

# Lecture 2

Pressureless perfect fluid  $\rightarrow$  fluctuations  $\rightarrow$  EOM in Fourier space.  
( $\delta, \theta$ )

Perturbative solution:

( $\Omega_m = 1$ , EdS approximation)  $D_+(t) = a$

$$\delta(\vec{k}, \tau) = \sum_{n=1}^{\infty} D_+^{(n)}(\tau) \cdot \delta^{(n)}(\vec{k})$$

$$\theta(k, \tau) = -\gamma_k(\tau) \sum_{n=0}^{\infty} D_{+}^n(\tau) \cdot \tilde{\theta}^{(n)}(k)$$

$$\delta^{(n)}(\mathbf{k}) = \prod_{m=1}^n \left\{ \int \frac{d^3 q_m}{(2\pi)^3} \delta^{(1)}(\mathbf{q}_m) \right\} F_n(\mathbf{q}_1, \dots, \mathbf{q}_n) (2\pi)^3 \delta^{(\text{D})}(\mathbf{k} - \mathbf{q}_{\perp}^n)$$

$$\tilde{\theta}^{(n)}(\mathbf{k}) = \prod_{m=1}^n \left\{ \int \frac{d^3 q_m}{(2\pi)^3} \delta^{(1)}(\mathbf{q}_m) \right\} G_n(\mathbf{q}_1, \dots, \mathbf{q}_n) (2\pi)^3 \delta^{(\text{D})}(\mathbf{k} - \mathbf{q}_1^n)$$

$$\downarrow$$

$$\vec{q}_a^b \equiv \vec{q}_a + \vec{q}_{a+1} + \dots + \vec{q}_b$$

$$F_n(\mathbf{q}_1, \dots, \mathbf{q}_n) = \sum_{m=1}^{n-1} \frac{G_m(\mathbf{q}_1, \dots, \mathbf{q}_m)}{(2n+3)(n-1)} \left[ (2n+1)\alpha(\mathbf{q}_1^m, \mathbf{q}_{m+1}^n) F_{n-m}(\mathbf{q}_{m+1}, \dots, \mathbf{q}_n) \right. \\ \left. + 2\beta(\mathbf{q}_1^m, \mathbf{q}_{m+1}^n) G_{n-m}(\mathbf{q}_{m+1}, \dots, \mathbf{q}_n) \right]$$

$$G_n(\mathbf{q}_1, \dots, \mathbf{q}_n) = \sum_{m=1}^{n-1} \frac{G_m(\mathbf{q}_1, \dots, \mathbf{q}_m)}{(2n+3)(n-1)} \left[ 3\alpha(\mathbf{q}_1^m, \mathbf{q}_{m+1}^n) F_{n-m}(\mathbf{q}_{m+1}, \dots, \mathbf{q}_n) \right. \\ \left. + 2n\beta(\mathbf{q}_1^m, \mathbf{q}_{m+1}^n) G_{n-m}(\mathbf{q}_{m+1}, \dots, \mathbf{q}_n) \right].$$

An example:

$$\delta^{(2)}(\vec{k}, \tau) = \int_{\vec{q}_1, \vec{q}_2} (2\pi)^3 \delta^D(\vec{k} - \vec{q}_1 - \vec{q}_2) F_2(\vec{q}_1, \vec{q}_2) \delta^{(1)}(\vec{q}_1, \tau) \delta^{(1)}(\vec{q}_2, \tau)$$

$$= \frac{5}{7} + \frac{1}{2} \frac{\vec{q}_1 \cdot \vec{q}_2}{q_1 q_2} \left( \frac{q_1}{q_2} + \frac{q_2}{q_1} \right) + \frac{2}{7} \frac{(\vec{q}_1 \cdot \vec{q}_2)^2}{q_1^2 q_2^2}$$

General cosmology :  $D_+^{\text{EdS}} \rightarrow D_+^{\Lambda\text{CDM}}$  is a very good approximation.

Properties of PT kernels:

$$1) \quad F_n(\underbrace{\vec{k}_1 + \vec{q}, \vec{k}_2 - \vec{q}, \vec{k}_3, \dots, \vec{k}_n}_{\text{how high } q \text{ affects low } k?}) \xrightarrow{q \gg k_i} \frac{k^2}{q^2}$$

Mass and momentum conservation fix it to be  $\sim k^2$ .



$p_A = p_B$  outside  
this region

$$(p_A - p_B)(\vec{k}) = \int_{\vec{x} \in R} d^3\vec{x} (p_A(\vec{x}) - p_B(\vec{x})) e^{-i\vec{k} \cdot \vec{x}}$$

on large scales  $kR \ll 1$ .

$$(p_A - p_B)(\vec{k}) = \underbrace{\int_{\vec{x} \in R} d^3\vec{x} (p_A - p_B)}_{=0, \text{ mass conservation}} - i\vec{k} \underbrace{\int_{\vec{x} \in R} d^3\vec{x} \cdot \vec{x} (p_A - p_B)}_{=0, \text{ momentum conservation}} + O(k^2 k^2)$$

$$2) \quad F_n(\vec{k}_1, \dots, \vec{k}_{n-1}, \vec{q}) \xrightarrow{q \ll k_i} \frac{\vec{q} \cdot \vec{k}_i}{q^2}$$

$$\text{An example: } F_2(\vec{k}_1, \vec{q}) = \frac{5}{7} + \frac{1}{2} \frac{\vec{k}_1 \cdot \vec{q}}{k_1 q} \left( \frac{k}{q} + \frac{q}{k} \right) + \frac{2}{7} \frac{(\vec{k}_1 \cdot \vec{q})^2}{k^2 q^2}$$

$$\text{When } q \ll k, \text{ then } F_2(\vec{k}_1, \vec{q} \ll k) \rightarrow \frac{1}{2} \frac{\vec{k}_1 \cdot \vec{q}}{q^2}$$

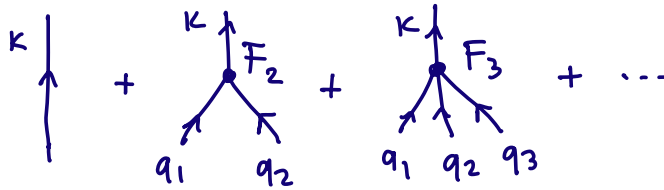
The origin of these terms is in large displacements or bulk flows.

$$(\int d\tau \cdot \vec{v}) \cdot \vec{\nabla} \delta \rightarrow \frac{\vec{\nabla}}{\nabla^2} \delta \cdot \vec{\nabla} \delta \xrightarrow{\text{in Fourier space}} \frac{\vec{q}_1 \cdot \vec{q}_2}{q_1^2} \delta(\vec{q}_1) \delta(\vec{q}_2)$$

$$\underbrace{\delta_{\text{short}}}_{\text{long mode } (\delta(\vec{q}), q \ll k)} \xrightarrow{\vec{v}} \delta_{\text{short}}(\vec{x} + \int d\tau \vec{v}) \simeq \delta_{\text{short}}(\vec{x}) + \underbrace{\int d\tau \vec{v} \cdot \vec{\nabla} \delta_{\text{short}}(\vec{x})}_{\text{wavy line}} + \dots$$

# Nonlinear correlation functions

$$\delta_{NL} = \delta^{(1)} + \delta^{(2)} + \delta^{(3)} + \dots$$



Example: bispectrum  $\langle \delta_{NL}(\vec{k}_1) \delta_{NL}(\vec{k}_2) \delta_{NL}(\vec{k}_3) \rangle$   
 $\equiv (2\pi)^3 \delta^D(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) B(\vec{k}_1, \vec{k}_2, \vec{k}_3)$

$$\langle \delta_{NL}(\vec{k}_1) \delta_{NL}(\vec{k}_2) \delta_{NL}(\vec{k}_3) \rangle =$$

$$\underbrace{\langle \delta^{(1)}(\vec{k}_1) \delta^{(1)}(\vec{k}_2) \delta^{(2)}(\vec{k}_3) \rangle}_{\text{three Wick contractions}} + 2 \text{ permutations (nonlinear } \delta^{(2)}(\vec{k}_i) \text{ and } \delta^{(2)}(\vec{k}_j))$$

$$\int \vec{q}_1 \int \vec{q}_2 (2\pi)^3 \delta^D(\vec{k}_3 - \vec{q}_1 - \vec{q}_2) \cdot F_2(\vec{q}_1, \vec{q}_2) \underbrace{\langle \delta^{(1)}(\vec{k}_1) \delta^{(1)}(\vec{k}_2) \delta^{(1)}(\vec{q}_1) \delta^{(1)}(\vec{q}_2) \rangle}_{\text{three Wick contractions}}$$

- 1)  $(2\pi)^3 \delta^D(\vec{k}_1 + \vec{k}_2) P(k_1) \cdot (2\pi)^3 \delta^D(\vec{q}_1 + \vec{q}_2) P(q_1) \rightarrow$  this is 0 since  $\vec{k}_1 \neq \vec{k}_2$
  - 2)  $(2\pi)^3 \delta^D(\vec{k}_1 + \vec{q}_1) P(k_1) \cdot (2\pi)^3 \delta^D(\vec{k}_2 + \vec{q}_2) P(k_2)$
  - 3)  $(2\pi)^3 \delta^D(\vec{k}_1 + \vec{q}_2) P(k_1) \cdot (2\pi)^3 \delta^D(\vec{k}_2 + \vec{q}_1) P(k_2)$
- } these give the same contribution.

$$B(\vec{k}_1, \vec{k}_2, \vec{k}_3) = 2F_2(\vec{k}_1, \vec{k}_2) P(k_1) P(k_2) + 2 \text{ permutations.}$$

Note:  $k_1 \ll k_2, k_3 \Rightarrow F_2(\vec{k}_1, \vec{k}_2) \sim \frac{1}{2} \frac{\vec{k}_1 \cdot \vec{k}_2}{k_1^2}$   
 the other permutation  $F_2(\vec{k}_1, \vec{k}_3) \sim \frac{1}{2} \frac{\vec{k}_1 \cdot \vec{k}_3}{k_1^2}$   
 $\vec{k}_2 \approx -\vec{k}_3 \rightarrow$  IR divergent terms cancel!

Nonlinear power spectrum:

$$\langle \delta_{NL} \delta_{NL} \rangle = \underbrace{\langle \delta^{(1)} \delta^{(1)} \rangle}_{\downarrow} + \underbrace{\langle \delta^{(2)} \delta^{(2)} \rangle}_{\downarrow} + \underbrace{\langle \delta^{(1)} \delta^{(3)} \rangle + \langle \delta^{(3)} \delta^{(1)} \rangle}_{\downarrow} + \dots$$

$$P_{NL}(k) = P_{\text{lin}}(k) + P_{22}(k) + P_{13}(k) + \dots$$

$$\begin{aligned}
 P_{22} &= \langle \text{diagram} \rangle \rightarrow \text{1-loop diagram} \\
 P_{13} &= 2 \langle \text{diagram} \rangle \rightarrow \text{1-loop diagram}
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} P_{22} \\ P_{13} \end{aligned}} \right\} \text{1-loop diagrams}$$

We have same number of integrals as  $\delta^D$  functions

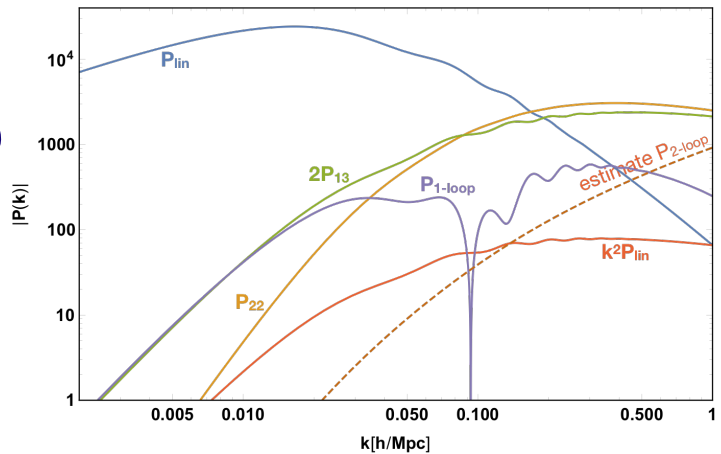
There is an overall  $\delta^D$  function  $\Rightarrow$  One momentum integral remains

We call this one-loop integral.

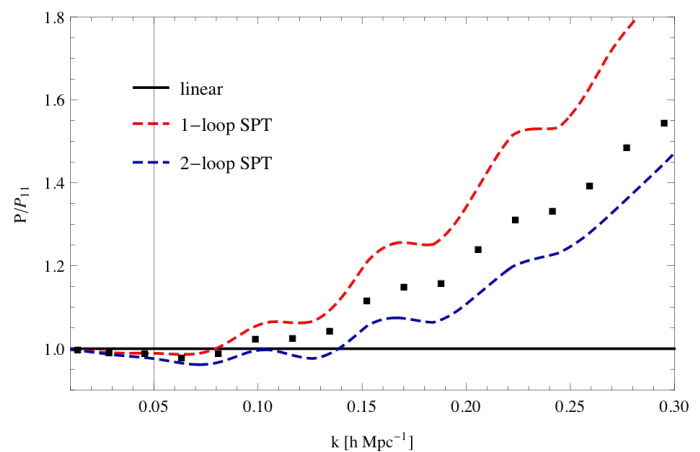
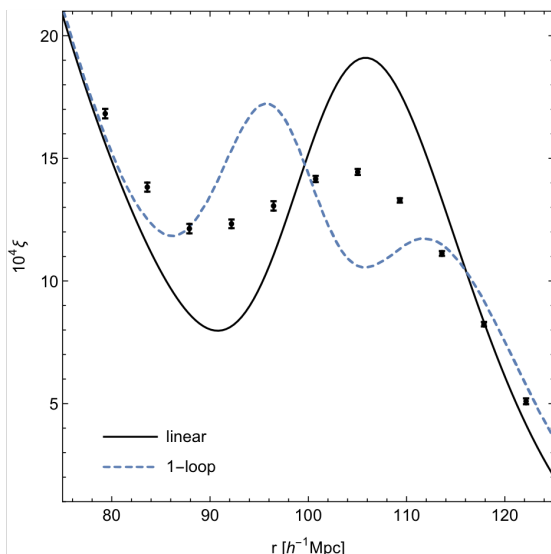
$$P_{22}(k, \tau) = 2 \int_{\vec{q}} F_2^2(\vec{q}, \vec{k} - \vec{q}) P_{\text{lin}}(\vec{q}, \tau) P_{\text{lin}}(\vec{k} - \vec{q}, \tau)$$

$$P_{13}(k, \tau) = 6 \int_{\vec{q}} F_3(\vec{q}, -\vec{q}, \vec{k}) P_{\text{lin}}(\vec{q}, \tau) P_{\text{lin}}(\vec{q}, \tau)$$

$$P_{22}, P_{13} \sim D_+^4$$



What went wrong?



What went wrong?

- 1) EOM were wrong
- 2) The solution we found was "incomplete"

Both of these things...

What are the correct equations of motion?

$$\frac{\partial f}{\partial \tau} + \frac{\vec{p}}{ma} \frac{\partial f}{\partial \vec{x}} - am \vec{\nabla} \phi \frac{\partial f}{\partial \vec{p}} = 0$$

$f(\vec{x}, \vec{p}, \tau)$  is the full phase-space distribution function.

But this system is collisionless!

However, the fluid-like description  
is not hopeless!

- 1) Mean free path set by the age of the Universe.

$$v_{DM} \sim O(100 \text{ km/s}) \quad \text{distance} \sim \text{few Mpc.}$$

- 2) DM forms DM halos.

$\sim \text{few Mpc.}$