Miniprojects

EAUMP–ICTP School on Enumerative Combinatorics in Collaboration with AMU, CIMPA and ISP

Arusha, Tanzania July 2023

Please form **teams of 3-4 people** to work on **one miniproject** of your choice. Please try as many parts of your chosen project as you can. If you cannot do one subpart, please still move on to the rest of the project.

Please prepare a **carefully written report of 5 pages maximum length**. Both hand writing and LaTeX are acceptable.

Deadline for submission: **10am Monday 24 July**, to Dr Makungu or by email to `bszendroi@gmail.com` and `makungu_j@yahoo.com`.

Miniprojects on basic combinatorics

— Per Alexandersson —

Project 1. Let $B_{n,k}$ be the set of binary words of length n with exactly k 1's. We can cyclically rotate a binary word ℓ steps to the right, so that the last ℓ bits are moved to the front of the word. A word is called **fixed under rotation by ℓ** if you get back the same word after rotation by ℓ steps.

(a) For each $k \in \{0, 1, 2, 4\}$ and each $\ell \in \{0, 1, 2, 4\}$, find how many words in $B_{8,k}$ are fixed under rotation by ℓ.
(b) Let n be a positive integer. For every divisor $\ell \mid n$, find the number of words in $B_{n,k}$ which are fixed under rotation by ℓ. The answer will of course depend on n, k and ℓ.
(c) Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ be positive integers with sum n. Let W_α be the set of words consisting of α_1 1s, α_2 2s, and so on. Compute $|B_\alpha|$.
(d) Let α be as above with sum n. Find a formula for how many words in B_α are fixed under rotation by ℓ, where $\ell \mid n$. The answer depends on the properties of the α_i.

Project 2. Let T_n be the set of triangulations of the regular n-gon in the plane. Recall that $|T_{n+2}| = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number.

A triangulation in T_n can be rotated by $\ell \in \{0, 1, 2, \dots, n-1\}$ steps counterclockwise, and we obtain a (potentially) new triangulation. A triangulation is called **fixed** under rotation by ℓ steps if you get back the *same* triangulation as you started with after the rotation.

(a) Draw all triangulations in T_6. For each $\ell \in \{1, 2, 3\}$, find how many triangulations in T_6 are fixed under rotation by ℓ steps.
(b) For each $\ell \in \{1, 2, 3\}$, find how many triangulations in T_8 and T_9 are fixed under by ℓ steps.
(c) Show that no triangulation in T_n is fixed under any rotation $\ell \in \{1, \dots, n-1\}$ if 2 or 3 is not a divisor of n.
(d) Let $2 \mid n$. Compute the number of triangulations in T_n invariant under rotation by $180°$ (same as rotation by $\ell = \frac{n}{2}$ steps).
(e) Let $3 \mid n$. Compute the number of triangulations in T_n invariant under rotation by $120°$ (same as rotation by $\ell = \frac{n}{3}$ steps).

Miniprojects on generating functions and partitions

— Balázs Szendrői —

Project 3.

(1) Consider the Fibonacci sequence with $f_0 = 0, f_1 = 1$ and
$$f_{n+1} = f_n + f_{n-1}.$$
Let
$$b_n = \sum_{m=0}^{n} f_m f_{n-m}$$
be the sequence of sums of products of Fibonacci numbers.
  (a) Using the product rule for generating functions, find a formula for the generating function
  $$B(x) = \sum_{n=0}^{\infty} b_n x^n$$
  of the sequence b_n.
  (b) Using your formula, write down a recursion satisfied by the sequence $[b_0, b_1, \ldots]$.
  (c) Using a partial fraction expansion, find an explicit formula for b_n in terms of n.

(2) Consider the recursively given sequence with $c_0 = 0, c_1 = 1$ and
$$c_{n+1} = c_n + c_{n-1} + n.$$
  (a) Using the methods of shifts and addition of generating functions, find a closed form for the generating function
  $$C(x) = \sum_{n=0}^{\infty} c_n x^n.$$
  (b) Using partial fractions and expansions, or otherwise, find an explicit formula for c_n.

Project 4.

(1) Define *partitions*, their *weight* and *Ferrers diagram*.
(2) Define the *Young graph* of partitions.
(3) For a fixed partition λ of weight n, let $S(\lambda)$ denote the set of partitions of $n+1$ that λ is connected to in the Young graph. Similarly, let $T(\lambda)$ be the set of partitions of $n-1$ that λ is connected to in the Young graph. For example, for $\lambda = (1)$, $S(\lambda)$ consists of the partitions (2) and $(1,1)$ of 2, while $T(\lambda)$ consists of one element, the empty partition.
   (a) Find $S(\lambda)$ and $T(\lambda)$ for the partition $\lambda = (3,2,1)$ of 6.
   (b) Prove that for any partition λ, $S(\lambda)$ has exactly one more element than $T(\lambda)$.
(4) Let F be the infinite-dimensional complex vector space with one basis vector $\mathbf{v}_\lambda$ for each partition λ, including the empty partition. Let A, B be the operators (linear maps) on F defined on the basis $\{\mathbf{v}_\lambda\}$ by the following formulae:

$$A\mathbf{v}_\lambda = \sum_{\mu \in S(\lambda)} \mathbf{v}_\mu,$$

$$B\mathbf{v}_\lambda = \sum_{\mu \in T(\lambda)} \mathbf{v}_\mu.$$

So for example,

$$\begin{aligned} A\mathbf{v}_{(1)} &= \mathbf{v}_{(2)} + \mathbf{v}_{(1,1)}, \\ B\mathbf{v}_{(1)} &= \mathbf{v}_\emptyset. \end{aligned}$$

   (a) Let again $\lambda = (3,2,1)$. Compute $A\mathbf{v}_\lambda$, $B\mathbf{v}_\lambda$, $BA\mathbf{v}_\lambda$ and $AB\mathbf{v}_\lambda$.
   (b) Prove that for any partition λ,

$$(BA - AB)(\mathbf{v}_\lambda) = \mathbf{v}_\lambda.$$

Deduce the identity of operators

$$BA - AB = \mathrm{Id}_F$$

on the vector space F.

The outcome of this project is that the vector space F, usually called (fermionic) Fock space, is a representation of the Heisenberg Lie algebra defined by the commutation relation

$$BA - AB = 1.$$

Miniprojects on analytic methods

— Stephan Wagner —

Project 5. Recall *Stirling's formula*:

$$n! \sim \frac{n^n}{e^n}\sqrt{2\pi n}.$$

In the lecture, we also proved the following form:

$$\ln n! = n\ln n - n + \frac{1}{2}\ln(2\pi n) + O(n^{-1}).$$

(1) Show that the Catalan numbers satisfy the following asymptotic formula:

$$C_n = \frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n^3}}$$

as $n \to \infty$.

(2) For a constant a, prove that

$$\ln\left((n/2 + a\sqrt{n})!\right) = \frac{n\ln(n/2)}{2} - \frac{n}{2} + a\sqrt{n}\ln(n/2) + a^2 + \frac{1}{2}\ln(\pi n) + O(n^{-1/2}).$$

(3) Use this to show that (for constant a)

$$\ln\binom{n}{n/2 + a\sqrt{n}} = n\ln 2 - 2a^2 - \frac{1}{2}\ln\left(\frac{\pi n}{2}\right) + O(n^{-1/2}).$$

(4) It follows that

$$\binom{n}{n/2 + a\sqrt{n}} \sim \sqrt{\frac{2}{\pi n}}\,2^n e^{-2a^2},$$

which can be seen as a special case of the *central limit theorem*.

Project 6. This project is concerned with the *Euler–Mascheroni constant*:

$$\gamma = \lim_{n\to\infty}(H_n - \ln n),$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

(1) Show by means of the Euler–Maclaurin formula that

$$H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \int_n^\infty B_3(\{x\})x^{-4}\,dx.$$

(2) Prove that the maximum and minimum of the Bernoulli polynomial $B_3(x)$ on the interval $[0,1]$ are $\frac{1}{12\sqrt{3}}$ and $-\frac{1}{12\sqrt{3}}$ respectively. Thus

$$|B_3(x)| \le \frac{1}{12\sqrt{3}}$$

for all $x \in [0,1]$.

(3) Conclude that

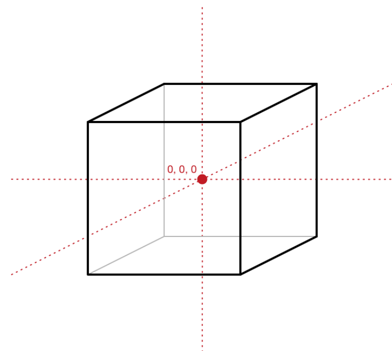
$$\left|\gamma - \left(H_n - \ln n - \frac{1}{2n} + \frac{1}{12n^2}\right)\right| \le \frac{1}{36\sqrt{3}n^3}.$$

(4) Take $n = 1000$ in this inequality to compute the first ten decimal places of γ (using mathematical software of your choice).

Miniproject on group actions

— Fernando Rodriguez Villegas —

Project 7. This project is concerned with the *rotational symmetry group G of the cube* $C \subset \mathbb{R}^3$ centered at the origin.



(1) Describe all elements of G as rotations around a fixed axis by a certain angle, both of which you should find.
(2) Count how many rotations are there of each kind, and compute the order of each rotation.
(3) Find the number of elements of the group G.
(4) Use Pólya's theory to count how many different ways are there to colour the faces of the cube red/blue, up to rotational symmetry.
(5) How about the number of colourings when we use k colours?
(6) Identify G as an abstract group.