Large Scale Structure of the Universe

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Dark Energy Survey & LSST

Legacy Survey of Space and

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Five lectures on Cosmology and Large Scale Structure

Lecture I: The average Universe

Lecture II: Distances and thermal history

Lecture III: The perturbed Universe

Lecture IV: Theoretical challenges and surve

Lecture V: Observational cosmology with LS

Plan for Lecture III:

III.1 – Growth of perturbations

III.2 – Statistics of perturbations

III.1- Growth of perturbations

1.1 – Introduction

1.2 – Physical degrees of freedom

1.3 – Newtonian perturbations

1.4 – Jeans instability

1.5 – Linearized newtonian growth of dark matter

1.6 – Transfer function

1.1 – Introduction

FLRW

ckground (or average) evolution of the Universe:

$$\boxed{\bar{G}_{\mu\nu} = 8\pi G\bar{T}_{\mu\nu}}$$

$$g_{\mu\nu}(x,t) = \bar{g}_{\mu\nu}(t)$$
$$\rho_i(x,t) = \bar{\rho}_i(t)$$
$$p_i(x,t) = \bar{p}_i(t) = w_i(t)$$

volution of perturbations in the Universe:

$$\boxed{\delta G_{\mu\nu} = 8\pi G\delta T_{\mu\nu}}$$

$$g_{\mu\nu}(x,t) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(x$$
$$\rho_i(x,t) = \bar{\rho}_i(t) + \delta\rho_i(x,t)$$
$$p_i(x,t) = \bar{p}_i(t) + \delta p_i(x,t)$$

i = different components
+ velocity perturbations

1.2 – Physical degrees of freedom

ere are 10 degrees of freedom in the Einstein tensor
ere are 10 degrees of freedom in the energy-momentum tensor
ere are 10 equations related them (Einstein's equation)
ere are 4 redundancy (gauge) relations: diffeomorphism transformations

nere are 10+10-10-4=6 physical degrees of freedom

is possible to find 6 gauge-invariant degrees of freedom: Bardeen variables.
ternatively, one can choose (fix) a particular gauge.
of these degrees of freedom are associated to gravitational waves – more on
urani's lectures. We will only worry with scalar perturbations.

.1.3 – Newtonian perturbations

e can gain some intuition about the growth of perturbations using a newtonian
proximation to GR.

wtonian approximation is valid for nonrelativistic matter (v<<c, p<<ρ) and for sca
ep inside the Hubble radius.

e will use newtonian fluid dynamics describing a fluid in a gravitational field.
is fluid is described by 3 equations:
Continuity equation (energy conservation)
Euler equation (force equation)
Poisson equation (gravity equation)

nsider a fluid element with mass density ρ and velocity v at position r at time t:

$$\partial_t \rho = -\vec{\nabla}_r \cdot (\rho \vec{v})$$

$$(\partial_t + \vec{v} \cdot \vec{\nabla}_r)\vec{v} = -\frac{\vec{\nabla}_r p}{\rho} - \vec{\nabla}_r \phi$$

$$\nabla_r^2 \phi = 4\pi G \rho$$

earize equations with small perturbations (consider only one component):

$$(r,t) = \bar{\rho}(t) + \delta\rho(r,t)$$

$$(r,t) = \bar{p}(t) + \delta p(r,t)$$

$$(r,t) = \bar{v}(t) + \delta v(r,t)$$

$$(r,t) = \bar{\phi}(t) + \delta\phi(r,t)$$

e the Universe is expanding, we want to consider the fluid equations with respec
oving coordinates x instead of physical coordinates r:

$$\vec{r}(t) = a(t)\vec{x}$$

locity is given by:

$$\vec{v}(t) = \dot{a}(t)\vec{x} + a(t)\dot{\vec{x}} = H\vec{r}(t) + \vec{u}$$

Background velocity
"Hubble flow"

Peculiar velocity
perturbation

ddition, will take derivatives with respect to x and time derivatives with fixed x:

$$\vec{\nabla}_r = \frac{1}{a}\vec{\nabla}_r \qquad \left(\frac{\partial}{\partial t}\right)_r = \left(\frac{\partial}{\partial t}\right)_x - H\vec{x}\cdot\vec{\nabla}_x$$

e will introduce the density contrast to parametrize the dens
erturbations:

$$\delta \equiv \frac{\delta\rho}{\bar{\rho}}$$

ntinuity equation can be written as:

$$\left[\partial_t - H\vec{x}\cdot\vec{\nabla}\right][\bar{\rho}(1+\delta)] + \frac{1}{a}\vec{\nabla}\cdot[\bar{\rho}(1+\delta)(Ha\vec{x}+\vec{u})] = 0$$

ntinuity equation to zeroth order in perturbations (δ=0, u=0):

$$\partial_t \bar{\rho} + 3H\bar{\rho} = 0$$

ntinuity equation for a nonreativistic matter background

ontinuity equation to first order in perturbations:

$$\partial_t \delta = -\frac{1}{a}\vec{\nabla} \cdot \vec{u}$$

ler equation to first order in perturbations:

$$\partial_t \vec{u} + H\vec{u} = -\frac{\vec{\nabla}\delta p}{a\bar{\rho}} - \frac{1}{a}\vec{\nabla}\delta\phi$$

oisson equation to first order in perturbations:

$$\nabla^2\delta\phi = 4\pi G a^2 \bar{\rho}\delta$$

possible to obtain one equation involving only δ:

ake time derivative of continuity equation

ake divergence of Euler equation

ssume a relation between pressure perturbation and density perturbation with t
oduction of the sound speed

e equation for the evolution of the density contrast at the linear level is:

$$\boxed{\ddot{\delta} + 2H\dot{\delta} - \left(\frac{c_s^2}{a^2}\nabla^2 + 4\pi G\bar{\rho}\right)\delta = 0}$$

$$\delta p \equiv c_s^2 \delta\rho$$

1.4 – Jeans instability

convenient to work in Fourier space:

$$\delta(\vec{x},t) = \int \frac{d^3k}{2\pi^3}\delta_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{x}}$$

e perturbation equation becomes:

$$\ddot{\delta}_{\vec{k}} + 2H\dot{\delta}_{\vec{k}} + c_s^2\left(\frac{k^2}{a^2} - k_J^2\right)\delta_{\vec{k}} = 0$$

ysical Jeans scale or wavenumber (as opposed to comoving):

$$k_J \equiv \sqrt{\frac{4\pi G\bar{\rho}(t)}{c_s^2}} \qquad \lambda_J = \frac{2\pi}{k_J}$$

small scales ($k/a >> k_J$) the solution is oscillating with a damped amplitude.
mping is due to the expansion of the Universe (H term – "Hubble friction").

large scales ($k/a << k_J$) one can neglect pressure perturbations ($c_s = 0$). We will
ve the equation next.
hout Hubble friction the perturbations would be unstable!

e: when baryons are coupled to radiation ($c^2_s = 1/3$) the Jeans length ~
bble horizon – perturbations do not grow. But perturbations in DM have negligibl
ns lenght and can grow.

.1.5 – Linearized newtonian growth of dark matter

s now obtain solutions of the linearized newtonian equation for the case of matter. For dark matter, the speed of sound of perturbations (and the correspon ns scale) can be neglected:

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}_m\delta_m = 0$$

Scale independent

will consider the growth in three regimes of the expansion of the Unvivers
atter domination
adiation domination
osmological constant domination

tter domination:

$$\propto t^{2/3} \Rightarrow H = \frac{2}{3t} \qquad H^2 = \frac{8\pi G}{3}\bar{\rho}_m \Rightarrow 4\pi G\bar{\rho}_m = \frac{3}{2}H^2 =$$

$$\ddot{\delta}_m + \frac{4}{3t}\dot{\delta}_m - \frac{2}{3t^2}\delta_m = 0$$

lution:

$$_m(t) = c_1 t^{-1} + c_2 t^{2/3}$$

Decaying mode

Growing mode

In matter-dominated era dark matt
perturbations grow as:

$$\boxed{\delta_m(t) \propto a(t)}$$

diation domination:

$$a(t) \propto t^{1/2} \Rightarrow H = \frac{1}{2t}$$

$$\ddot{\delta}_m + \frac{1}{t}\dot{\delta}_m = 0$$

Last term can be neglecte

olution:

$$_m(t) = \underbrace{c_1}_{\text{Constant mode}} + \underbrace{c_2 \ln t}_{\text{Growing mode}}$$

In radiation-dominated era dark mat
perturbations grow as:

$$\boxed{\delta_m(t) \propto \ln a(t)}$$

Slower growth

smological constant domination:

$$a \propto e^{Ht}$$

H(t) = H = const.

$$\ddot{\delta}_m + 2H\dot{\delta}_m = 0$$

Last term can be neglecte

olution:

$$(t) = c_1 + c_2 e^{-2Ht}$$

Constant mode

decaying mode

In Λ-dominated era dark matter perturbations **do not grow** s convenient to introduce the **linear growth function D(t)** that scribes the linear growth of modes inside the horizon (but larger thar e Jeans scale), where the growth is independent of scale:

$$\delta_{\vec{k}}(t) \equiv D(t)\delta_{\vec{k}}(t_0)$$

II.1.6 – Transfer function

wth of perturbations that are larger than the Hubble horizon can depend on its
oving scale characterized by the wavenumber k (even at the linear level).

 introduces the transfer function to account for this possibility, changing the prev
ation to:

$$\delta_{\vec{k}}(t) = D(t)T(\vec{k})\delta_{\vec{k}}(t_0)$$

r perturbations that never leave the horizon, T(k)=1.

ow do perturbations grow outside the horizon?
his is computed in full-fledged GR perturbations.
ere I just give the answer, which is simple enough:

$$\delta \propto \begin{cases} a^2 & \text{radiation dominated} \\ a & \text{matter dominated} \end{cases}$$

erturbations produced during inlfation are very long wavelenght and outside the ubble horizon initially (see Mehrdad's lectures).

ventually the enter the horizon.

hen does a perturbation crosses the horizon?
hat's when the comoving wavevector is caught up by the comoving Hubble scale

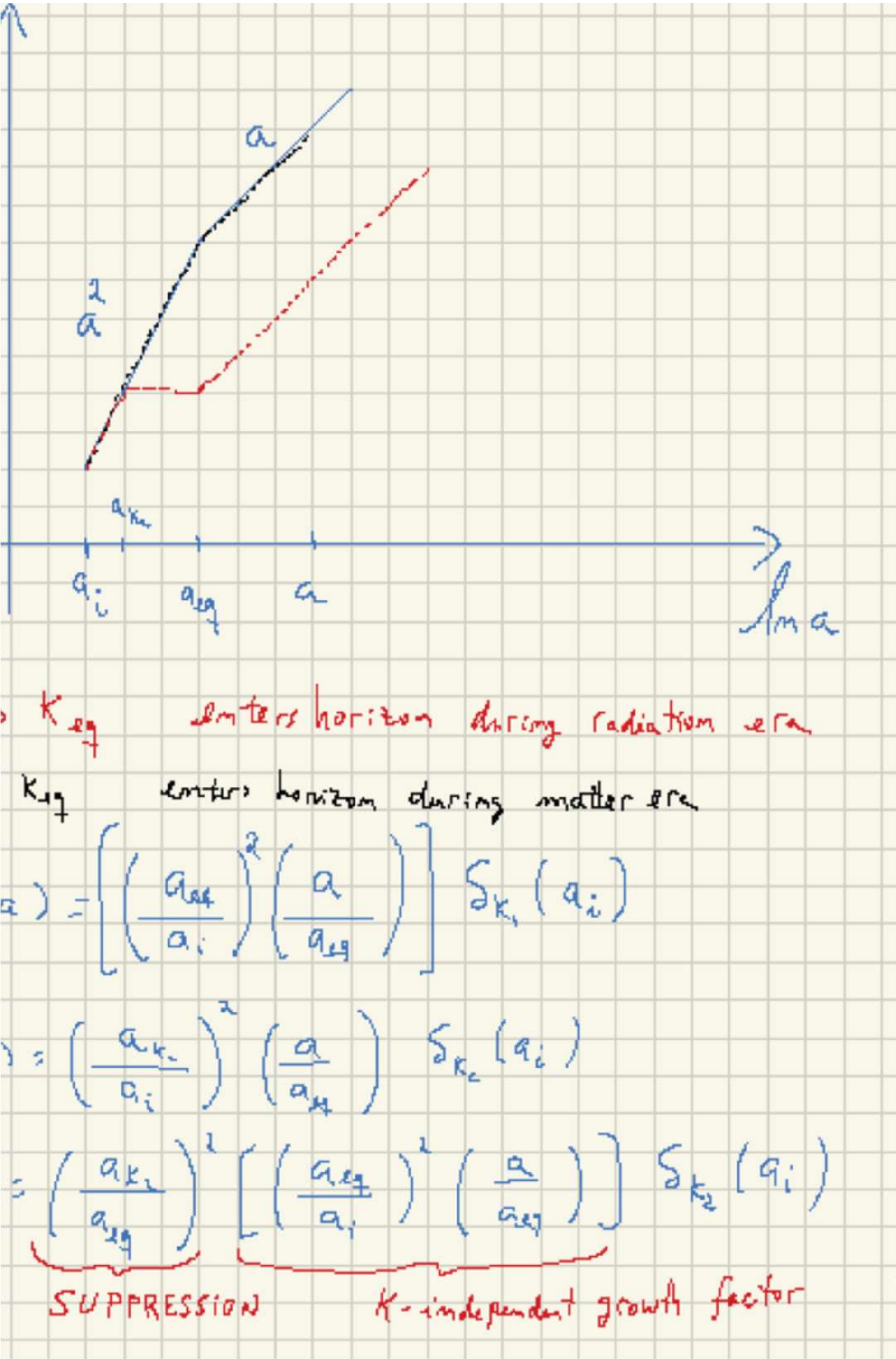
$$k_{h.c.} = a(t)H(t)$$

he crucial point is that perturbations can cross the horizon either in the adiation-dominated or matter-dominate era.

$$k_{eq} = a(t_{eq})H(t_{eq})$$

hort wavelenght modes ($k>k_{eq}$) enters the horizon in radiation era.
ong wavelenght modes ($k<k_{eq}$) enters the horizon in matter era.

his different behaviour introduces a scale dependence in the growth of perturbati
at is encapsulated in the transfer function.



Horizon crossing at:

$$k_{h.c.} = a(t)H(t)$$

Radiation era: H α a^{-2}

$$k_{h.c.} \propto a(t_{h.c.})^{-1}$$

Therefore the transfer function is:

$$T(k) = \begin{cases} 1 & k<k_{eq} \\ \dfrac{k_{eq}^2}{k^2} & k>k_{eq} \end{cases}$$

III.2- Statistics of perturbations

2.1 – Initial perturbations

2.2 – Summary statistics

2.3 – Power spectrum

2.4 – Primordial power spectrum

2.5 – Linear matter power spectrum today

2.6 – Higher order statistics

.2.1 – Initial perturbations

al perturbations are generated by small quantum fluctuations during the inflation
se of the Universe (see Mehrdad’s lectures).

y are random variables.

ory predicts a probability distribution for the initial perturbations.

st models of inflation predict a gaussian probability distribution.

 Universe is one possible realization of the random perturbations.

.2.2 – Summary statistics

babilty distributions are characterized by moments of the distribution – these are
nmary statistics”
rage, variance, asymmetry, kurtosis, etc

$$\langle\delta(x)\rangle, \quad \langle\delta^2(x)\rangle, \quad \langle\delta^3(x)\rangle, \quad \langle\delta^4(x)\rangle, \quad \cdots$$

aussian distribution is fully characterized by its first 2 moments:
rage and variance.

.2.3 – Power spectrum

density perturbations one expects zero average: $\langle\delta(\vec{x})\rangle = 0$

two-point correlation function defines the spatial correlation function ξ(r):

Two-point spatial
correlation function

$$(\vec{x}_1)\delta(\vec{x}_2)\rangle = \xi(\vec{x}_1 - \vec{x}_2) = \underbrace{\xi(|\vec{x}_1 - \vec{x}_2|)}_{\text{Homogeneity and isotropy}} = \xi(r)$$

Homogeneity and isotropy

e: since one can't average over different Universes the averages are over different **locations**
erent patches of the Universe can be tought of as coming from different realizations).

erpretation of 2 pt. correlation function: excess (or deficit) of clustering over dom at a given scale r

$$dP_{1,2} = (1 + \xi(r))dV_1 dV_2$$

random

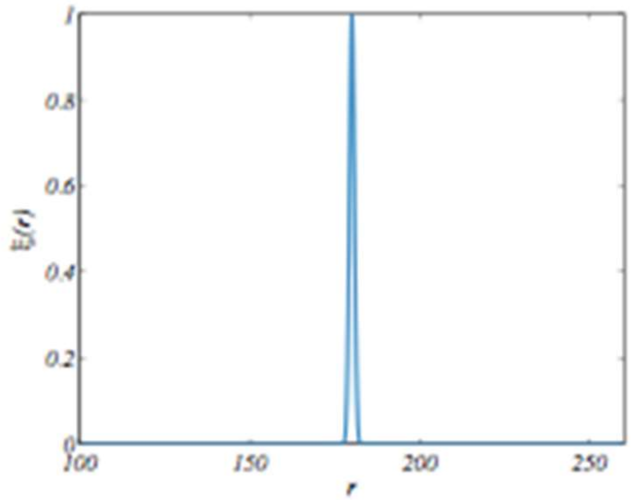
can also define the power spectrum as Fourier transform of the correlation func

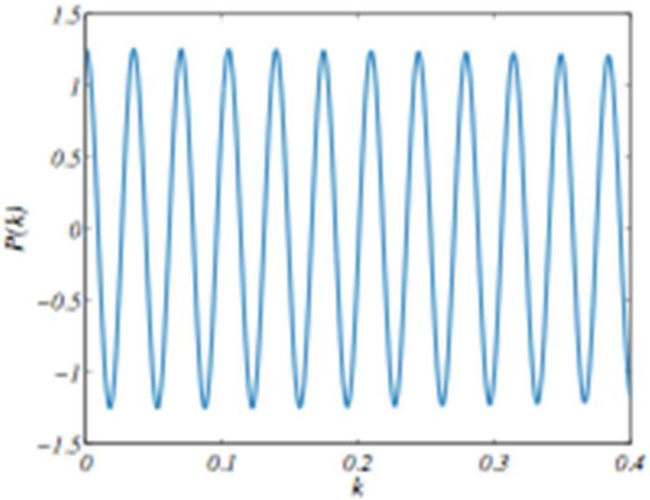
$$P(k) = \int d^3 r \xi(r) e^{i\vec{k}.\vec{r}}$$

It’s possible to work with either spatial correlation function or power spectrum – advantages and disadvantages.

Sharp peak in correlation results in oscillations in the power spectrum:

$$\xi(r) \approx \delta(r - r_*) \qquad P(k) \approx e^{ikr_*}$$





In terms of the Fourier transform of the density perturbations:

$$\langle \delta_{\vec{k}} \delta_{\vec{k'}} \rangle = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') P(k)$$

Dimensioless power spe

$$\Delta^2(k) \equiv \frac{k^3}{2\pi^2}$$

Recalling:

$$\delta_{\vec{k}}(t) = D(t) T(\vec{k}) \delta_{\vec{k}}(t_0)$$

One finds:

$$P(k,t) = D(t)^2 T(k)^2 P(k)_{ini}$$

2.4 – Primordial power spectrum

nordial power spectrum of scalar perturbations is generated during inflation.

ne simplest models it can be parametrized with an amplitude and a spectral inde

$$P(k)_{ini} = A_s k^{n_s}$$

e amplitude A_s and spectral index n_s are free parameters of the ΛCDM model.

nplest models of inflation **predict** n_s close to 1.
viations are related to small so-called slow-roll parameters – see Mehrdad's lect

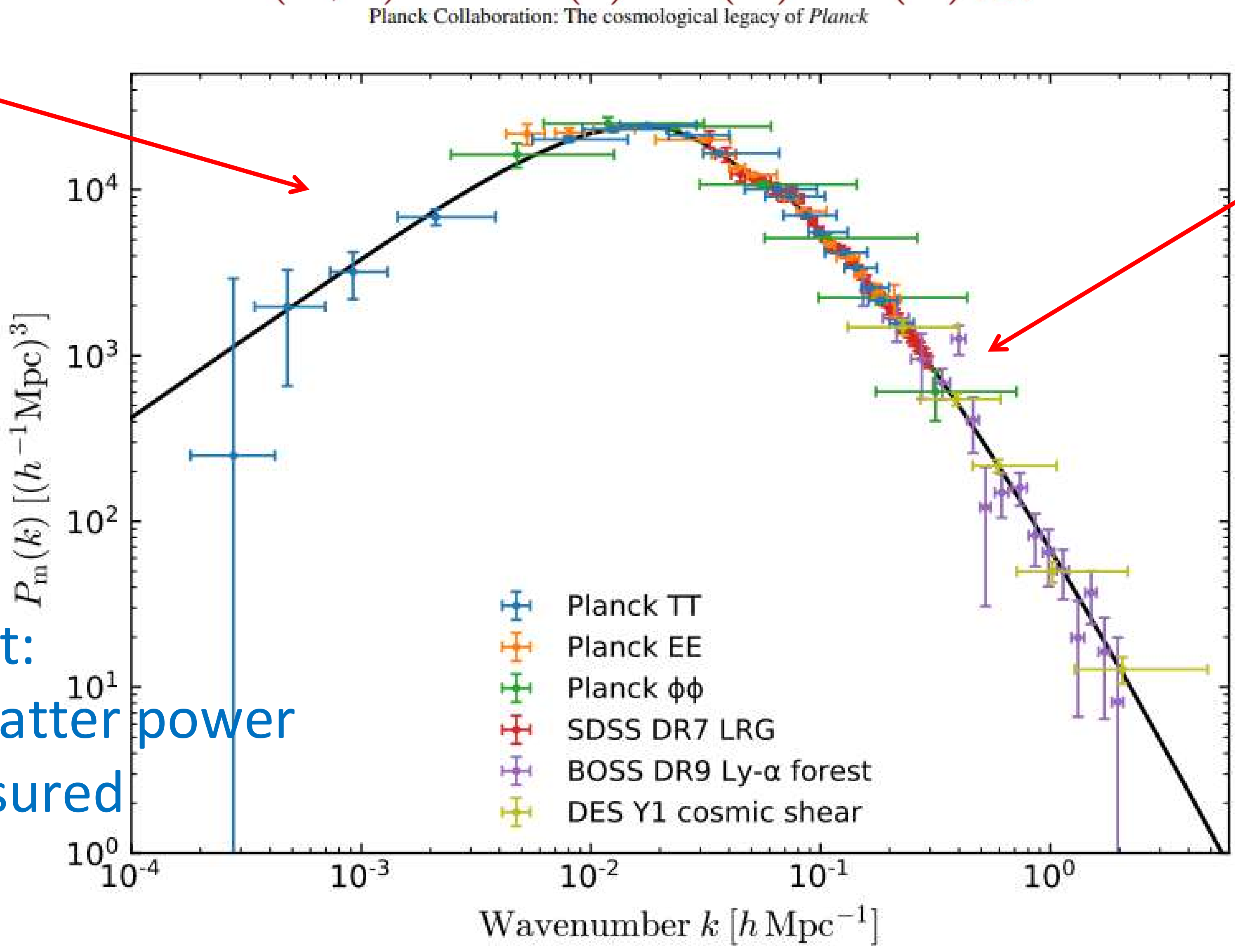
nck 2018: n_s=0.9649 $\pm$ 0.0042 at 68% CL - ~10σ away from 1!

2.5 – Linear matter power spectrum today

$$P(k,t) = D(t)^2 T(k)^2 P(k)_{ini}$$

P α k

P α



d of funny plot:

rred linear matter power

directly measured

2.6 – Higher-order statistics

e can also define higher-order statistics, such as the bispectrum B(k_1, k_2, k_3):

$$\langle \delta_{\vec{k}_1} \delta_{\vec{k}_2} \delta_{\vec{k}_3} \rangle \equiv (2\pi)^3 \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) B(\vec{k}_1, \vec{k}_2, \vec{k}_3)$$

ngaussian perturbations can be studied by measuring the bispectrum.

ce GR is nonlinear one expects nongaussian perturbations to develop from al gaussian ones. The detection of nongaussian **primordial** perturbation is a ve ve area. Measurements are difficult.