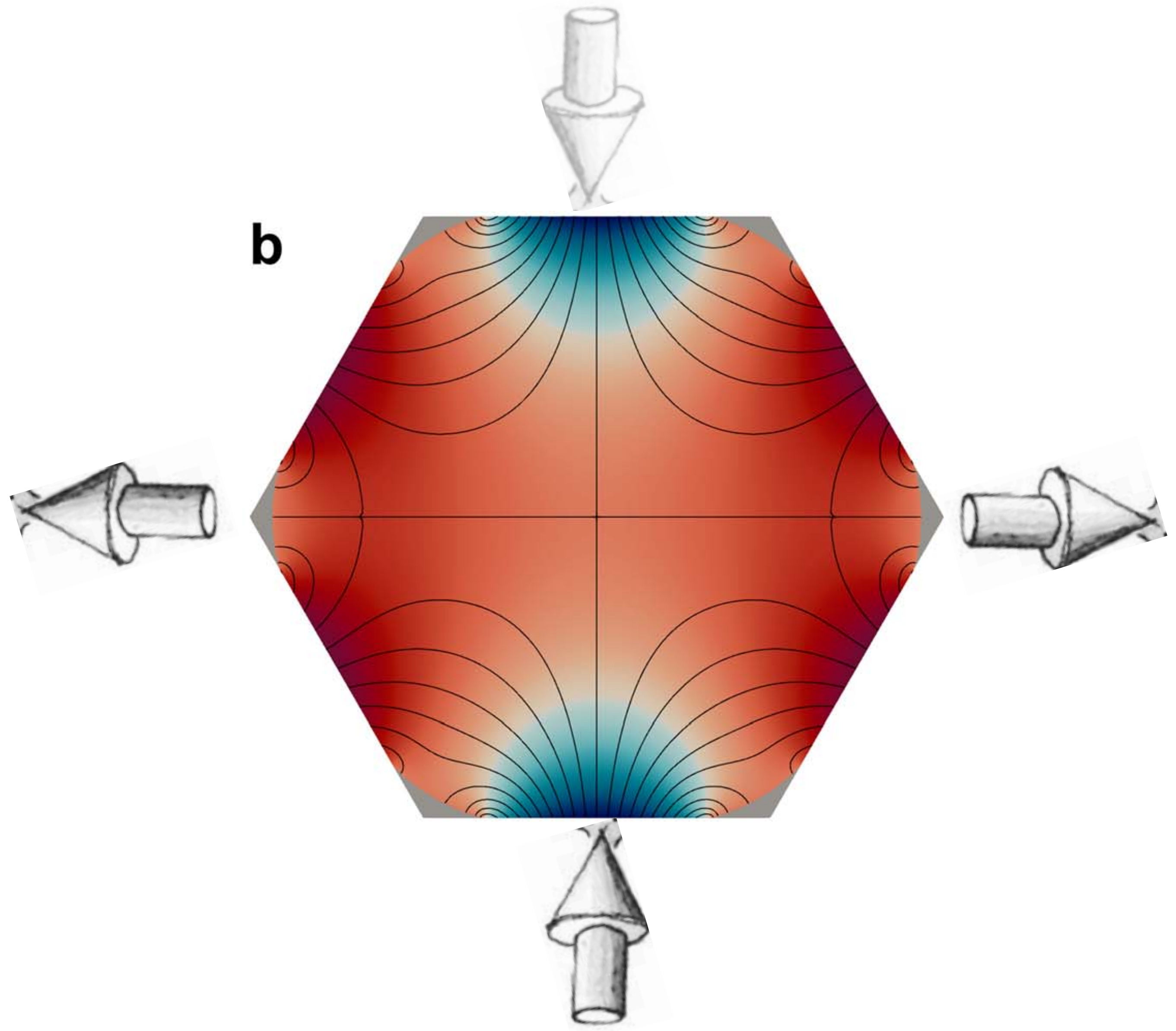
pressure force | viscous stress | body force

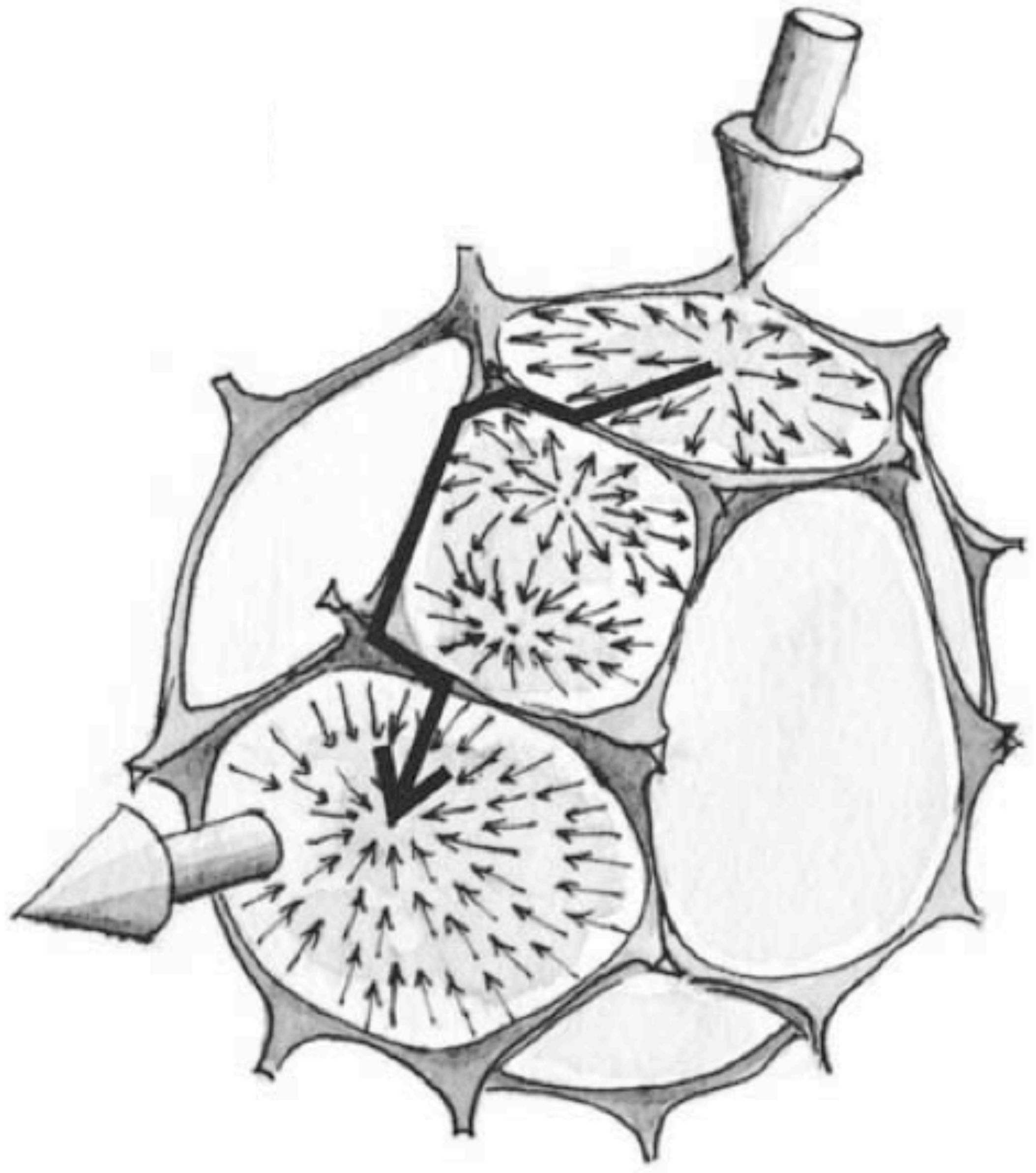
$$-\rho g \sim \eta v / R^2 - \rho_b g$$

$$v \sim \frac{\Delta \rho g R^2}{\eta}$$

Stokes settling speed

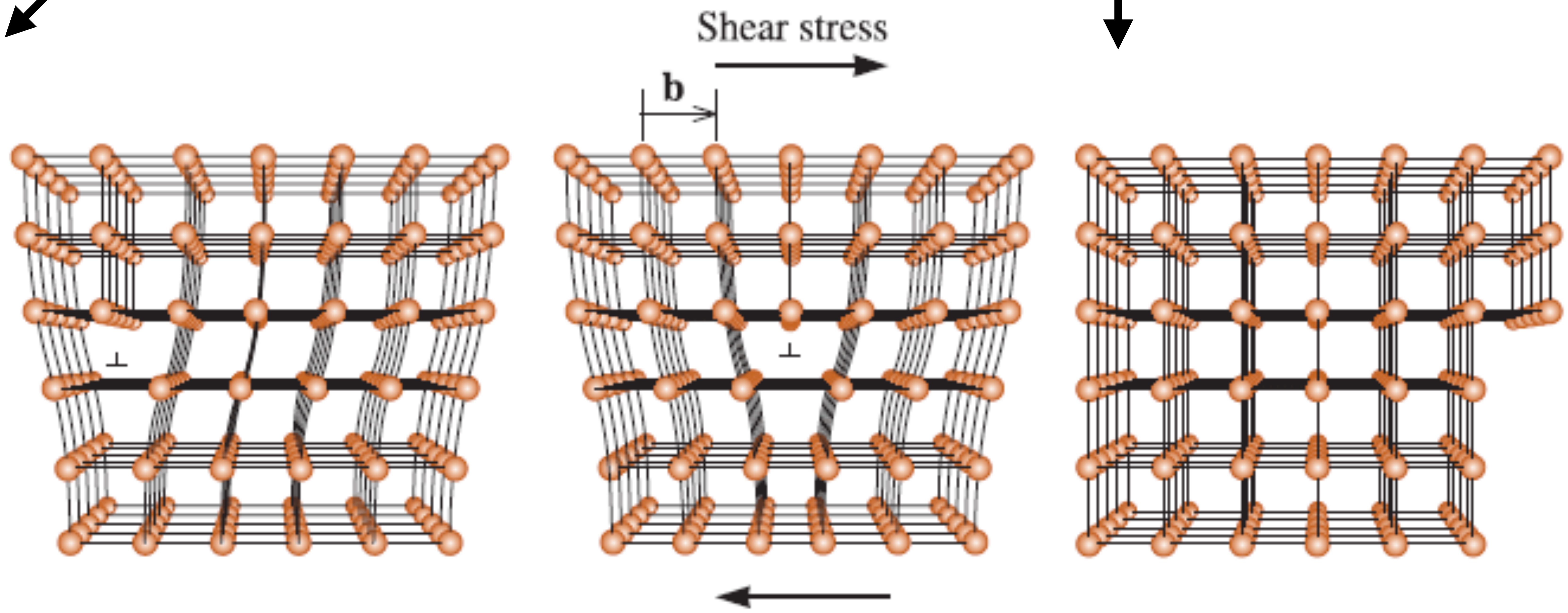
Why do high-temperature rocks flow (creep)?

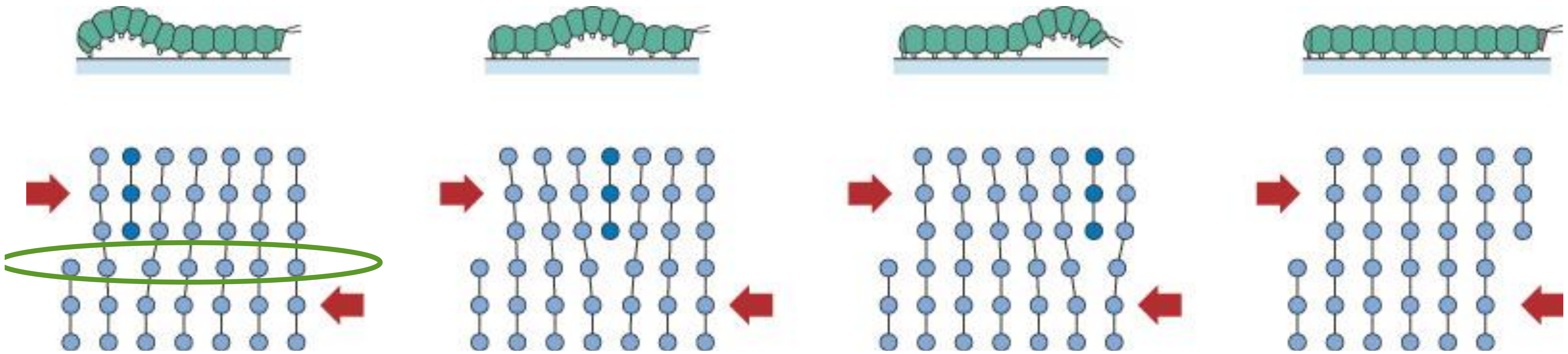




← Diffusion creep

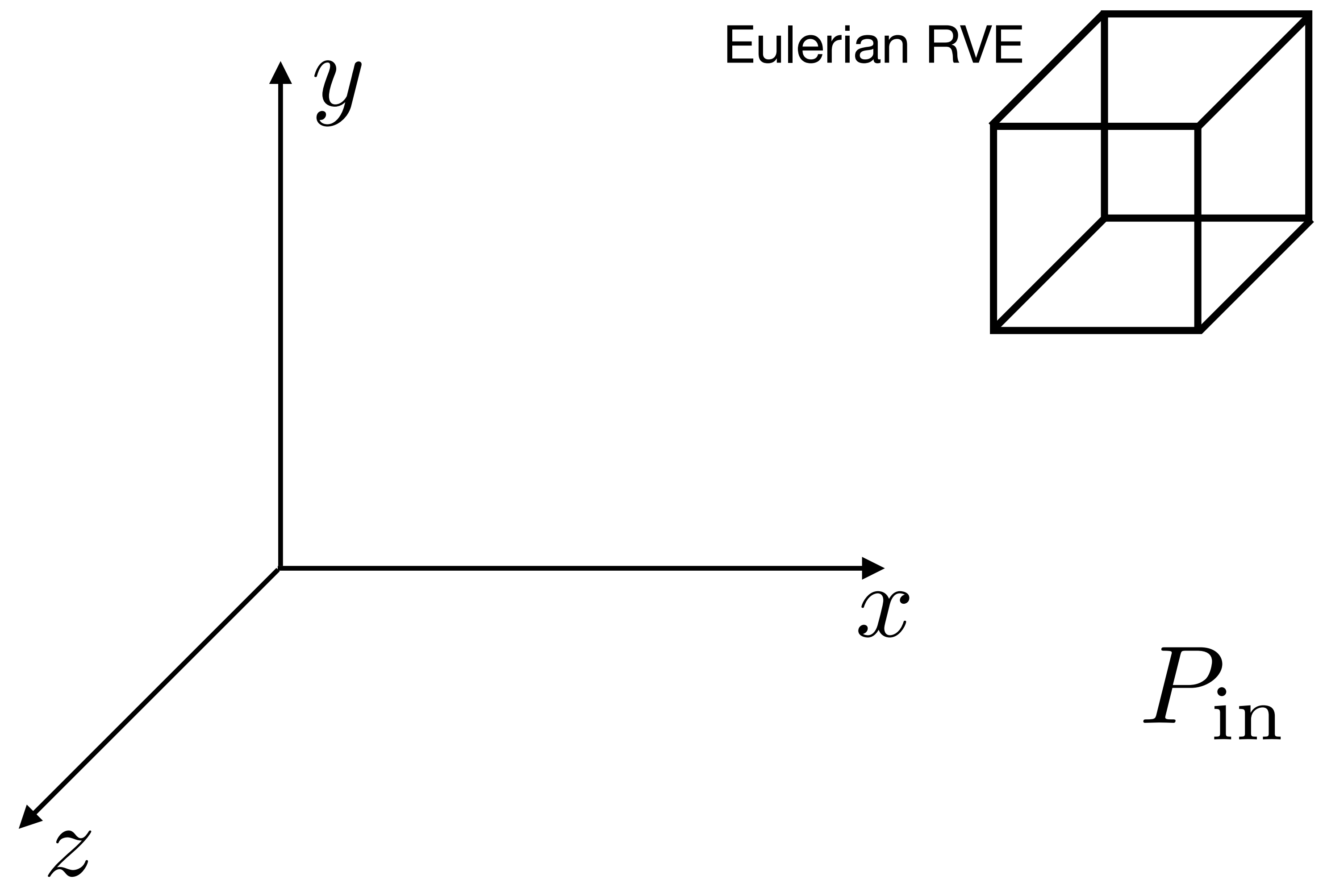
Dislocation creep





Work, Energy & Power

The First law of thermodynamics



$$d\mathcal{E} = (P_{\text{in}} + Q_{\text{in}})dt$$

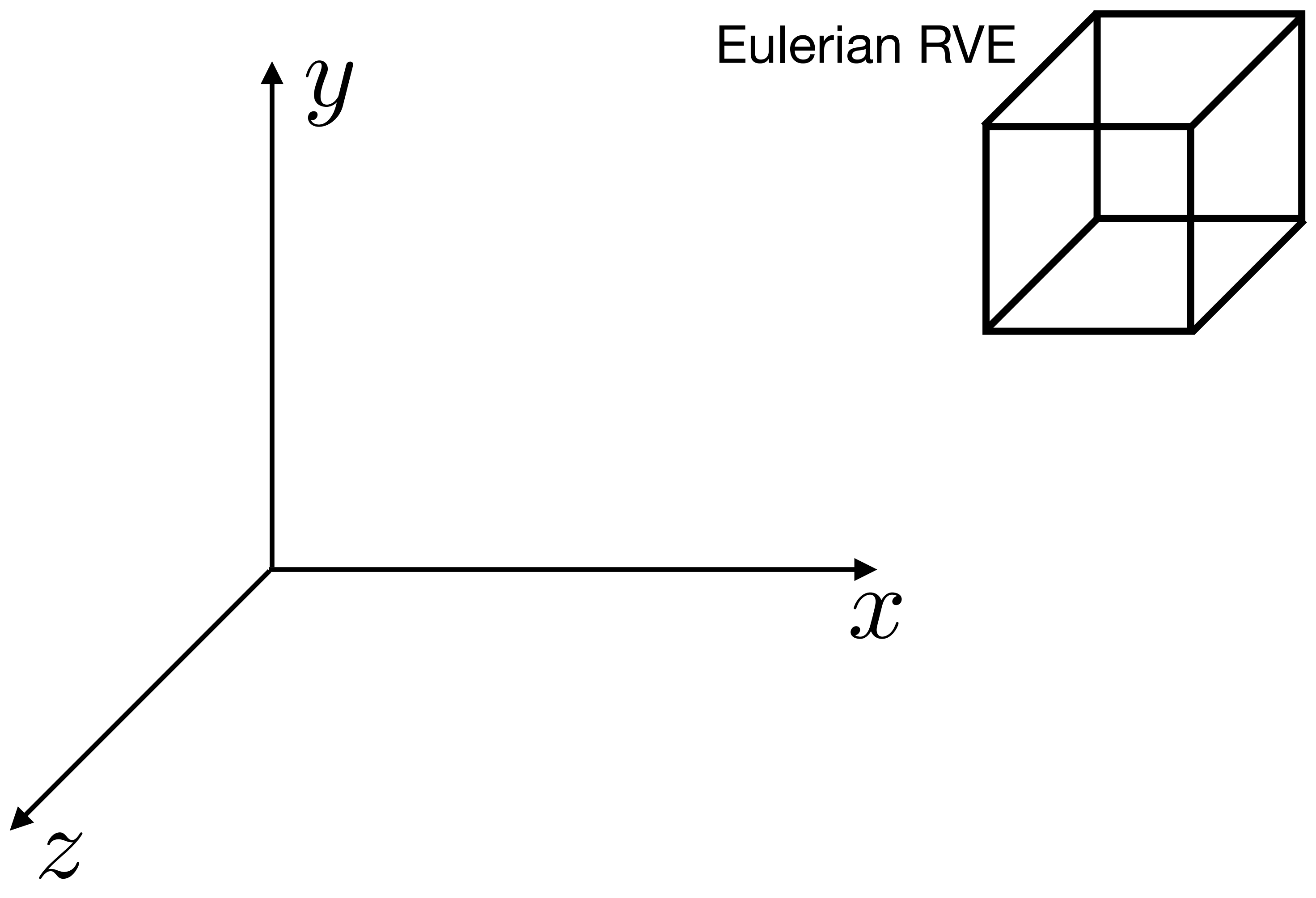
$$P = \frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v}$$

$$\begin{aligned}
P_{\text{in}} &= \int_{\partial \text{RVE}} \mathbf{T} \cdot \mathbf{v}\, dS + \int_{\text{RVE}} \rho \mathbf{g} \cdot \mathbf{v}\, dV, \\
&= \int_{\text{RVE}} \left[\mathbf{v} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \rho \mathbf{g}) + \boldsymbol{\sigma} : \boldsymbol{\nabla} \mathbf{v}\right] dV, \\
&= \int_{\text{RVE}} \left[\rho \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} + \boldsymbol{\sigma} : \boldsymbol{D}\right] dV \\
&= \int_{\text{RVE}} \left[\frac{\rho}{2} \frac{D}{Dt} \mathbf{v} \cdot \mathbf{v} + \boldsymbol{\sigma} : \boldsymbol{E}\right] dV \\
&= \frac{D}{Dt} \int_{\text{RVE}} \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v}\, dV + \int_{\text{RVE}} \boldsymbol{\sigma} : \boldsymbol{E}\, dV
\end{aligned}$$

$$\mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n}$$

$$\boldsymbol{D} = \boldsymbol{E} + \boldsymbol{\Omega}$$

The First law of thermodynamics



$$d\mathcal{E} = (P_{\text{in}} + Q_{\text{in}})dt$$

$$\begin{aligned} Q_{\text{in}} &= -\int_{\partial\text{RVE}} \mathbf{q}\cdot\mathbf{n}\, dS + \int_{\text{RVE}} \rho H\, dV, \\ &= \int_{\text{RVE}} (\rho H - \boldsymbol{\nabla}\cdot\mathbf{q})\, dV, \end{aligned}$$

$$\frac{D}{Dt}\underbrace{\int_{\text{RVE}} \left(\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} - \rho\mathbf{g}\cdot\mathbf{z} + \rho u\right) dV}_{\mathcal{E}} = \frac{D}{Dt}\int_{\text{RVE}} \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}\, dV + \int_{\text{RVE}} \boldsymbol{\sigma} : \boldsymbol{E}\, dV + \int_{\text{RVE}} (\rho H - \boldsymbol{\nabla}\cdot\mathbf{q})\, dV$$

KE PE IE

The energy equation and the temperature equation

$$\rho \frac{Du}{Dt} = \rho \mathbf{g} \cdot \mathbf{v} - \boldsymbol{\nabla} \cdot \mathbf{q} + \rho H + \boldsymbol{\sigma} : \boldsymbol{E}$$

Maxwell relations of thermodynamics

$$du = c_p\, dT - P\, d(1/\rho) - \alpha T\, dP/\rho$$

Fourier's law

$$\mathbf{q} = -k \boldsymbol{\nabla} T$$

$$\rho c_p \frac{DT}{Dt} = \alpha T \mathbf{g} \cdot \mathbf{v} + k \nabla^2 T + \rho H + \boldsymbol{\sigma} : \boldsymbol{E}$$

The temperature equation (from conservation of energy)

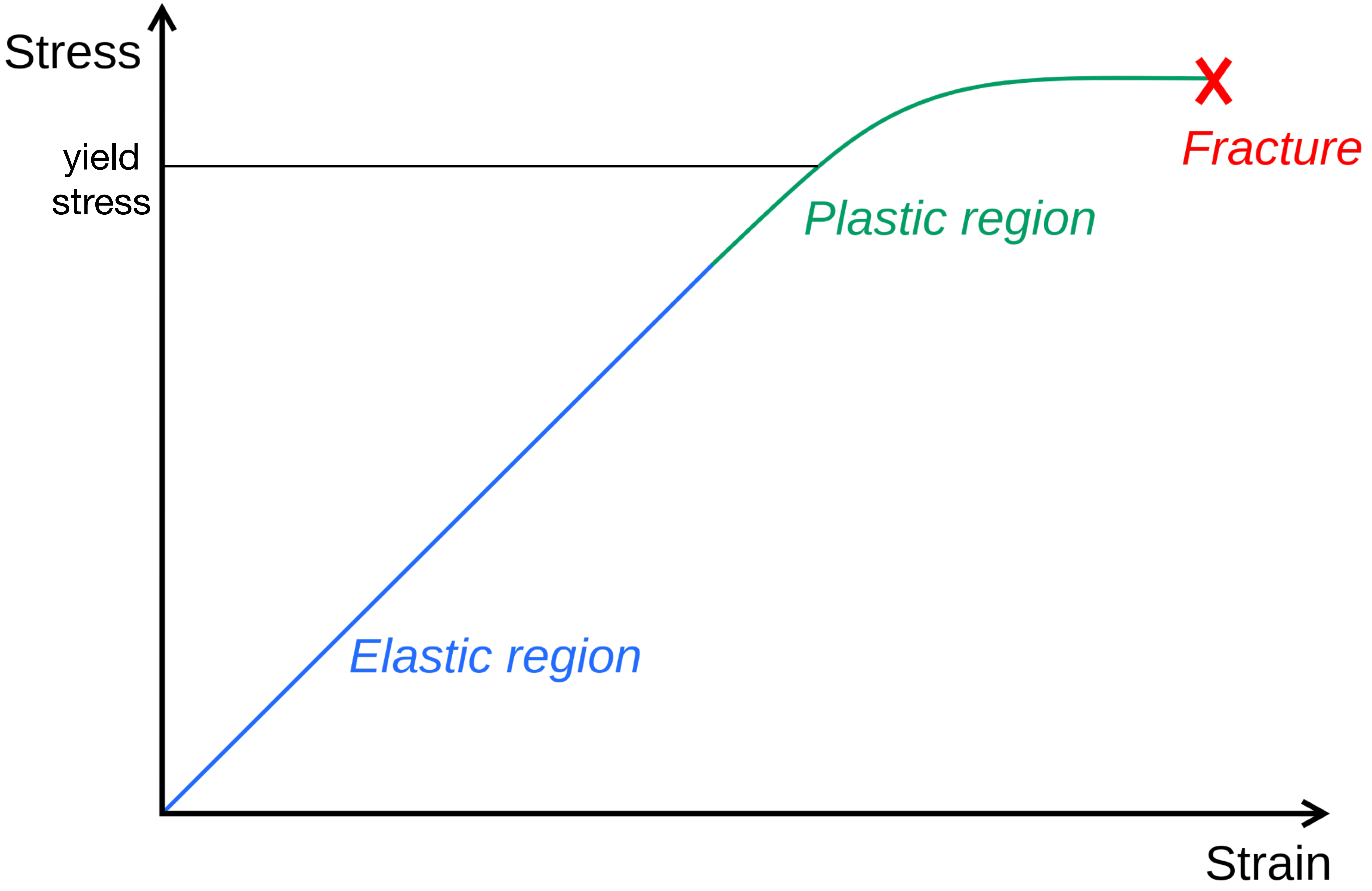
$\boldsymbol{\sigma} : \boldsymbol{E}$ energy storage (elastic) and energy dissipation (viscous)

Plasticity

Definition

Plasticity is a change in rheology that occurs when the stress exceeds a *yield stress*

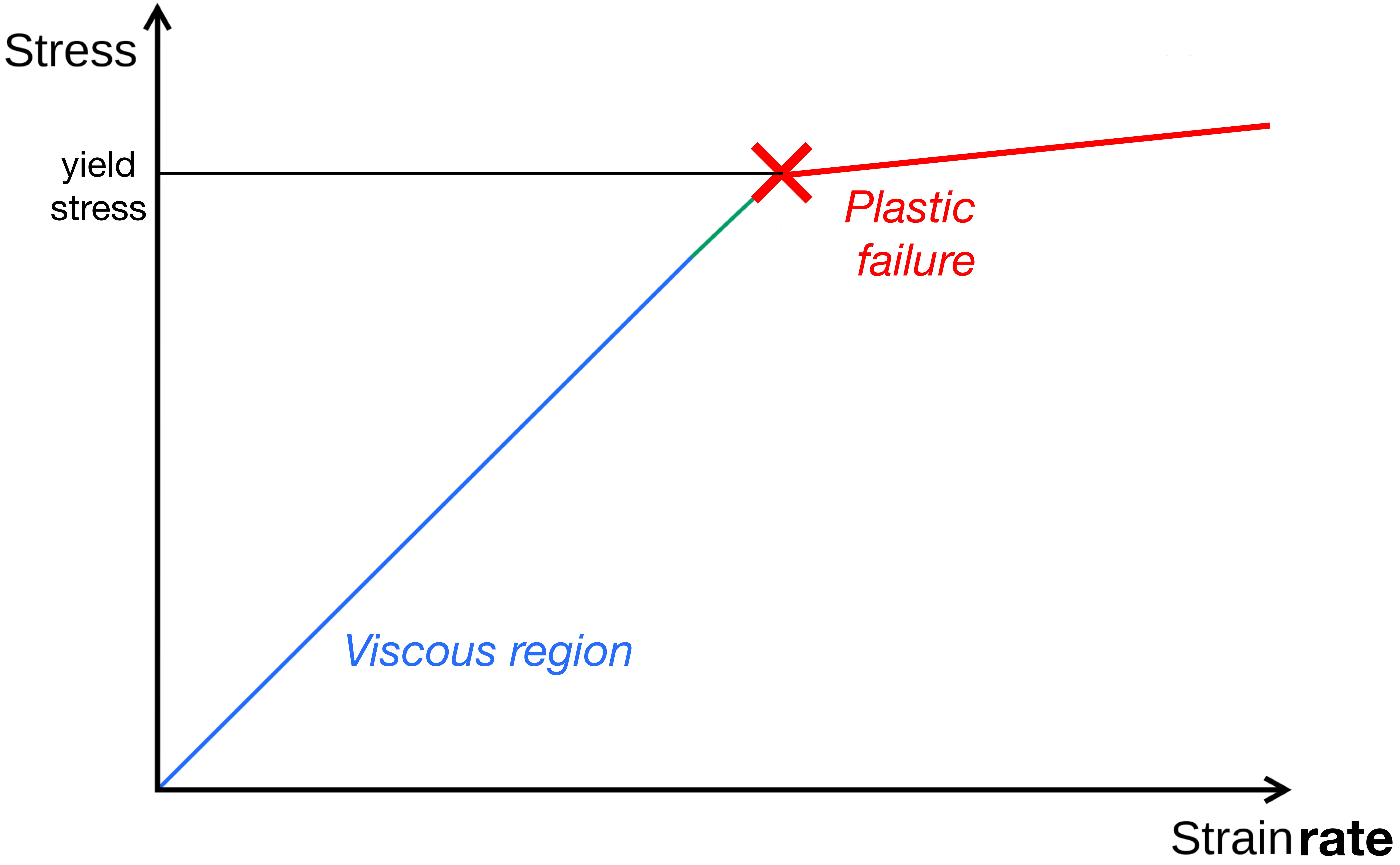
Engineering view:



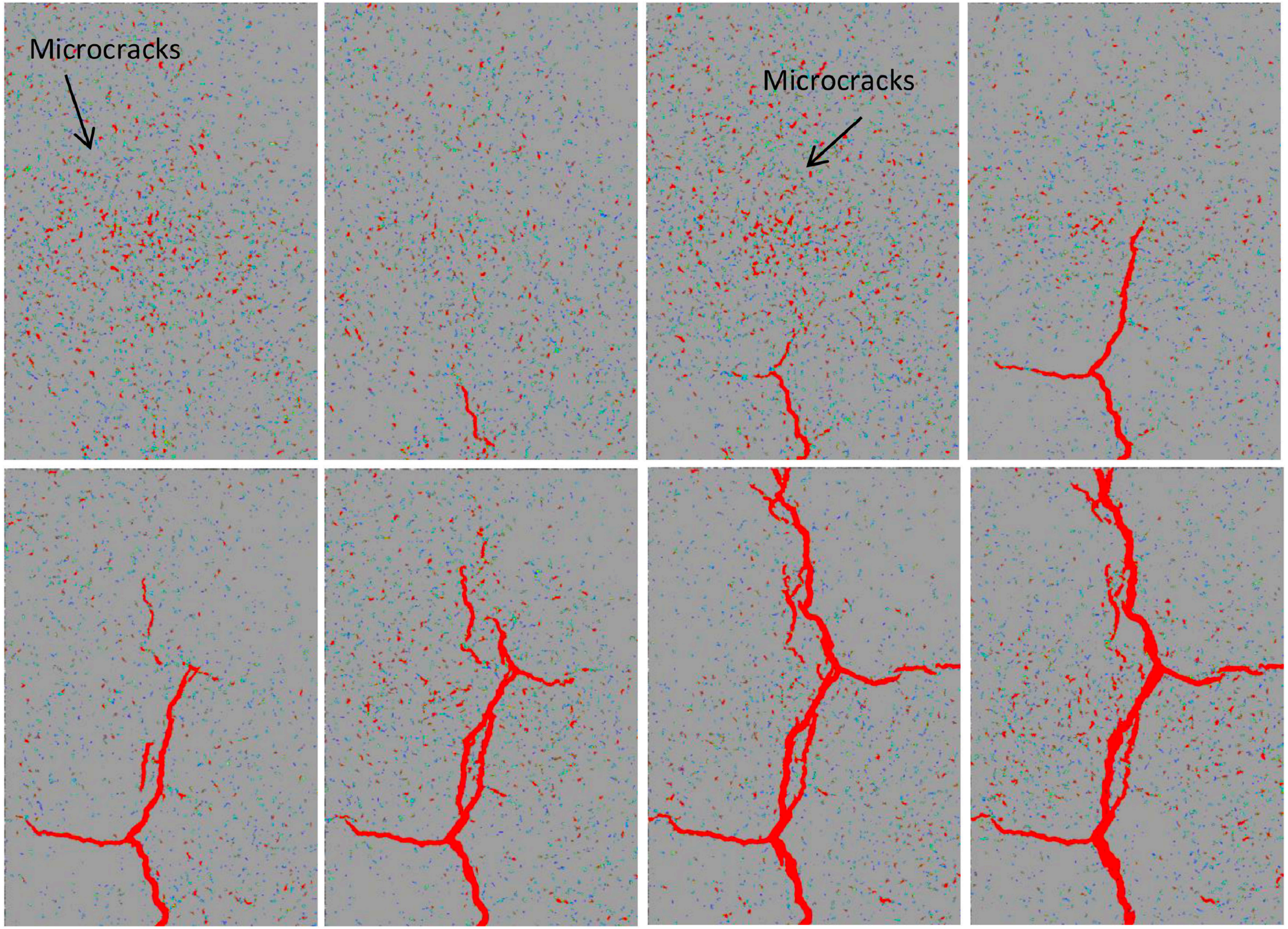
Definition

Plasticity is a change in rheology that occurs when the stress exceeds a *yield stress*

Geodynamics view:

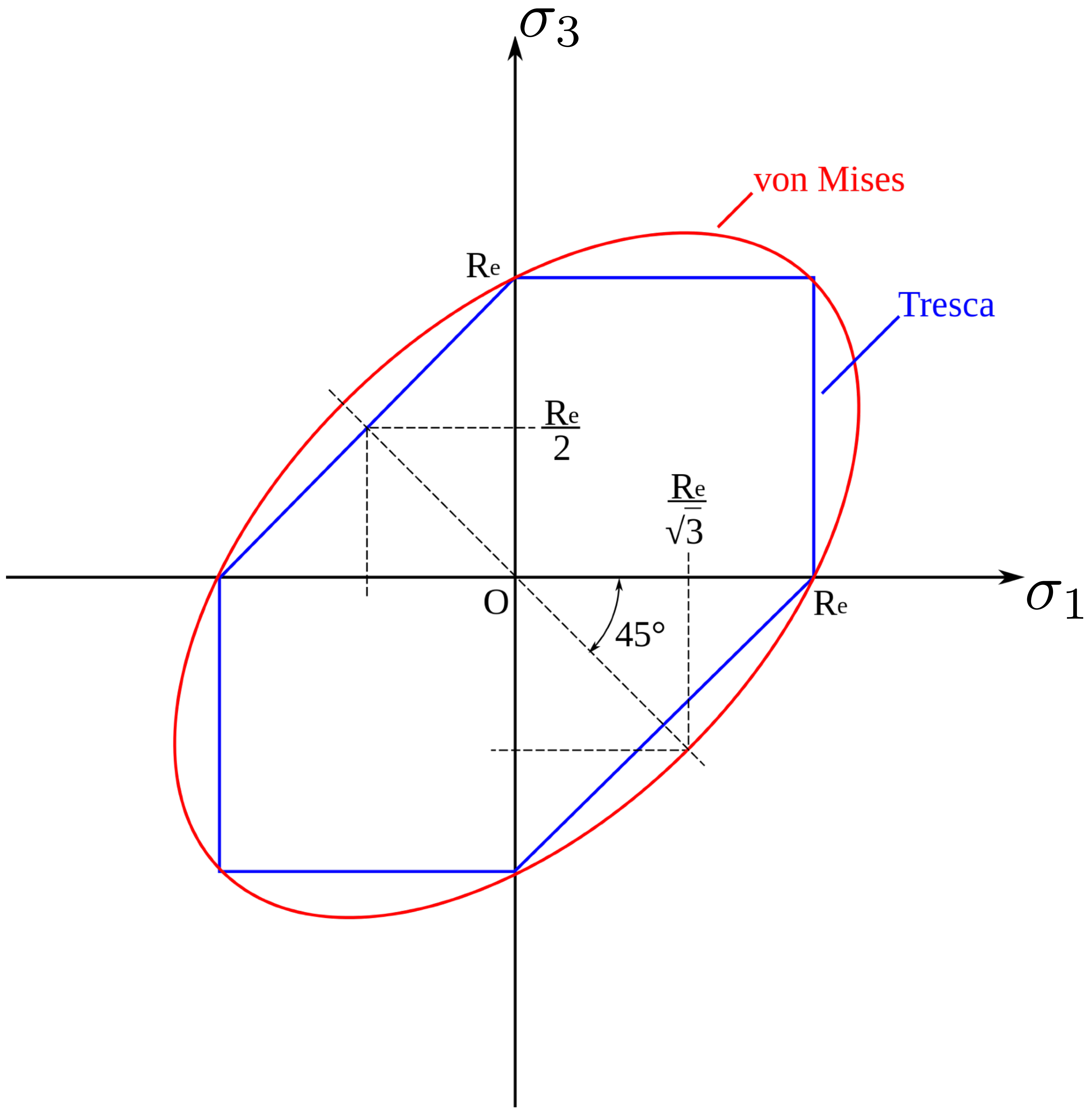


What is plasticity, really?

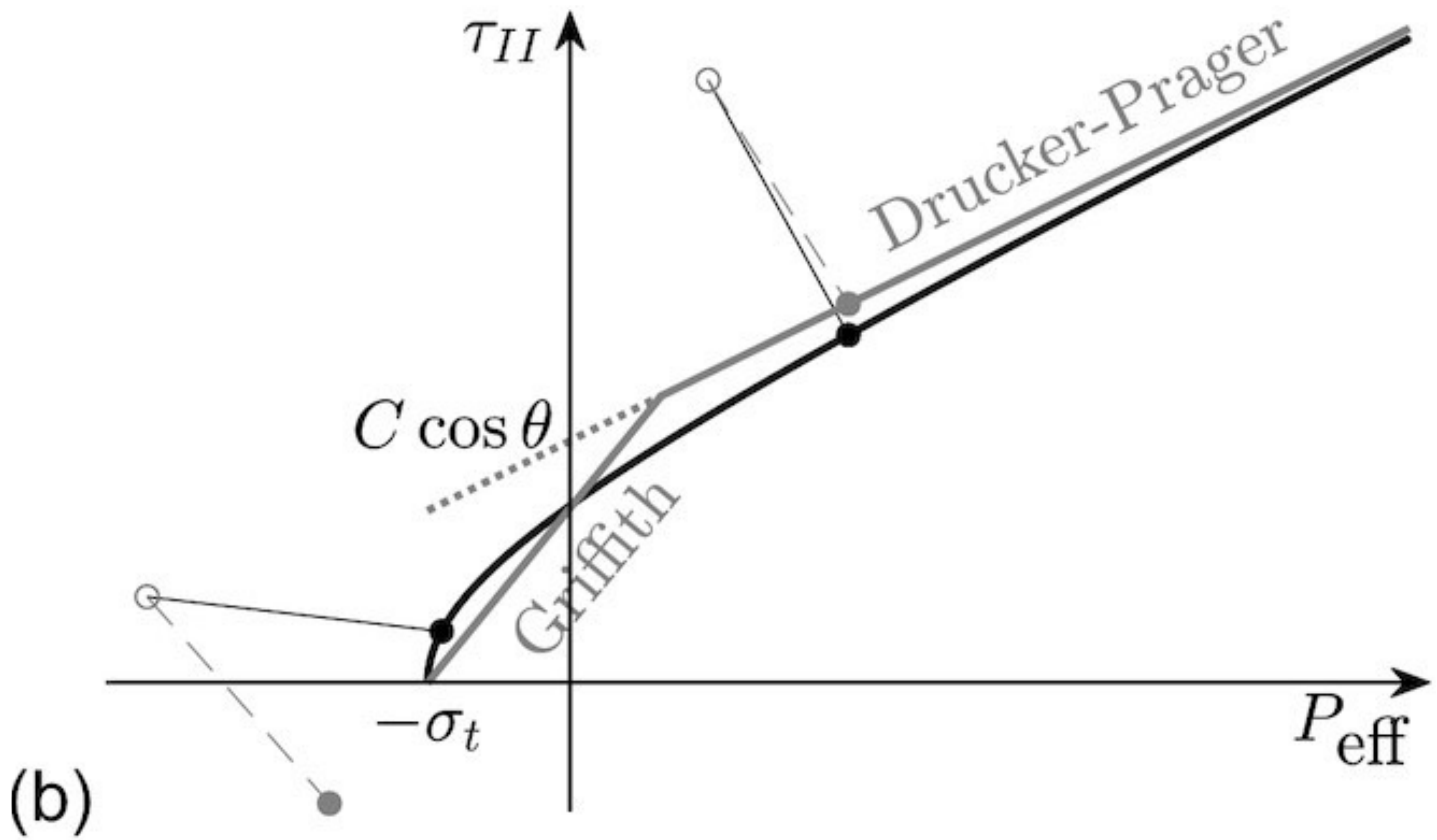


Alam & Loukili, *Int J Solids & Structures* 2020

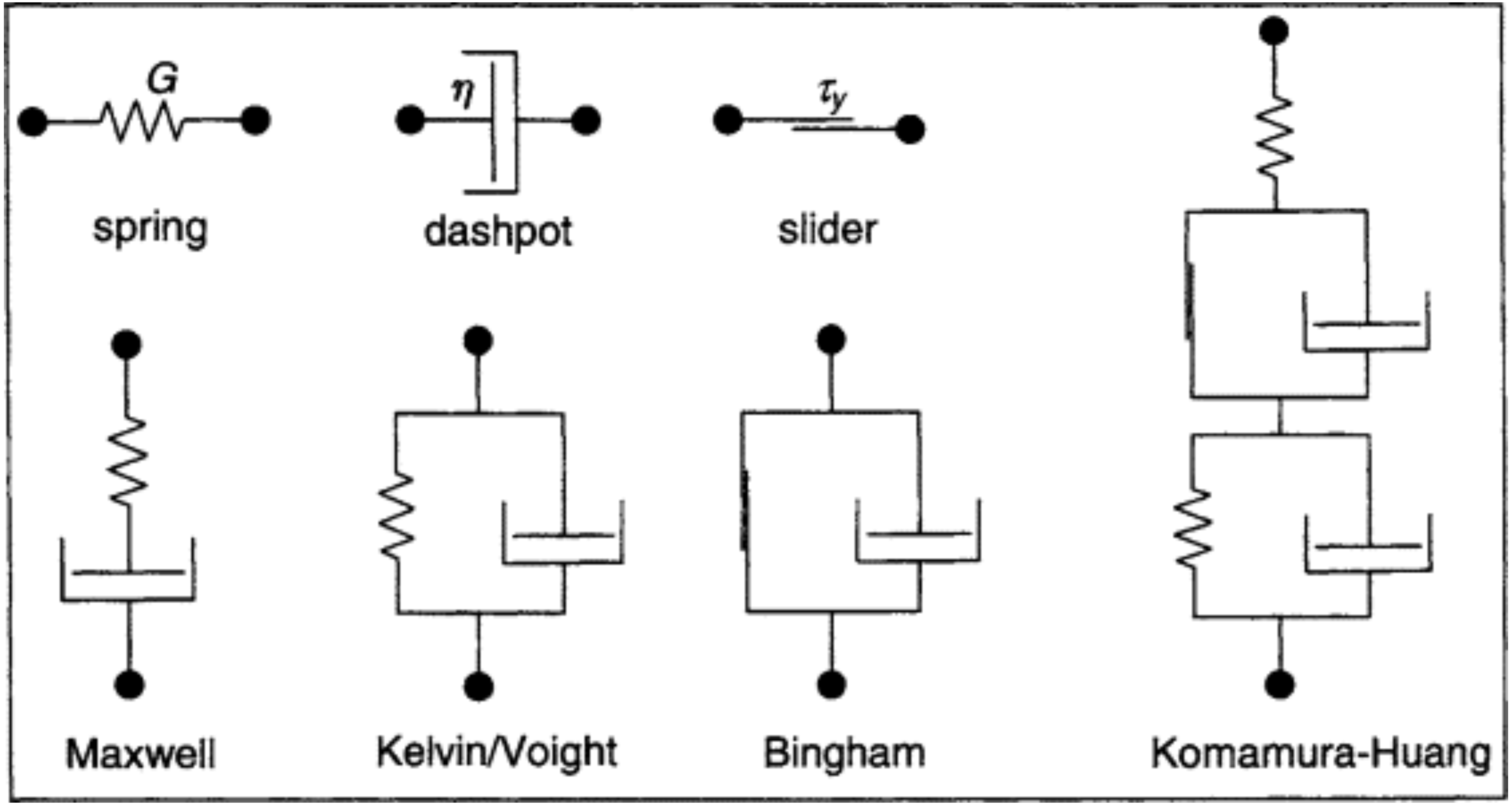
The yield criterion



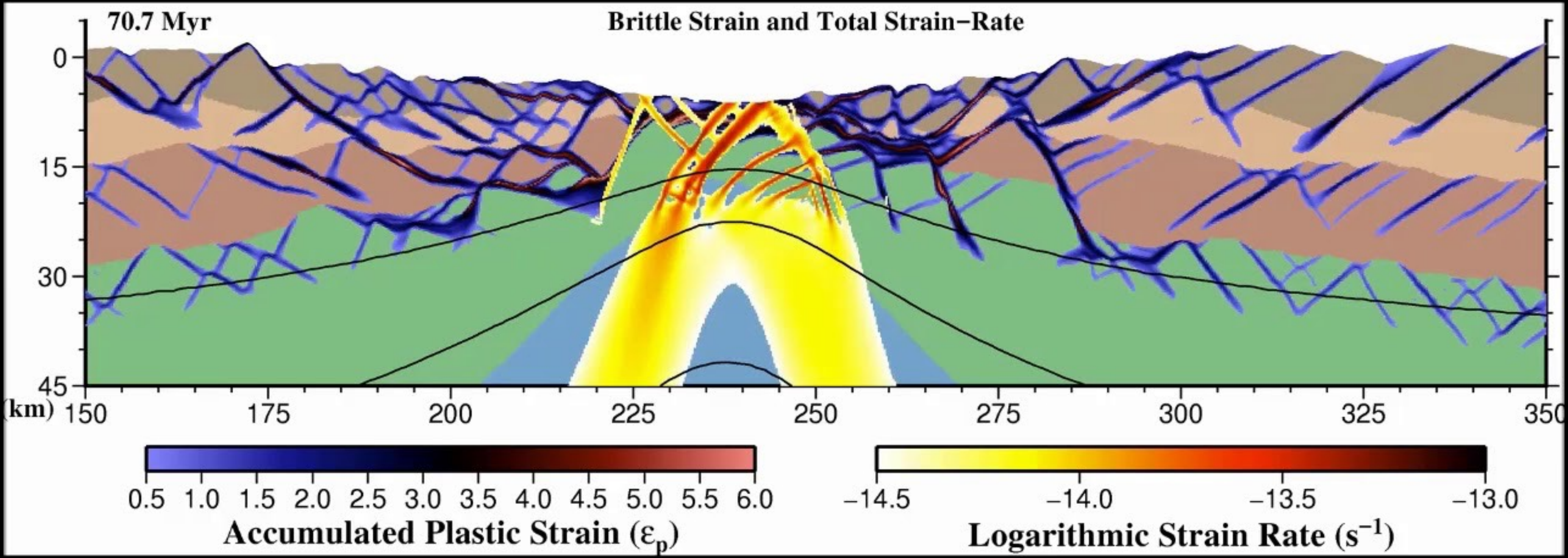
Plastic failure envelope:



Other composite rheologies



Faulting & frictional slip modelled as plastic deformation



Naliboff et al *Nature Comms* 2017