1 Survey of GW experiments and sources

1.1 Experiments: Frequency bands and sensitivities

Sensitivity plots are from [1].

1.2 Binary Systems

1.2.1 Dimensional Analysis

Will use $G = c = 1$, thus velocities are dimensionless and masses have units of distance.

$$r_s = \frac{2GM}{c^2} = 2M$$

1.2.2 Newtonian Formulas

Definitions:

$$\begin{aligned} M_T = &\quad m_1 + m_2 \quad \text{[Total mass]} \\ \eta = &\quad m_1 m_2/(m_1 + m_2)^2 \text{ [Symmetric mass ratio]} \\ \mu = &\quad \eta M_T \text{ [Reduced mass].} \end{aligned}$$

Circular Orbits:

$$\begin{aligned} |E| = &\quad \tfrac{1}{2} M_T \eta v^2 \\ \Omega^2 = &\quad M_T/a^3 \\ \Omega M_T = v^3 & \end{aligned}$$

These are Newtonian formulas, no c when putting dimensions back.

Important points:

1. LIGO & LISA require compact objects
2. There is a maximum frequency for a given binary, inversely proportional to the total mass.
3. The minimum frequency of an experiment sets the maximum mass of a system that can be detected. LIGO can only see neutron stars and black holes. LISA can observe SMBHs but not the heaviest ones.

1.2.3 GW emission

The dominant emission is quadrupole, so the emitted frequency is

$$f = 2 \times (\Omega/2\pi)$$

Quadrupole formula:

$$\begin{aligned} h \sim &\quad \ddot{Q}/d \\ \sim &\quad M_T \eta v^2/d \\ \sim &\quad M_T \eta (\Omega M_T)^{2/3}/d \\ \sim &\quad \mathcal{M}_c^{5/3} \Omega^{2/3}/d, \end{aligned}$$

with

$$\mathcal{M}_c = \eta^{3/5} M_T.$$

Emitted power (P_{GW} is dimensionless):

$$\begin{aligned} P_{GW} \sim & \quad \dot{h}^2 d^2 \\ \sim & \quad (\Omega M_T \eta v^2)^2 \\ \sim & \quad \eta^2 v^{10} \end{aligned}$$

This does not depend on the mass. Total radiated energy does depend as the timescale scales proportionally to the mass. Notice that you are emitting a fixed fraction of the rest mass in a timescale proportional to $1/M$. This is the maximum luminosity one can expect.

Exercise: Compare the GW luminosity with the Eddington Luminosity. Compare the order of magnitude and the scaling with physical constants and parameters including the mass. Repeat for stellar luminosity.

Energy loss timescale:

$$\begin{aligned} t_{GW} \sim & \quad E/P_{GW} \\ \sim & \quad M_T \eta^{-1} v^{-8} \\ \sim & \quad \mathcal{M}_c^{-5/3} \Omega^{-8/3} \end{aligned}$$

up to a constant gives how things are evolving. Another useful concept is the relative change of quantities during one orbit: $\dot{Q}/QP$, where Q is any quantity and P is the period.

$$\begin{aligned} \dot{E}/E \times P \sim & \quad P_{GW}/E \times P \sim P_{GW}/\Omega E \\ \sim & \quad \eta^2 v^{10}/\eta v^5 \\ \sim & \quad \eta v^5 \end{aligned}$$

A useful dimensionless is the relative perior change in one period:

$$\dot{P}/P \times P = \dot{P} = 3/2 \ \dot{E}/E \times P \sim \eta v^5 \sim \eta (M_T \Omega)^{5/3}$$

with all the factors:

$$\dot{P} = -(2\pi)\frac{96}{5}(\mathcal{M}_c \Omega)^{5/3}$$

$\dot{P} \sim 1$ period changes by order one in one period. This happens at merger.

Exercise: Consider the Binary pulsar. Derive the functional form for the cumulative shift of the periastron. Compute $\dot{P}$ using the formulas for cirular binaries and compare with the measured value.

1.2.4 Frequency evolution

$$\begin{aligned} \dot{\Omega}/\Omega \sim & \quad P_{GW}/E \sim \eta v^8 / M_T \\ \sim & \quad \eta (\Omega M_T)^{8/3}/M_T, \\ \sim & \quad \mathcal{M}_c^{5/8} \Omega^{8/3}/M_T \end{aligned}$$

integrating in time:

$$\Omega \sim \mathcal{M}_c^{-5/8} \tau^{-3/8}$$

Number of GW cycles until merger:

$$N = \int f d\tau \sim (\mathcal{M}_c/\tau)^{-5/8} \sim (\mathcal{M}_c f)^{-5/3}$$

1.2.5 Time vs Frequency domain

In the time domain:

$$h(t) \sim h_0(t) \cos(\Phi(t)),$$

$h_0(t)$ changes slowly, on a timescale t_{GW} while $\Phi(t) = 2\Omega t$ and:

$$h_0(t) \sim \mathcal{M}_c^{5/3} \Omega^{2/3}/d$$

We are also interested in the Fourier transform of the strain $h(f)$,

$$h(f) = \int dt e^{-i2\pi f t} h(t),$$

$h(f)$ has units of time.

Here it would make a difference whether the events is chirping during the observed period or the frequency is constant.

Constant frequency

$$\begin{aligned} h(t) \sim &\quad h_0 e^{i2\pi f_{GW} t} \\ h(f) \sim &\quad \delta(f - f_{GW}) h_0 \\ \sim &\quad T \times h_0 \text{ [over freq. width } \sim 1/T] \end{aligned}$$

Chirping signal:

$$h(t) \sim h_0(t) e^{i\Phi(t)}$$

Integrate using stationary phase approximation. Contribution peaks where $\dot{\Phi}|_{t_f} = f$. Expand in phase around that point:

$$\begin{aligned} \dot{\Phi}(t) \approx &\quad \Phi_0 + f(t - t_f) + 1/2\dot{f}(t - t_f)^2 + \dots \\ \approx &\quad \Phi_0 + f(t - t_f) + 1/2\dot{P} f^2 (t - t_f)^2 + \dots, \end{aligned}$$

so that:

$$\begin{aligned} h(f) \sim &\quad \tfrac{h_0}{\dot{P}^{1/2} f} \\ \sim &\quad \tfrac{M_T \eta (\Omega M_T)^{2/3}/d}{(\mathcal{M}_c \Omega)^{5/6} f} \\ \sim &\quad \mathcal{M}_c^{5/6} f^{-7/6}/d \\ \sim &\quad (f\mathcal{M}_c)^{5/6}/(f^2 d) \end{aligned}$$

1.2.6 SNR formulas

$$SNR^2 = 4\int \frac{|h(f)|^2}{S(f)}df = 4\int \frac{|fh(f)|^2}{fS(f)}\frac{df}{f} \equiv 4\int \frac{|h_c(f)|^2}{fS(f)}\frac{df}{f}$$

For monochromatic signal:

$$SNR^2 \sim \frac{|h_0|^2 fT}{fS(f)}$$

For Chirping signal:

$$\frac{dSNR^2}{d\ln f} \sim \frac{|h_0|^2/\dot{P}}{fS(f)} \sim \frac{|h_0|^2 f t_{GW}}{fS(f)}$$

1.3 Sources in different frequency bands

Simple worked out example for LIGO (GW150914) [2]. Original binary pulsar [3]. Binary WD [4].Systematic search for LISA sources (ZTF) [5].

1.4 Cosmological backgrounds

Assume there is a process that create Gravitational waves in the early Universe when the temperature was T. At that time the energy density in the GW was Ω_w, it is interesting to compute the energy density and frequency of those GW today. Formulas from [6].

Assume you have a stochastic background of GW so that

$$\langle h^*_{p_1}(f_1)h^*_{p_2}(f_2)\rangle = \frac{1}{2}\delta_{p_1,p_2}\delta^D(f_1 - f_2)S_h(f_1)$$

Energy density is given by:

$$\begin{aligned}\rho = & \tfrac{1}{16\pi}\int df(2\pi f)^2|h(f)|^2 \\ = & \tfrac{\pi}{4}\int d\ln f\ f^2\ fS_h(f) = \tfrac{\pi}{4}\int d\ln f\ f^2\ h_c^2\end{aligned}$$

So the energy density:

$$\Omega_{GW}(f) = \quad \tfrac{\pi}{4}\,\tfrac{f^2 h_c^2}{\rho_c}$$

Estimate for frequency:

$$\begin{aligned}H^2(T)M_{pl}^2 &= g_\star T^4 \\ f = & \ xH(T)a_0/a \\ a/a_0 = & \ (S_0/S)^{1/3} = (g_{\star 0}/g_\star(T))^{1/3}T_0/T \\ f \sim & \ xTT_0/M_{pl}(g_\star(T)/g_{\star 0})^{1/6}\end{aligned}$$

$$\Omega_{GW}/\Omega_r = \quad \tfrac{\rho_p}{\rho_r}(a_0/a_p)^4(\tfrac{1}{\rho_p}\tfrac{d\rho_{GW}}{d\ln f})_{t_p}$$

$$\rho_p/\rho_r(a_0/a_p)^4 = \quad (g_{\star 0}/g_\star(T))^{-1/3}$$

Putting some numbers:

$$
\begin{aligned}
f = & \quad 2.6 \times 10^{-8}\text{Hz } x \times (\tfrac{g_\star(T_p)}{100})^{1/6} \tfrac{T}{\text{GeV}} \\
h^2\Omega_{GW}(f) = & \quad 1.6 \times 10^{-5} (\tfrac{g_\star(T_p)}{100})^{-1/3} (\tfrac{1}{\rho_p} \tfrac{d\rho_{GW}}{d\ln f})_{t_p}
\end{aligned}
$$

2 Detecting GW

Typical change in length induced by GW:

$$\Delta L \sim L \times h \sim 10^{-18}\mathrm{m} \times (\frac{L}{\mathrm{km}}) \times (\frac{h_0}{10^{-21}}) \sim 10^{-3}\mathrm{fm} \times (\frac{L}{\mathrm{km}}) \times (\frac{h_0}{10^{-21}})$$

Notice that h is a dimensionless number, so you are trying to measure a 10^{-21} effect. Electron magnetic moment, one part in ten trillion 10^{-13}. Frequencies of record setting clocks 10^{-19}.

Wavelength of GW and laser:

$$\begin{aligned} \lambda_{GW} = &\quad c/f_{GW} = 3 \times 10^3\mathrm{km} \times (\tfrac{100\mathrm{Hz}}{f_{GW}}) \\ \lambda_{Laser} = &\quad 1064\mathrm{nm} \end{aligned}$$

So that:

$$\frac{\Delta L}{\lambda_{Laser}} \sim 10^{-12} \times (\frac{L}{\mathrm{km}}) \times (\frac{h_0}{10^{-21}}) \times (\frac{1064\mathrm{nm}}{\lambda_{Laser}})$$

2.1 Time delay

Redshift induced by GW:

$$Z(t) = \frac{\nu(t) - \nu_0}{\nu_0} = \frac{1}{2}\hat{p}^i\hat{p}^j \int_{t_0}^{t} dt' \frac{\partial h_{ij}(t', x')}{\partial t'},$$

which leads to:

$$Z(t) = \frac{1}{2}\frac{\hat{p}^i\hat{p}^j}{(1 - k \cdot p)}\Delta h_{ij},$$

In a round trip:

$$\frac{\nu_2 - \nu_0}{\nu_0} = \frac{\nu_2 - \nu_1}{\nu_0} + \frac{\nu_1 - \nu_0}{\nu_0} = \frac{1}{2}\frac{\hat{p}^i\hat{p}^j}{(1 + k \cdot p)}\Delta h_{ij}|_{21} + \frac{1}{2}\frac{\hat{p}^i\hat{p}^j}{(1 - k \cdot p)}\Delta h_{ij}|_{10}$$

To get a phase shift (or the associated time delay) we can simply integrate Z which oscillates with frequency f_{GW},

$$\delta\phi = 2\pi \int \nu dt = 2\pi\nu_0 \int Z dt,$$

Exercise Do the integral in the case $k \cdot p = 0$:

$$t_2 - t_0 = \int Z dt = \frac{2L}{c} + \frac{L}{c}h(t_0 + L/c)\frac{\sin(2\pi f_{GW} L/c)}{2\pi f_{GW} L/c}.$$

If you are feeling brave do the general case and find the angular response.

2.2 Michelson Interferemeter:

See more information in [7, 8].

$$E_{refl} = \quad [t_{BS} r_x t_{BS} e^{2ikL_x} + r_{BS} r_y r_{BS} e^{2ikL_y}] E_{in}$$

$$E_{as} = \quad [-r_{BS} r_x t_{BS} e^{2ikL_x} + t_{BS} r_y r_{BS} e^{2ikL_y}] E_{in}$$

If for example $r_{BS}^2 = 1/2$ and $r_x \sim r_y \sim 1$:

$$E_{refl} = \quad [e^{2ikL_x} + e^{2ikL_y}]/2 E_{in} = e^{\bar{\phi}} \cos(\delta\phi) E_{in}$$

$$E_{as} = \quad [-e^{2ikL_x} + e^{2ikL_y}]/2 E_{in} = -e^{\bar{\phi}} i \sin(\delta\phi) E_{in}$$

with $\bar{\phi} = 2k(L_x + L_y)/2$ and $\delta\phi = 2k(L_x - L_y)/2$. The power is given by:

$$P_{refl} = \quad \cos^2(\delta\phi) P_{in}$$

$$P_{as} = \quad \sin^2(\delta\phi) P_{in}$$

Exercises:

- Consider the case when $\Delta L = L_x - L_y = \Delta L(f) e^{-2\pi f t}$. Calculate the response in P_{refl} and P_{as}.
- Assume that the laser input to the interferometer is not perfect, so either the amplitude or the frequency of the laser fluctuate with frequency f. Calculate the response in P_{refl} and P_{as}. Why is the asymmetric port used to detect GW rather than the reflected power?

2.3 Fabry-Perot Cavity:

Now replace mirrors with Fabry-Perot Cavity. The reflected electric field and the circulating field are given by:

$$E_{refl} = \quad \frac{-r_i + (r_i^2 + t_i^2) r_e e^{2ikL}}{1 - r_i r_e e^{2ikL}} E_{in} \equiv r_{FP}(L) E_{in}$$

$$E_{circ} = \quad \frac{t_i}{1 - r_i r_e e^{2ikL}} E_{in}$$

Assume you are are resonance $e^{2ikL} = e^{2n\pi + \delta\phi}$ and $(r_i^2 + t_i^2) = 1$,

$$r_{FP} = \frac{-r_i + r_e e^{i\delta\phi}}{1 - r_i r_e e^{i\delta\phi}} \approx \frac{-r_i + r_e}{1 - r_i r_e} + i \frac{r_e t_i^2}{(1 - r_i r_e)^2} \delta\phi,$$

for $r_e \approx 1$

$$r_{FP} = 1 + i \frac{1 + r_i}{1 - r_i} \delta\phi \approx e^{i \frac{1+r_i}{1-r_i} \delta\phi} \approx e^{iG\delta\phi}.$$

For $r_i \sim 1$ then $G >> 1$. Now r_x and r_y of the Michelson interferometer are replaced by a phase so effectively it is like a much longer single arm Michelson interpherometer. However notice that when G is large also the circulating power becomes large.

$$P_{circ} = \frac{t_i^2}{(1-r_i)^2} P_{in} = \frac{(1+r_i)}{(1-r_i)} P_{in}$$

Notice that the average number travel around the Fabry Perot is (Exercise derive it):

$$\langle N \rangle = r_i^2/(1-r_i^2).$$

What happens is $\delta\phi$ is $\delta\phi(t)$ with frequency f_{GW}? Now the reflected field will have side bands with frequencies $f_{Laser} \pm f_{GW}$. The corresponding reflection coefficient will be:

$$r_{FP}(f_{GW}) = e^{iGC(f_{GW})\delta\phi(f_{GW})}.$$

where we defined a frequency response $C(f_{GW})$:

$$C(f_{GW}) = \frac{1-r_i}{1-r_i\exp(i4\pi f_{GW}L)},$$

so that:

$$|C|^2 = \frac{(1-r_i)^2}{1+r_i^2-2r_i\cos\alpha} \approx \frac{1}{1+(f_{GW}/f_\star)^2},$$

with $\alpha = 4\pi f_{GW} L$

$$f_\star = \frac{(1-r_i)}{4\pi L}.$$

Because $G >> 1$, the frequency where you start to notice a decrease in sensitivity is correspondingly lower.

One convenient way to derive $C(f)$ is to compute E_{refl} as the sum of rays that went around several number of times and use that the overall shift of each contribution is the sum of many the contributions in each loop and take into account that the contribution is time dependent. To make contact with GW, consider a varying time needing to make a loop. So if you did N loops the total time

$$\Delta t_N = \sum_i^N \delta t_i,$$

with $\delta t_i = 2L/c + \delta t_i^{GW}$ with $\delta t_i^{GW} \propto e^{i2\pi f_{GW} t}$. One can write:

$$\begin{aligned} E_{refl}(t) = &\quad -r_i E_I(t) + t_i r_e t_i E_I(t-\Delta t_1) + t_i r_e r_i r_e t_i E_I(t-\Delta t_2) + \cdots \\ = &\quad -r_i E_I(t) + t_i^2 r_e \textstyle\sum_{n=1}^{\infty} (r_e r_i)^{n-1} E_I(t-\Delta t_n). \end{aligned}$$

Exercise: Do the sum and find the response to a frequency f_{GW}.

For LIGO $L \sim 4$km and $G \sim 250$ so $L_{eff} \sim 10^3$km. Still,

$$\frac{\Delta L}{\lambda_{Laser}} \sim 10^{-9} \times (\frac{L}{10^3\text{km}}) \times (\frac{h_0}{10^{-21}}) \times (\frac{1064\text{nm}}{\lambda_{Laser}}),$$

one needs a very precise measurement.

2.4 Shot noise

$$\begin{aligned} P_{as} = & \quad P_{in}\sin^2(\phi) \\ \frac{P_{as}}{\Delta x}(f_{GW}) = & \quad kP_{in}\sin(2\phi) \end{aligned}$$

Analogy with cosmology, signal is density of photons (density in time or rate):

$$\begin{aligned} \mu = & \quad \text{Number of photons per unit time} \\ \mu = & \quad \bar{\mu} + \delta\mu_{GW} + \delta\mu_{noise} \\ \bar{\mu} = & \quad (P_{in}/h\nu)\sin^2(\phi) \\ \delta\mu_{GW} = & \quad (P_{in}/h\nu)\sin(2\phi)k\Delta x \\ \langle \delta\mu^*_{noise}\delta\mu_{noise}\rangle = & \quad \delta^D(f_1 - f_2)\bar{\mu} \end{aligned}$$

So that:

$$SNR^2 = \frac{fT((P_{in}/h\nu)\sin(2\phi)k\Delta x)^2}{f(P_{in}/h\nu)\sin^2(\phi)}$$

or equivalently:

$$SNR = (T(P_{in}/h\nu))^{1/2}2k\Delta x|\cos(\phi)| = (N_{photons})^{1/2}2k\Delta x|\cos(\phi)|.$$

Basically what get you the remaining orders of magnitude is the large number of photons dues to the large power of the laser:

$$SNR = (N_{photons})^{1/2} \times \frac{4\pi L h}{\lambda} \times |\cos(\phi)|.$$

For LIGO $\dot{N} \sim 4.3 \times 10^{21}$Hz. Notice best signal to noise comes when $\cos(\phi) = 1$, which corresponds to a dark asymmetric port.

Equivalently the PSD is given by:

$$PSD_{shot} = (\frac{\lambda}{4\pi L|\cos(\phi)|})^2 \frac{1}{P_{in}/h\nu}$$

If we now consider the Fabry-Perot cavity, the same GW ampliude produces a much larger signal but the shot noise fluctuations are the same so:

$$PSD_{shot} \sim (\frac{\lambda}{4\pi LG})^2 \frac{1}{P_{in}/h\nu}$$

2.5 Radiation pressure

Consider the radiation pressure force on the mirrors, $F = 2P/c$ where P is the incident power. If the power fluctuates the mirrors experience a fluctuating force, which leads to a displacement and thus a signal. As we increase the power to make the shot noise smaller we increase the size of this fluctuating force. The spectrum of the force is then:

$$PSD_{Force} = (\frac{2}{c})^2 PSD_{Power}$$

But we have:

$$PSD_{Power} = (h\nu)^2 \bar{\mu}$$

On the other hand the fluctuating force at frequency f leads to a displacement

$$(2\pi f)^2 M \delta x = \delta F,$$

where M is the mass of the mirror. Thus the PSD for the measurement of h induced by this displacement is

$$PSD_{RP} = \frac{PSD_{Force}}{(2\pi f)^4 M^2 L^2} = \frac{(2/c)^2 P_{in}(h\nu)}{(2\pi f)^4 M^2 L^2}$$

If we now consider the Fabry-Perot cavity, although the GW response is larger by a factor of G so is the response to a δx caused from the radiation pressure. So that factor cancels. However in addition, the power in the cavity is amplified with respect to the power in beam splitter by a factor G. So fluctuations in the beam splitter power are amplified by a factor of G in the cavity. Thus the PSD of the force fluctuations are up by a factor of G^2. In total:

$$PSD_{RP} \sim \frac{(2/c)^2 P_{in}(h\nu) G^2}{(2\pi f)^4 M^2 L^2}$$

Notice that

$$PSD_{RP} \propto P_{in} G^2 \qquad PSD_{shot} \propto \frac{1}{P_{in} G^2}$$

So there is an optimum laser power, which depends on the frequency. In fact notice that:

$$L^4 \times PSD_{RP} \times PSD_{shot} = \frac{\hbar^2}{M^2 (2\pi f)^4}$$

2.6 Quantum Limit

You are measuring a position x on intervals of time Δt in order to compute the Fourier components of the displacement. Measurements at different times are independent. The power spectrum of the shot noise in the x measurement is given by:

$$P_{\delta x} = \sigma_x^2 \Delta t$$

A measurement with uncertainty σ_x results in a minimum uncertainty in the momentum $\sigma_x \sigma_p \sim \hbar$. This momentum noise leads to a force, $\delta F \sim \delta p / \Delta t$. So that

$$\sigma_F^2 \Delta t = \frac{\hbar^2}{\sigma_x^2 \Delta t},$$

both x and F have white noise power spectrum. Equivalently:

$$P_{\delta F} = \frac{\hbar^2}{P_{\delta x}}$$

The equation of motion for the mass is

$$M\delta\ddot{x}_F = \delta F,$$

so the power spectra of the induced δx

$$P_{\delta x_F} = \frac{1}{M^2(2\pi f)^4}\frac{\hbar^2}{P_{\delta x}},$$

or

$$P_{\delta x_F}P_{\delta x} = \frac{\hbar^2}{M^2(2\pi f)^4}.$$

The total power

$$P_{total} = P_{\delta x} + \frac{\hbar^2}{M^2(2\pi f)^4}\frac{1}{P_{\delta x}},$$

so that:

$$P^{min}_{total} = \frac{2\hbar}{M(2\pi f)^2}$$

Important: This is only happening because we are using a position measurement, which induces a momentum change which then leads to a position change through the equations of motion. If we measure p, the induced uncertainty in the position does not feed back to the momentum through the equations of motion. So there would not be an effect analog to the one presented here. So in a sense this limit is only fundamental for this particular meausrement technique.

3 Pulsar timing arrays

Additional information [9, 10, 11].

3.1 Individual sources

Amplitude of time domain signal:

$$h_0 = 2\mathcal{M}_c^{5/3}\Omega^{2/3}/d = 1.3\times 10^{-15}\ (\frac{\mathcal{M}_c}{10^9 M_\odot})^{5/3}(\frac{f_{GW}}{6\mathrm{nHz}})^{2/3}(\frac{150\mathrm{Mpc}}{d}).$$

To determine the strain at a detector one needs to include geometric factors related to the inclination of the system in the plane of the sky and the direction of observation relative to the detector.

The period is changing very slowly:

$$-\frac{P}{\dot{P}} = 4.6\times 10^5\mathrm{yrs}\ (\frac{\mathcal{M}_c}{10^9 M_\odot})^{-5/3}(\frac{f_{GW}}{6\mathrm{nHz}})^{-8/3}$$

3.2 Stochastic Background

Assume you have a certain density of sources per comoving volume emitting GW. Want to compute the energy density in GW today. We have:

$$\frac{d\rho_{GW}}{d\ln f} = \int dz \frac{dn_{comoving}}{dz}\frac{1}{1+z}\frac{dE}{d\ln f}|_{emitted}$$

For the emitted energy:

$$\frac{dE}{d\ln f} = f\frac{\dot{E}}{\dot{f}} = \frac{2}{3}E \propto M_T\eta v^2 \propto \mathcal{M}_c^{5/3} f_{GW}^{2/3}$$

Notice that the observed frequency is redshifted. Putting all the constants etc:

$$\frac{d\rho_{GW}}{d\ln f} = \frac{\pi^{2/3}}{3}\frac{(G\mathcal{M}_c)^{5/3}}{G}\int dz \frac{dn_{comoving}}{dz}\frac{1}{(1+z)^{1/3}}$$

We can connect this to the amplitude of the GW background as follows:

$$\rho_{GW} = \frac{c^2}{16\pi G}(\dot{h}_+^2 + \dot{h}_\times^2).$$

For a stochastic background

$$\langle h_P^* h_{P\prime}\rangle = \delta^D(f - f\prime)\ \delta_{PP\prime}\ S(f)/2$$

so that

$$\langle\rho_{GW}\rangle = \frac{\pi c^2}{4G}\int d\ln f\ f^2 h_c^2,$$

with $h_c^2 = fS(f)$. So

$$h_c^2 = \frac{4}{3\pi^{1/3}c^2}\frac{(G\mathcal{M}_c)^{5/3}}{f^{4/3}}n_0\langle(1+z)^{-1/3}\rangle$$

An estimate gives:

$$h_c = 1.35 \times 10^{-16}(\frac{\mathcal{M}_c}{10^8 M_\odot})^{5/3} \times (\frac{1\text{yr}^{-1}}{f_{GW}})^{4/3} \times \frac{n_0}{1\ \text{Mpc}^{-3}} \times (\frac{\langle(1+z)^{-1/3}\rangle}{0.7})$$

Exercise: Assume there is a Poisson distribution of sources and that each source emits gravitational waves at a fixed frequency during the observational window. Use the formula for $h(f)$ applicable in this case to calculate the power spectrum and $h_c^2(f)$ (This should be an analogous calculation to the one halo term in a halo model calculation of the density power spectrum). Show that $h_c^2(f) \propto \int dz f|h_0|^2 \bar{n} dV/dzdt/df$. Here dt/df gives the time a source spends emitting at frequency f. Note that $|h_0|^2 dt/df$ is also the square of the strain for a chirping signal, $\propto \mathcal{M}_c^{5/3} f^{-7/3}$.

3.3 Signal

Following conventions of [9].

Gravitational waves produce an oscillating change in frequency of the arrival of the pulses as well as its associated delay in the arrival times:

$$\begin{aligned} \Delta\nu/\nu \equiv & \quad Z \\ r = & \quad \textstyle\int_0^t Z dt \end{aligned}$$

with:

$$Z_{GW}(t) = \frac{1}{2}\frac{\hat{p}^i\hat{p}^j}{(1+k\cdot p)}\Delta h_{ij} = Z_{GW}^{earth} + Z_{GW}^{pulsar},$$

Z_{GW}^{earth} correlates the signal at each pulsar while Z_{GW}^{pulsar} are uncorrelated.

We can think of Z_{GW}^{earth} as a scalar quantity in the sky, whose value depends on the frequency and the direction in the sky. It is a complex number encoding amplitude and phase. For a single source in the z with the axis of the orbit oriented along the coordinate axis:

$$z(\theta,\phi) = \frac{1}{2}(1+cos\theta)(\cos 2\phi h_+ - \sin 2\phi h_x).$$

If we are interested in the correlations on the sky produced by the GW background we can expand the map of z in spherical harmonics and compute the C_l for one source (or GW coming from a single direction). The C_l are a scalar, so we can then superimpose the C_l for all sources. The result is:

$$C_l = \frac{2\pi(h_+^2 + h_x^2)}{(l+2)(l+1)(l)(l-1)} \quad (l \geq 2)$$

The Courier transform of this, the correlation function is called the Hellings & Downs curve:

$$\begin{aligned} C(\theta) = &\ \textstyle\sum_l (\frac{2l+1)}{4\pi} C_l Pl(\cos\theta) \\ C_l = &\ \frac{6\pi}{(l+2)(l+1)(l)(l-1)} \quad (l \geq 2) \\ C(\theta) = &\ \tfrac{1}{2}\{1 + \tfrac{3}{2}(1-\cos\theta)[\ln(\tfrac{1-\cos\theta}{2}) - \tfrac{1}{6}]\} \end{aligned}$$

Model used in the Nonograv paper for the covariance matrix of the timing residuals:

$$\begin{aligned} \langle c_{ai} c_{nj} \rangle = &\ \delta_{ij}(\delta_{ab}\phi_{ai} + \Phi_{ab,i}) \\ \phi_{ai} = &\ \frac{A_i^2}{12\pi^2 f_{ref}^3 T}(\frac{f_i}{f_{ref}})^{-\gamma_a} \\ \Phi_{ab,i} = &\ \Phi_{HD,i}[C(\theta_{ab}) + \delta_{ab}/2] \\ \Phi_{HD,i} = &\ \frac{A_{HD}^2}{12\pi^2 f_{ref}^3 T}(\frac{f_i}{f_{ref}})^{-\gamma_{HD}} \end{aligned}$$

with a, b running over pulsars and i, j running over frequencies. The δ_{ab} piece of $\Phi_{ab,i}$ comes from the pulsar term which is uncorrelated between pulsars and ϕ_{ai} represents an intrinsic red noise contribution free for each pulsar.

Take

$$h_c^2 = A_{HD}^2 (\frac{f_i}{f_{ref}})^{-4/3}.$$

We want to compute the power spectrum of

$$r = Z/(i2\pi f),$$

then

$$\langle r_{ai} r_{bj} \rangle = \frac{\delta_{ij}}{T} \frac{1}{3(2\pi)^2 f^3} h_c^2 C(\theta_{ab})$$

where the $(2\pi f)^2$ comes from the conversion from Z to time delay residual, a factor of 3 comes from the convention in the Hellings & Downs curve and the extra factor of f compes from the definition $h_c^2 = fS(f)$. In this convention the Fourier transform to get r_{ai} is done doing the integral dt/T so that r_{ai} has units of time. Because of this convention one has a prefactor $\delta^D(f_1 - f_j)/T^2$ wich is the same as δ_{ij}/T.

In total we get:

$$\langle r_{ai} r_{bj}^* \rangle = \frac{\delta_{ij}}{T} \frac{A_{HD}^2}{12\pi^2 f_{ref}^3} (\frac{f_i}{f_{ref}})^{-4/3-3} C(\theta_{ab}),$$

which explains the conventions in the Nanograv paper. So the expectation is a slope $13/3 = 4/3 + 3$.

4 B modes

4.1 Thomson Scattering

Linear polarization is described by two Stokes parameters Q and U. Thomson scattering produces linearly polarized light if the incident radiation has a quadrupole anisotropy.

$$Q + iU \propto \int d\hat{n}' (\hat{m} \cdot \hat{n}')^2 T(n'),$$

where $\hat{m} = \hat{e}_1 + i\hat{e}_2$ and $(\hat{e}_1, \hat{e}_2, \hat{n})$ are orthogonal. Equivalently

$$\begin{array}{rl} Q \propto & \int d\hat{n}' \; [(\hat{e}_1 \cdot \hat{n}')^2 - (\hat{e}_2 \cdot \hat{n}')^2] T(n') \\ U \propto & \int d\hat{n}' \; [2(\hat{e}_1 \cdot \hat{n}')(\hat{e}_2 \cdot \hat{n}')] T(n'), \end{array}$$

4.2 Anisotropies

Consider a density fluctuation or a gravitational wave with $\hat{k} = \hat{z}$. The temperature anisotropy created as a photon travels between the last two scatterings is given by

$$\frac{\delta T}{T} = \frac{\delta \nu}{\nu}$$

Consider a photon moving along direction $\hat{n}'$, the frequency change is due to the Doppler shift

$$\frac{\delta \nu}{\nu} = \hat{n}'_i [v_i(x_0 + \hat{n}'_i \lambda_T) - v_i(x_0)] = \hat{n}'_i \hat{n}'_j \lambda_T \partial_i v_j$$

which implies:

$$Q + iU \propto \lambda_T \hat{m}_i \hat{m}_j \partial_i v_j \propto (\hat{m} \cdot \hat{k})^2 k \lambda_T v \cos^2 \theta' k \lambda_T v.$$

Equivalently:

$$\begin{array}{rl} Q \propto & \sin^2 \theta \; k \lambda_T v \\ U = & 0, \end{array}$$

On the other hand, for tensor modes:

$$\frac{\delta \nu}{\nu} = \frac{1}{2} \lambda_T \hat{n}'_i n'_j \dot{h}_{ij},$$

so that:

$$\begin{array}{rl} Q \propto & \lambda_T [\hat{e}_{1i} \hat{e}_{1j} - \hat{e}_{2i} \hat{e}_{2j}] \dot{h}_{ij} \\ U \propto & \lambda_T [\hat{e}_{1i} \hat{e}_{2j} + \hat{e}_{2i} \hat{e}_{1j}] \dot{h}_{ij}. \end{array}$$

It is best to define:

$$h^{\pm} = h_+ \pm i h_\times = [(\hat{x}_i \hat{x}_j - \hat{y}_i \hat{y}_j) \pm i(\hat{x}_i \hat{y}_j + \hat{y}_i . \hat{x}_j)] h$$

For these modes we get:

$$
\begin{aligned}
Q \propto &\quad (1+\cos^2\theta)e^{\pm i2\phi}h^{\pm} \\
U \propto &\quad \pm 2i\cos\theta e^{\pm i2\phi}h^{\pm}.
\end{aligned}
$$

U in this coordinate system is not zero, so there are B modes. U brakes parity, but sign is opposite for $h^{\pm}$. Parity symmetry is imposed by demanding $h^{\pm}$ have the same power. Can have non-zero B modes in any given realization.

Also keep in mind that v and $h^{\pm}$ are proportional to e^{ikz}.

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