Books:

- *The Young Universe: Primordial Cosmology,*
  edited by R. Taillet (John Wiley & Sons, 2022) ISBN : 1789450322
  → Chapter 2: Cosmological Microwave Background, by JL

- *Neutrino cosmology,*
  JL, G. Mangano, G. Miele, S. Pastor (Cambridge University Press 2013)
  → Chapter 5: Cosmological Microwave Background, by JL

Notes from Master course on advanced Cosmology:

- *The Ingredients of the Universe,* -> link on Indico page of this school
  JL, course at RWTH Aachen University
  1. Recalls on homogeneous cosmology
  2. Thermal history of the Universe
  3. Linearised gravity
  4. Inflation
  5. CMB anisotropies
  6. Large Scale Structure

Summary of yesterday: perturbed degrees of freedom and equations of motion

Degrees of freedom	Equation of motion
Gravitational potential $\psi(\eta, \vec{x})$	Einstein 00: $\phi, \psi \leftrightarrow \delta$
Scale factor distortion $\phi(\eta, \vec{x})$	Einstein ij: $(\phi - \psi) \longrightarrow \sigma$
CDM density/velocity $\delta_\mathrm{c}(\eta, \vec{x})\,, \theta_\mathrm{c}(\eta, \vec{x})$	continuity δ_c' + Euler θ_c'
Baryon density/velocity $\delta_\mathrm{b}(\eta, \vec{x})\,, \theta_\mathrm{b}(\eta, \vec{x})$	Continuity δ_b' + Euler θ_b' (incl. Thomson)
Photons $f_\gamma(\eta, \vec{x}, p, \hat{n})$	Boltzmann $\frac{d}{d\eta} f_\gamma =$ Thomson
[Neutrinos $f_\nu(\eta, \vec{x}, p, \hat{n})$]	[Boltzmann $\frac{d}{d\eta} f_\nu = 0$]

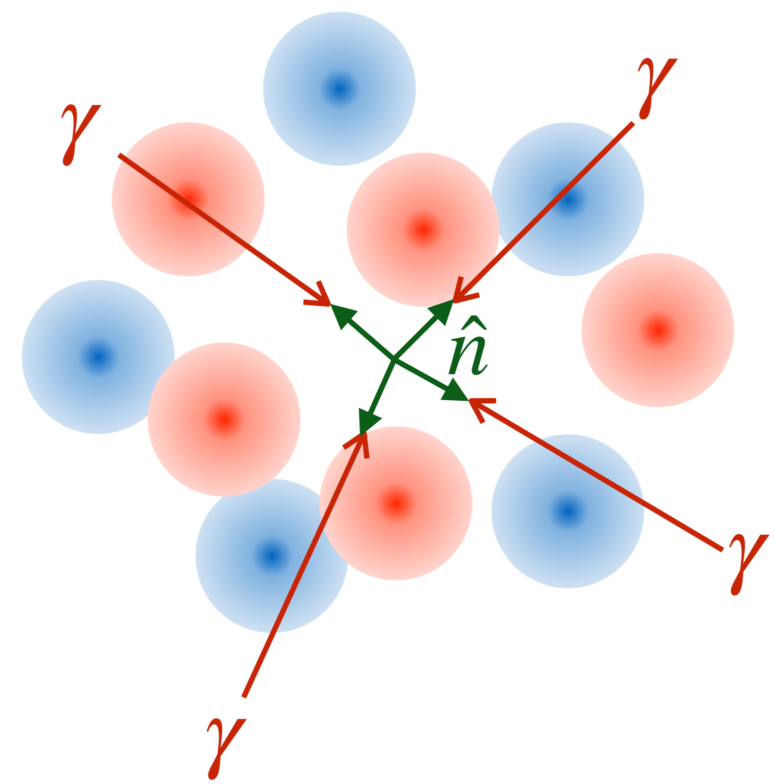
Photon phase-space distribution

Blackbody shape: $$f_\gamma(\eta,\vec{x},p,\hat{n}) = \frac{1}{e^{\frac{p}{T(\eta,\vec{x},\hat{n})}}-1}$$

Up to very good approximation: preserved even when leaving thermal equilibrium, but becomes direction-dependent due to gravitational interactions:

redshifting along geodesics:

$$\frac{d\ln(a\,p)}{d\eta} = \underbrace{\phi'}_{\text{dilation}} - \underbrace{\hat{n}\cdot\vec{\nabla}\psi}_{\text{gravitational Doppler}}$$



Then: $T(\eta,\vec{x},\hat{n}) = \bar{T}(\eta)\,\big(1+\Theta(\eta,\vec{x},\hat{n})\big)$

Linearised Boltzmann equation

Temperature fluctuation: $T(\eta, \vec{x}, \hat{n}) = \bar{T}(\eta) \left(1 + \Theta(\eta, \vec{x}, \hat{n})\right)$

Monopole and dipole of Θ account for local density & bulk velocity:

$$\Theta(\eta, \vec{x}, \hat{n}) = \frac{1}{4}\delta_\gamma(\eta, \vec{x}) + \hat{n} \cdot \vec{v}_\gamma(\eta, \vec{x}) + \text{higher multipoles}$$

Linearised Boltzmann:

$$\Theta' + \hat{n} \cdot \overrightarrow{\nabla}\Theta - \phi' + \hat{n} \cdot \overrightarrow{\nabla}\psi = -\Gamma_\gamma \Big(\hat{n} \cdot (\vec{v}_\gamma - \vec{v}_\mathrm{b}) + \text{higher multipoles}\Big)$$

dilation

gravitational Doppler

Thomson scattering

Thomson scattering wants to align velocity of photons vs. electron/baryons, and to wash out higher multipoles!

Boltzmann hierarchy

Start from linearised Boltzmann and perform:

1. Fourier transformation
2. Legendre expansion $\Theta(\eta, \vec{k}, \hat{n}) = \sum_l (-i)^l (2l+1)\, \Theta_l(\eta, \vec{k})\, P_l(\hat{k} \cdot \hat{n})$

relates to Θ_2

$$\delta'_\gamma + \frac{4}{3}\theta_\gamma - 4\phi' = 0$$

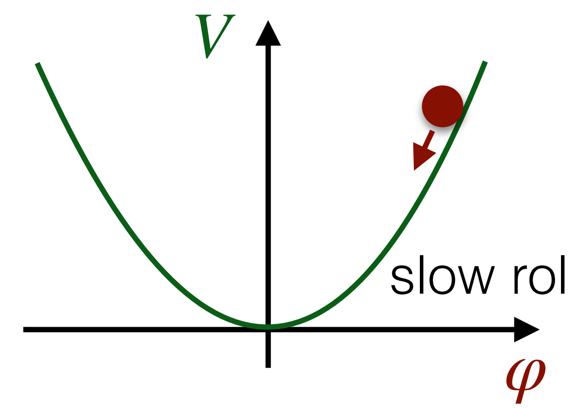
$$\theta'_\gamma + k^2 \left(-\frac{1}{4}\delta_\gamma + \sigma_\gamma \right) - k^2\psi = \tau'(\theta_\gamma - \theta_\mathrm{b})$$

$$\Theta'_l - \frac{kl}{2l+1}\Theta_{l-1} + \frac{k(l+1)}{2l+1}\Theta_{l+1} = \tau'\Theta_l \quad \forall l \geq 2$$

$\Rightarrow$ Solved together with previous equations by Einstein-Boltzmann solvers (CMBFAST, CAMB, CLASS…)

Stochastic theory of cosmological perturbations

Initial conditions



Canonical single-field inflation guarantees:

A. stochastic perturbations with independent Fourier modes

B. gaussian statistics for each Fourier mode / each d.o.f.

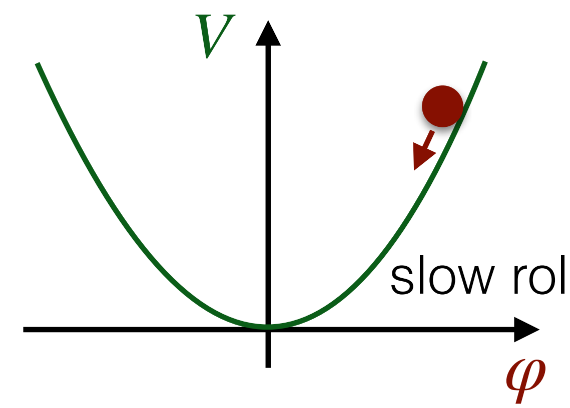
   ⇒ described by variance(wavenumber) = power spectrum

C. for each Fourier mode, all d.o.f. related to each other (fully correlated) on super-Hubble scales: “adiabatic initial conditions”

e.g. during RD: $$-2\psi = -2\phi \underset{\substack{\uparrow \\ \text{Einstein eq.}}}{=} \delta_\gamma = \delta_\nu = \frac{4}{3}\delta_\mathrm{b} = \frac{4}{3}\delta_\mathrm{c} \underset{\substack{\uparrow \\ \text{Einstein eq.}}}{=} \mathrm{constant}$$

(Comes from $A(\eta,\vec{x}) = \bar{A}(\eta + \delta\eta(\vec{x})) = \bar{A}(\eta) + \underbrace{\bar{A}'(\eta)\,\delta\eta(\vec{x})}_{\text{perturbation } \delta A(\eta,\vec{x}) \text{ in adiabatic case}}$)

Primordial power spectrum



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$\Rightarrow$ need power spectrum for single degree of freedom, e.g. curvature perturbation $\mathcal{R} \equiv \phi - \frac{a'}{a}\frac{v_{\rm tot.}}{a^2}$ in Newt. Gauge

$\Rightarrow$ Primordial spectrum: $\langle \mathscr{R}(\eta_i, \vec{k}) \mathscr{R}^*(\eta_i, \vec{k}') \rangle = \delta_D(\vec{k}' - \vec{k})\ P_{\mathscr{R}}(k)$

D. Power law, nearly scale-invariant spectrum: $P_{\mathscr{R}}(k) = \frac{2\pi^2}{k^3} A_s \left(\frac{k}{k_*}\right)^{n_s - 1}$

Transfer functions

For each Fourier mode $\vec{k}$:

- all perturbations → system of linear coupled differential equations
- adiabatic ICs → single constant of integration $\mathcal{R}(\eta_{\rm ini},\vec{k})$
- $\forall A \in \{\phi, \psi, \delta_X, \theta_X, \Theta_\ell, \dots\}$ $\quad A(\eta, \vec{k}) = T_A(\eta, k)\, \mathscr{R}(\eta_i, \vec{k})$

stochastic Fourier mode

stochastic IC

Deterministic solution of e.o.m. normalised to $\mathscr{R}=1$

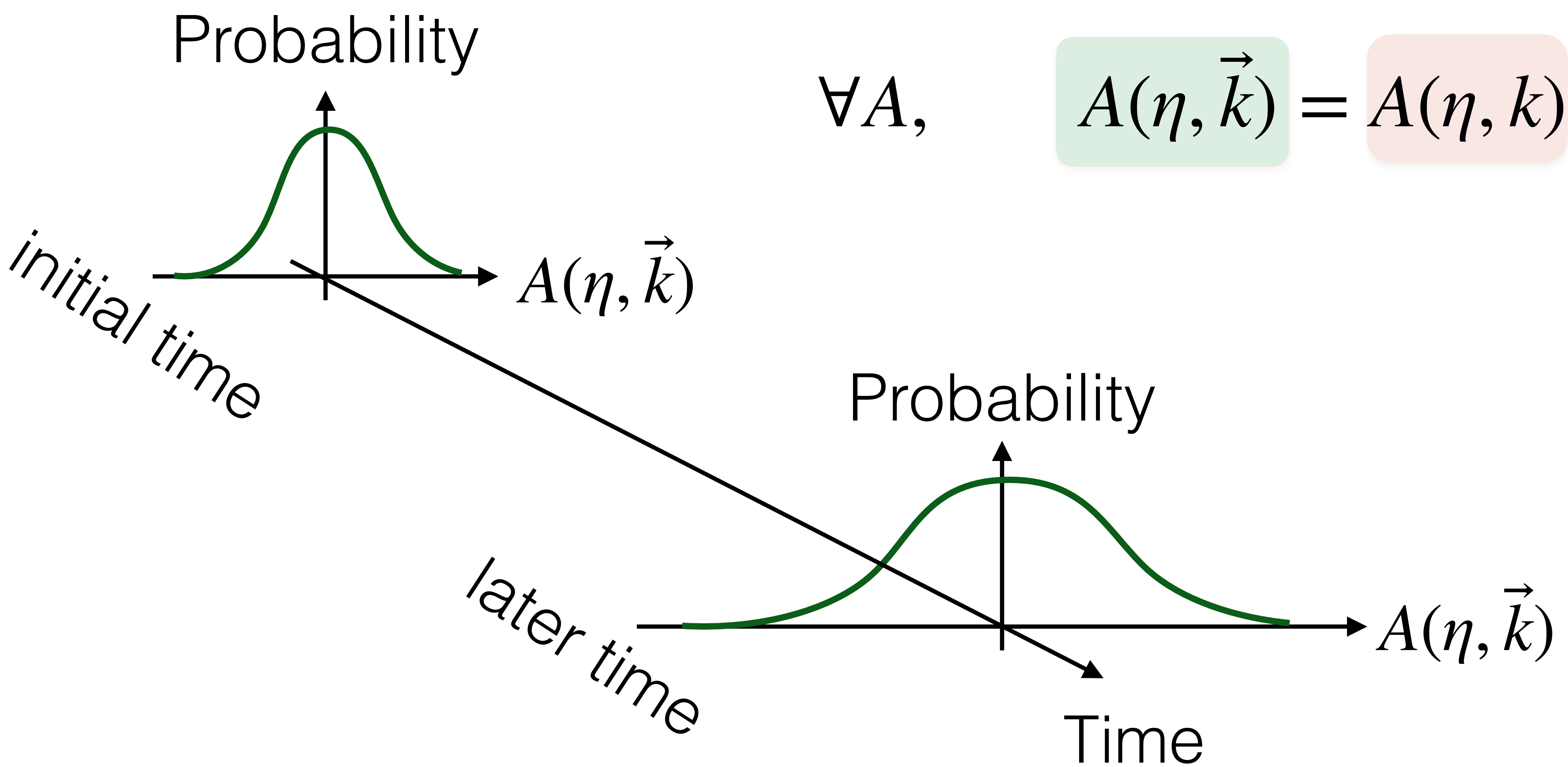
= transfer function of A

Isotropic background ⇒ depends only on k

⇒ denoted later as $A(t, k)$

Linear transport of probability

$$\forall A, \qquad A(\eta, \vec{k}) = A(\eta, k)\, \mathscr{R}(\eta_i, \vec{k})$$



Linearity of solutions ⇒ probability shape always preserved

(standard model: Gaussian)

⇒ variance evolves like square of transfer function

Power spectrum

Adiabatic initial conditions

$\Rightarrow$ for <u>any</u> perturbation at <u>any</u> time:

$$\langle A(\eta,\vec{k})\, A^*(\eta,\vec{k}')\rangle = A(\eta,k)\, A^*(\eta,k')\, \langle \mathscr{R}(\eta_\mathrm{i},\vec{k})\, \mathscr{R}^*(\eta_\mathrm{i},\vec{k}')\rangle$$

$$= \boxed{\underbrace{|A(\eta,k)|^2}_{\text{transfer function of } A}\;\underbrace{P_{\mathscr{R}}(k)}_{\text{primordial curvature spectrum}}}\;\delta_D(\vec{k}-\vec{k}')$$

transfer function of A

power spectrum $P_A(\eta,k)$ of A at η

primordial curvature spectrum

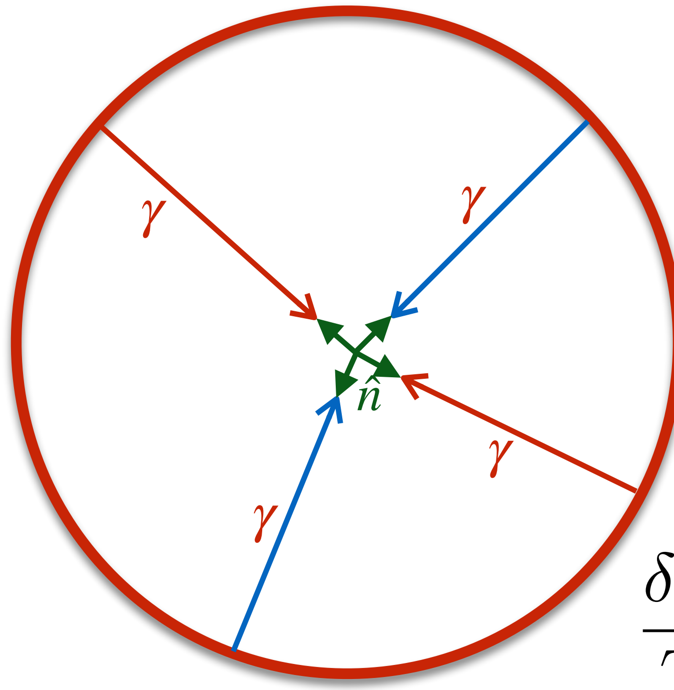
Spectrum of temperature anisotropies

Temperature multipoles

$g(\eta)$ very peaked at $\eta_{\rm dec}$

$\Downarrow$

last scattering sphere



$$\frac{\delta T}{\bar{T}}(\hat{n}) = \Theta(\eta_0, \vec{o}, -\hat{n}) = \sum_{lm} a_{lm} Y_{lm}(\hat{n})$$

inversion + Fourier + Legendre $\Rightarrow$ $a_{lm} = (-i)^l \int \frac{d^3\vec{k}}{2\pi^2} Y_{lm}(\hat{k}) \Theta_l(\eta_0, \vec{k})$

stochastic, Gaussian $\longleftarrow$ stochastic, Gaussian

correlation/variance $\Rightarrow$ $\langle a_{lm} a^*_{l'm'} \rangle = \delta^K_{ll'} \delta^K_{mm'} \left[\frac{2}{\pi} \int dk\, k^2 \Theta_l^2(\eta_0, k) \; P_{\mathscr{R}}(k) \right]$

photon transfer function

primordial spectrum

Temperature power spectrum

Defined as: $$C_l = \langle a_{lm} a^*_{lm} \rangle = \frac{2}{\pi} \int dk\, k^2 \Theta_l^2(\eta_0, k)\; P_{\mathscr{R}}(k)$$

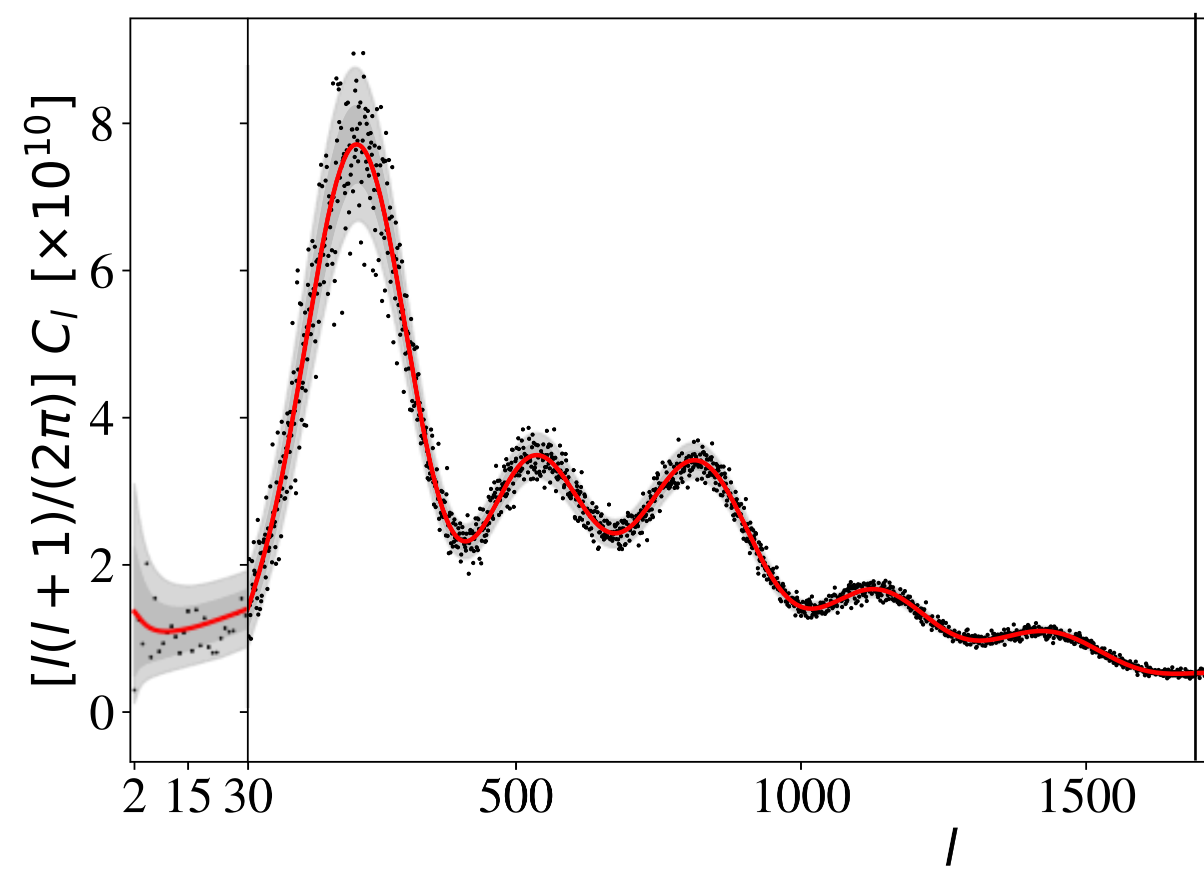
photon transfer function

primordial spectrum

theory $\longleftrightarrow$ observations

Estimator: $$\hat{C}_l(a_{lm}) \equiv \frac{1}{2l+1} \sum_{-l \leq m \leq l} |a_{lm}|^2$$

Cosmic variance: $$\langle (\hat{C}_l - C_l)^2 \rangle = \frac{2}{2l+1} C_l^2$$



Physics of temperature anisotropies

“Line-of-sight” integral in Fourier space

Boltzmann hierarchy ⇒ formal solution Zaldarriaga & Harari astro-ph/9504085:

$$\Theta_l(\eta_0,\vec{k}) = \int_{\eta_{\rm ini}}^{\eta_0} d\eta \, \{ g \, (\Theta_0 + \psi) \; j_l(k(\eta_0-\eta)) + g \, k^{-1}\theta_{\rm b} \; j_l'(k(\eta_0-\eta)) + e^{-\tau}(\phi' + \psi') \, j_l(k(\eta_0-\eta)) \}$$

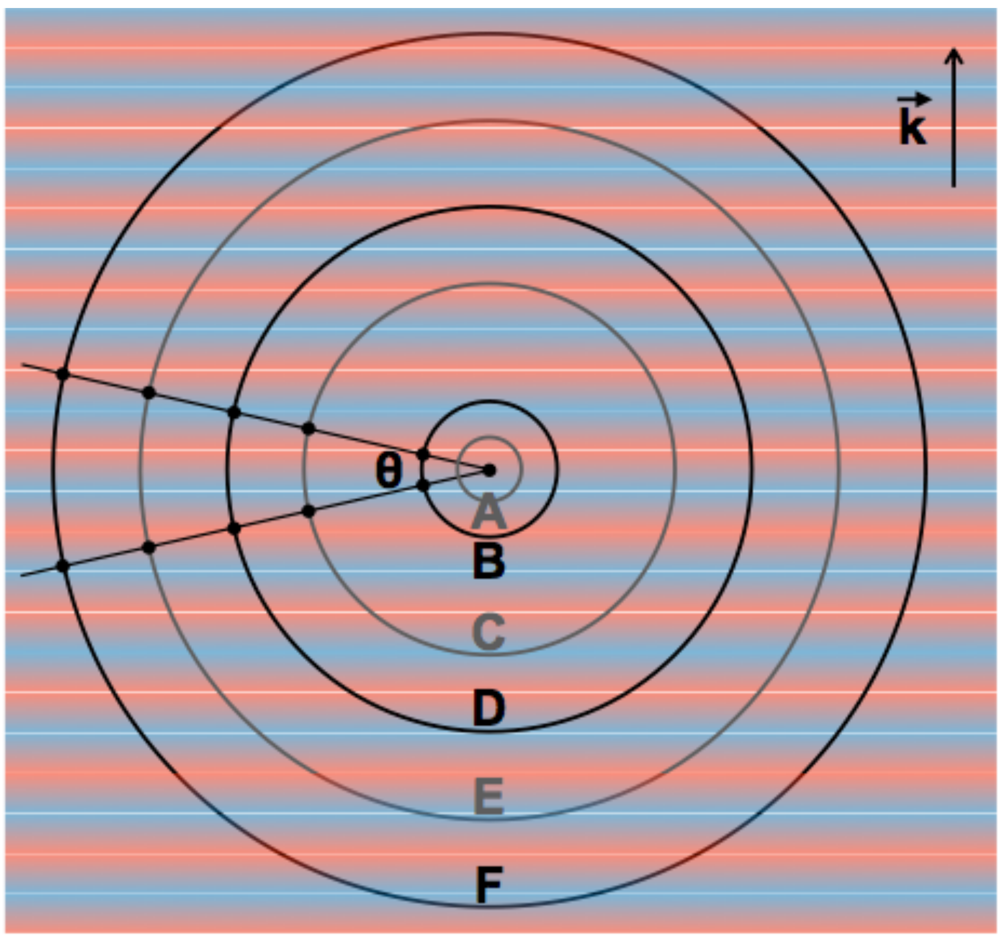
valid both for single mode $\vec{k}$ or transfer function with k

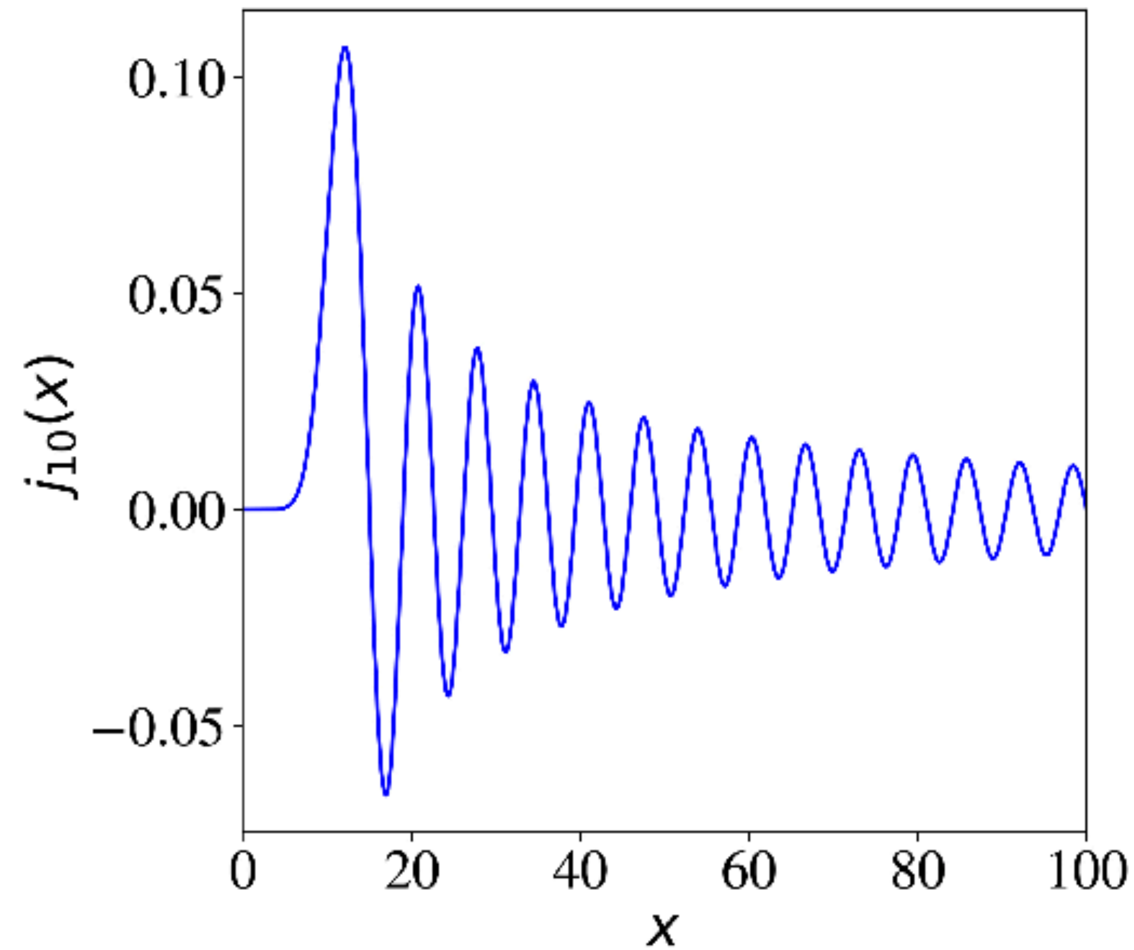
structure: $\int d\eta \; f(\eta) \; A(\eta, \vec{k}) \; j_\ell(k(\eta_0 - \eta))$

“Physical effects relevant at times described by $f(\eta)$ imprint CMB photon anisotropies described in Fourier space by $A(\eta,\vec{k})$, that project to multipole space according to $j_\ell(k(\eta_0-\eta))$”

Angular projection of Fourier modes

Role of $j_\ell(k(\eta_0 - \eta))$?





Main contribution: $\theta = \frac{\pi}{l} = \frac{\lambda/2}{d_{\mathrm{a}}} = \frac{a(\eta)\,\pi/k}{a(\eta)\,(\eta_0 - \eta)} \quad \Leftrightarrow \quad l = k(\eta_0 - \eta)$

Other contributions: harmonics